Baire Category

(November, 2015 version)

G. Eric Moorhouse, University of Wyoming

The Baire Category Theorem was proved by René-Louis Baire in his PhD dissertation in 1899. It is a powerful tool for showing that a certain set is 'large' when other approaches seem inadequate even to show the given set is nonempty. After presenting and proving the theorem itself, we give two reasonably standard applications—one in real analysis, and one in group theory.

1. Complete Metric Spaces

We begin with a characterization of complete metric spaces. Let X be a metric space with distance function $d : X \times X \to [0, \infty)$. The *diameter* of a bounded nonempty subset $A \subset X$ is defined as

$$\operatorname{diam}(A) = \sup\{d(a,b) \,:\, a, b \in A\}.$$

If A is unbounded, then $\operatorname{diam}(A) = \infty$. Also if A is empty, we may define its diameter to be zero. Note that in a metric space, $\operatorname{diam} B_\varepsilon(x) \leqslant \operatorname{diam} \overline{B_\varepsilon(x)} \leqslant 2\varepsilon$.

1.1 Lemma. Consider a chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ of nonempty closed sets in a complete metric space X, with $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$. Then $\bigcap_n K_n \neq \varnothing$.

Proof. Choose points $x_n \in K_n$. For all $m \geqslant 0$ we have

$$d(x_{n+m}, x_n) \leqslant \operatorname{diam}(K_n) \to 0 \quad \text{as } n \to \infty$$

so the sequence $x_1, x_2, x_3, \ldots$ is Cauchy; it must converge to a point $x \in X$. Clearly $x \in K_n$ for all n and so $x \in \bigcap_n K_n$. □

The latter result has a valid converse, which we will not need here: In a metric space, if every chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ with $\operatorname{diam}(K_n) \to 0$ has nonempty intersection, then the space is complete. Note that the condition $\operatorname{diam}(K_n) \to 0$ is essential, since in $\mathbb{R}$ we have $\bigcap_n [n, \infty) = \varnothing$.

We will also use the following, which is easily proved.

1.2 Lemma. Every closed subspace of a complete metric space is complete. □

Next, some more preliminaries for general topological spaces.

2. Density

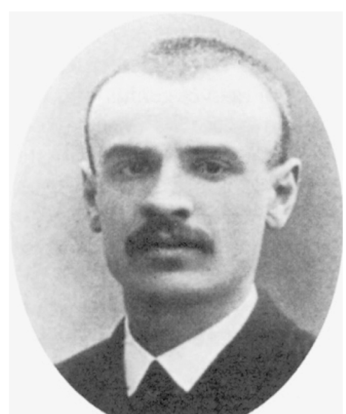
Recall: If A is a subset of a topological space X, the *closure of* A (denoted $\overline{A}$) is the smallest closed set containing A; the *interior of* A (denoted A°) is the largest open set contained in A. It is not hard to see that $\overline{X \setminus A} = X \setminus (A^\circ)$. Note that in a metric space, we have $\overline{B_\delta(x)} \subseteq B_\varepsilon(x)$ whenever $\delta < \varepsilon$.

We say A is *dense in* X, or simply *dense*, if $\overline{A} = X$, i.e. if every nonempty open set contains a point of A. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$. We say A is *somewhere dense* if its closure contains a nonempty open set; equivalently, $\overline{A}$ has nonempty interior, i.e. $\overline{A}^{\,\circ} \neq \varnothing$. If the closure of A has empty interior, i.e. $\overline{A}^{\,\circ} = \varnothing$, we say that A is *nowhere dense*. Thus $\mathbb{Q} \cap [0,1]$ is somewhere dense in $\mathbb{R}$, whereas $\left\{\frac{1}{n} : n = 1, 2, 3, \ldots\right\}$ is nowhere dense. If A has nonempty interior, then A is somewhere dense, since $\overline{A}^{\,\circ} \supseteq A^\circ \neq \varnothing$.

3. Baire Category

Let X be a topological space. We say X is of *first category* if X is a countable union of nowhere dense sets. Otherwise, X is of *second category*. For example, $\mathbb{Q}$ is of first category, since it is a countable union of singleton sets, each of which is nowhere dense in $\mathbb{Q}$. The first result is that this cannot happen in a complete metric space like $\mathbb{R}$.

3.1 Baire Category Theorem. Let X be a complete metric space. Then X is of second category. That is, if $X = A_1 \cup A_2 \cup A_3 \cup \cdots$, then for some n, the set A_n is somewhere dense in X.



René-Louis Baire
(1874–1932)

Proof. Suppose, on the contrary, that each A_n is nowhere dense in X. We recursively construct a sequence of points $x_1, x_2, x_3, \ldots$ in X as follows. First choose $x_1 \notin \overline{A_1}$; this is easy since $\overline{A_1}$ has empty interior and so must be a proper subset of X. There exists $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x_1) \cap \overline{A_1} = \varnothing$. Without loss of generality, we have $\varepsilon_1 < 1$.

Set $U_1 = B_{\varepsilon_1/2}(x_1)$, so that $\overline{U_1} \subseteq B_{\varepsilon_1}(x_1)$ and $\overline{U_1} \cap \overline{A_1} = \varnothing$. Since U_1 is a nonempty open set, it cannot be contained in $\overline{A_2}$. So we can choose $x_2 \in U_1 \setminus \overline{A_2}$. The latter set is open, so there exists $\varepsilon_2 > 0$ such that $B_{\varepsilon_2}(x_2) \subseteq U_1 \setminus \overline{A_2}$. Without loss of generality, $\varepsilon_2 < \frac{1}{2}$.

Set $U_2 = B_{\varepsilon_2/2}(x_2)$, so that $\overline{U_2} \subseteq B_{\varepsilon_2}(x_2)$ and $\overline{U_2} \cap \overline{A_1} = \varnothing$. Since U_2 is a nonempty open set, it cannot be contained in $\overline{A_3}$. Choose $x_3 \in U_2 \setminus \overline{A_3}$. Again, $U_2 \setminus \overline{A_3}$ is open, so there exists $\varepsilon_3 > 0$ such that $B_{\varepsilon_3}(x_3) \subseteq U_2 \setminus \overline{A_3}$. We may assume that $\varepsilon_3 < \frac{1}{3}$.

Continuing in this way, we obtain a sequence of points $x_1, x_2, x_3, \dots$ and open balls $U_n = B_{\varepsilon_n/2}(x_n)$ where $0 < \varepsilon_n < \frac{1}{n}$, satisfying

$$\overline{U_{n+1}} \;=\; \overline{B_{\varepsilon_{n+1}/2}(x_{n+1})} \;\subseteq\; B_{\varepsilon_{n+1}}(x_{n+1}) \;\subseteq\; U_n \smallsetminus \overline{A_{n+1}} \;\subseteq\; \overline{U_n} \;=\; \overline{B_{\varepsilon_n/2}(x_n)}$$

and $\operatorname{diam}(\overline{U_n}) \leqslant \frac{\varepsilon_n}{2} < \frac{1}{2n}$. By Lemma 1.1, there exists $x \in \bigcap_n \overline{U_n}$. But $x \in A_n$ for some n, and $x \in \overline{U_n} \cap \overline{A_n}$, a contradiction. □

3.2 Corollary. Let X be a complete metric space. Then a countable intersection of dense open subsets of X is again dense. That is, if $U_1, U_2, U_3, \dots$ are dense open subsets of X, then the intersection $D = U_1 \cap U_2 \cap U_3 \cap \cdots$ satisfies $\overline{D} = X$.

Proof. Let $x \in X$ and $\varepsilon > 0$. We must show that $B_\varepsilon(x) \cap D \neq \varnothing$. Define

$$Y = \overline{B_{\varepsilon/2}(x)} \;\subseteq\; B_\varepsilon(x);$$

we will show that in fact $Y \cap D \neq \varnothing$. By Lemma 1.2, Y is a complete metric space. Set $A_n = Y \smallsetminus U_n$, so that A_n is closed both in X and in Y. We first verify that A_n is nowhere dense in Y.

Suppose A_n is somewhere dense in Y. Then there exists $y \in Y$ and $\delta > 0$ such that

$$B_\delta(y) \cap Y \;\subseteq\; \overline{A_n} \cap Y \;=\; A_n\,.$$

This implies that $B_\delta(y) \cap (Y \smallsetminus A_n) = \varnothing$. But $y \in Y = \overline{B_{\varepsilon/2}(x)}$, so there exists a point $z \in B_\delta(y) \cap B_{\varepsilon/2}(x)$. Now the open set $B_\delta(y) \cap B_{\varepsilon/2}(x)$ is nonempty, so it contains a point of the dense set U_n, i.e. there exists a point

$$z' \;\in\; U_n \cap B_\delta(y) \cap B_{\varepsilon/2}(x) \;\subseteq\; U_n \cap B_\delta(y) \cap Y \;\subseteq\; U_n \cap A_n \;=\; \varnothing,$$

a contradiction.

Thus A_n is nowhere dense in Y as claimed. By Theorem 3.1, there exists a point

$$u \;\in\; Y \smallsetminus \bigcup_n A_n \;=\; \bigcap_n (Y \smallsetminus A_n) \;=\; \bigcap_n (Y \cap U_n) \;=\; Y \cap \bigcap_n U_n \;=\; Y \cap D$$

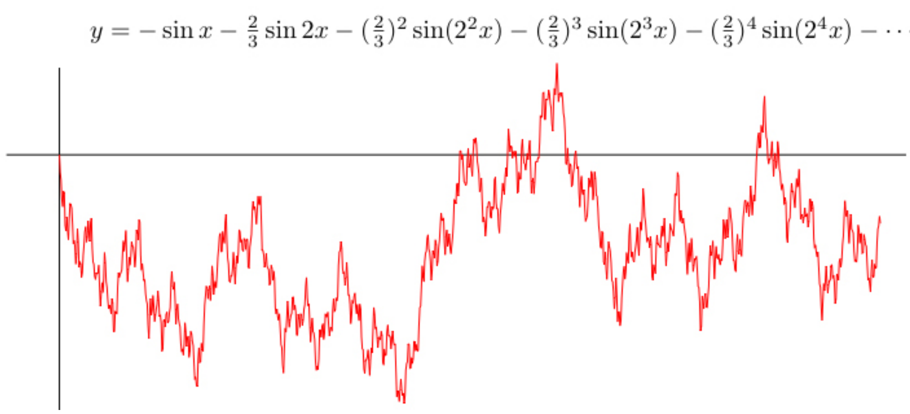
as required. □

As an application of Theorem 3.1, let $X = \mathbb{R}$ or $\mathbb{R}^2$. Suppose we colour all the points of X using a countable (finite or infinite) set of colours. We conclude that for some n, the set of points of colour n is somewhere dense in X.

As an application of Corollary 3.2, let $U_\ell = \mathbb{R}^2 \smallsetminus \ell$, the subset of the Euclidean plane formed by deleting a line ℓ. Note that U_ℓ is a dense open subset of the plane. If we take a countable collection $\mathcal{L}$ of lines in the plane, then the countable intersection $D = \bigcap_{\ell \in \mathcal{L}} U_\ell$ must be dense in the plane. This is not so trivial when $\mathcal{L}$ is (countably) infinite, in which case D may not actually be open.

4. Nowhere Differentiable Functions

Corollary 3.2 is typically used to give nonconstructive proofs of existence for various pathological objects, for example to show the existence of a function $f : [0,1] \to \mathbb{R}$ which is everywhere continuous but nowhere differentiable. The first explicit examples of such functions were given by Weierstrass (1872); an example of such a function, illustrated by Monge-Álvarez [A], is shown here:



Prior to the examples of Weierstrass, it was generally thought that continuous functions were differentiable except at isolated points. If one wants only to show the existence of a continuous but nowhere-differentiable function, it turns out to be easier to prove the stronger statement (see Theorem 4.2 below) that in the sense of Baire category, *most* continuous functions $\mathbb{R} \to \mathbb{R}$ are nowhere differentiable. For convenience, here we consider just continuous functions $[0,1] \to \mathbb{R}$; but the case for domain $\mathbb{R}$ works almost as easily.

Let $V = \mathcal{C}([0,1])$, the real vector space of all continuous functions $[0,1] \to \mathbb{R}$. For $f \in V$, define $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Note that the supremum exists and is finite, since $[0,1]$ is compact. Then $\|\cdot\|$ is a norm on V, and V is a complete metric space (i.e. a Banach space). For $m, n \geqslant 1$, define $A_{m,n}$ to be the set of all $f \in V$ such that for some $x \in [0,1]$, we have

$$(*) \qquad \left| \frac{f(y) - f(x)}{y - x} \right| \leqslant m \quad \text{whenever } \ 0 < |y - x| < \tfrac{1}{n}\,.$$

4.1 Lemma.
(a) If f is differentiable at some point $x \in [0,1]$, then $f \in A_{m,n}$ for some $m, n \geqslant 1$.
(b) Each $A_{m,n}$ is closed and nowhere dense in V.

Proof. (a) Suppose f is differentiable at $x \in [0,1]$. There exists $n \geqslant 1$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < 1 \text{ whenever } 0 < |y - x| < \tfrac{1}{n}\,.$$

For all such y, we have $\left|\frac{f(y)-f(x)}{y-x}\right| \leqslant m$ where $m = \lceil |f'(x)| + 1 \rceil$.

(b) To show that $A_{m,n}$ is closed in V, consider a sequence $f_1, f_2, f_3, \ldots$ in $A_{m,n}$ and suppose that $f_k \to f \in V$. Since f is a uniform limit of continuous functions f_k (recall that the topology of V is defined by the sup norm), f is itself continuous. By definition, for each k there exists $x_k \in [0,1]$ such that

$$\left| \frac{f_k(y) - f_k(x_k)}{y - x_k} \right| \leqslant m \quad \text{whenever} \quad 0 < |y - x_k| < \tfrac{1}{n}\,.$$

By the Bolzano-Weierstrass Theorem, the sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence. We may suppose (after replacing the sequence $(x_k)_k$ by this convergent subsequence, if necessary) that $x_k \to x \in [0,1]$. Now suppose that $0 < |y - x| < \frac{1}{n}$. Then there exists K such that $0 < |y - x_k| < \frac{1}{n}$ for all $k > K$; and for all such k we have

$$(\dagger) \qquad \left| \frac{f_k(y) - f_k(x_k)}{y - x_k} \right| \leqslant m.$$

Now as $k \to \infty$ we have

$$|f_k(x_k) - f(x)| \;\leqslant\; |f_k(x_k) - f(x_k)| + |f(x_k) - f(x)| \;\leqslant\; \|f_k - f\| + |f(x_k) - f(x)| \;\to\; 0$$

since $f_k \to f$, $x_k \to x$ and f is continuous; so letting $k \to \infty$ in ($\dagger$) we obtain (*), i.e. $f \in A_{m,n}$. Thus $A_{m,n}$ is closed in V as required.

Now it is easy to see that $A_{m,n} \subset V$ is nowhere dense. For otherwise, there is an open ball $B_\delta(f) \subseteq \overline{A_{m,n}} = A_{m,n}$ for some $f \in V$ and $\delta > 0$. In this case there exists $x \in [0,1]$ such that (*) holds. Consider the continuous function $g \in V$ defined by

$$g(y) = f(y) + \tfrac{\delta}{2} \sin\left[\tfrac{4m}{\delta}(y - x)\right]$$

so that $\|g - f\| \leqslant \frac{\delta}{2}$. However if $g \in A_{m,n}$ then

$$\frac{\delta}{2} \left| \frac{\sin\left[\frac{4m}{\delta}(y - x)\right]}{y - x} \right| \leqslant m \quad \text{whenever} \quad 0 < |y - x| < \tfrac{1}{n}\,;$$

and letting $y \to x$ we obtain $2m \leqslant m$, a contradiction. Thus $A_{m,n}$ is nowhere dense in V as claimed. $\square$

In view of (b), Theorem 3.1 shows that the countable union $\bigcup_{m,n} A_{m,n}$ is a proper subset of V; therefore there exists $f \in V$ such that f is not contained in any of the sets $A_{m,n}$. By (a), such a function f must be nowhere differentiable.

Let $U_{m,n} = V \smallsetminus A_{m,n}$. Then $U_{m,n}$ is open. Also $U_{m,n}$ is dense since

$$\overline{U_{m,n}} = \overline{V \smallsetminus A_{m,n}} = V \smallsetminus A^\circ_{m,n} = V,$$

using the fact that $A^\circ_{m,n} = (\overline{A_{m,n}})^\circ = \varnothing$. By Corollary 3.2, the countable intersection $\bigcap_{m,n} U_{m,n}$ is also dense in V. But $\bigcap_{m,n} U_{m,n}$ consists of nowhere differentiable functions. We have:

4.2 Theorem. The set of nowhere differentiable functions is dense in V. $\square$

5. Infinite Permutation Groups

Much of this topic is drawn from Cameron [C] who points out (see [C, §2.2]) that for model-theoretic reasons, many fundamental questions about permutations of an infinite set Ω reduce to the case $|\Omega| = \aleph_0$. So without wasting time, we will take $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $G = \operatorname{Sym}\mathbb{N}$, the group of all permutations of $\mathbb{N}$, i.e. the set of all bijections $\mathbb{N} \to \mathbb{N}$ under composition. This group has cardinality

$$|G| = |\mathbb{R}| = |C| = 2^{\aleph_0}$$

where C is the Cantor space consisting of all sequences $(a_1, a_2, a_3, \ldots)$ with each $a_i \in \{0, 1\}$. To see this, first note that $|G| \geqslant |C| = 2^{\aleph_0}$ since we have an injection

$$C \to G, \quad (a_1, a_2, a_3, \ldots) \mapsto (1,2)^{a_1}(3,4)^{a_2}(5,6)^{a_3}(7,8)^{a_4}\cdots.$$

Here we use cycle notation for permutations; thus the permutation on the right maps

$$2k-1 \mapsto \begin{cases} 2k-1, & \text{if } a_k = 0; \\ 2k, & \text{if } a_k = 1; \end{cases} \qquad 2k \mapsto \begin{cases} 2k, & \text{if } a_k = 0; \\ 2k-1, & \text{if } a_k = 1. \end{cases}$$

On the other hand, $|G| \leqslant |\mathbb{R}| = 2^{\aleph_0}$ since we have an injection

$$G \to \mathbb{R}, \qquad f \mapsto f(1) + \cfrac{1}{f(2) + \cfrac{1}{f(3) + \cfrac{1}{f(4) + \cfrac{1}{f(5) + \cdots}}}};$$

here we use the fact that continued fraction expansions of real numbers are unique (at least in the case of nonterminating expansions).

Note that G has a subgroup of order $n!$ permuting $\{1, 2, 3, \ldots, n\}$ and fixing every $k > n$; this subgroup is usually denoted S_n. Also G has a subgroup $S_\infty < G$ consisting of all permutations $f \in G$ of *finite support*, i.e.

$$S_\infty = \{f \in G : \text{for some } N,\ f(k) = k \text{ whenever } k \geqslant N\} = \bigcup_{n=1}^{\infty} S_n.$$

Since S_∞ is a countable union of finite sets, S_∞ is countably infinite:

$$|S_\infty| = \aleph_0.$$

This says that S_∞ is a relatively small subgroup of G; evidently most permutations of $\mathbb{N}$ have infinite support. By Lagrange's Theorem, every subgroup $H \leqslant G$ satisfies $2^{\aleph_0} = |G| = [G : H]|H|$, so either the order or the index of H (possibly both) must equal $2^{\aleph_0}$. (The product of two infinite cardinals is equal to their maximum.) In our case S_∞ must have index $2^{\aleph_0}$: there are $2^{\aleph_0}$ cosets of S_∞ in $G = \operatorname{Sym} \mathbb{N}$.

We introduce an ultrametric on G as we have seen how to do on any set of infinite sequences: given $f \neq g$ in G define

$$d_0(f, g) = 2^{-n} \quad \text{where } n \text{ is the smallest integer such that } f(n) \neq g(n);$$

also define $d_0(f, f) = 0$. Since $f(k) = g(k)$ iff $(g^{-1} \circ f)(k) = k$, we immediately have

$$d_0(f, g) = d_0(g^{-1} \circ f, \iota) = d_0(f^{-1} \circ g, \iota)$$

where $\iota \in G$ is the identity, i.e. $\iota(k) = k$ for all $k \in \mathbb{N}$. As nice as this is, we can do better:

$$d(f, g) = \max\{d_0(f, g),\ d_0(f^{-1}, g^{-1})\}$$

defines a nicer metric on G which satisfies

5.1 Theorem. The space G is complete with respect to the metric d. □

The proof is left as an exercise (see [C, §2.4]). We remark that the space G is *not* complete for the metric d_0; for example the sequence

$$\iota,\ (1, 2),\ (1, 2, 3),\ (1, 2, 3, 4), \ldots$$

defined by $f_1 = \iota$ and $f_n = (1, 2, \ldots, n)$ for $n \geqslant 2$ satisfies

$$d_0(f_m, f_n) = 2^{-m} \text{ whenever } m < n$$

giving a Cauchy sequence for d_0, without any limit in G. (The sequence converges in $\mathbb{N}^{\mathbb{N}}$ to the unilateral shift $k \mapsto k+1$, but this is not a permutation of $\mathbb{N}$.) The problem does not arise with the new metric d since the sequence is no longer Cauchy; indeed $d(f_m, f_n) = \frac{1}{2}$ whenever $m \neq n$ since $f_m^{-1}(1) = m$ whereas $f_n^{-1}(1) = n$. Nevertheless the two metrics d_0 and d define the same topology on G (see [C, §2.4]). Rather than formally proving this, we give an example expressing an open ball for d_0 in terms of open balls for d:

$$\begin{aligned}\big\{f \in G : d_0(f,(1,3,5)) < \tfrac{1}{2}\big\} &= \big\{f \in G : f(1) = 3 \text{ and } f(2) = 2\big\} \\ &= \bigcup_{k=3}^{\infty} \big\{f \in G : f(1) = 3,\ f(k) = 1 \text{ and } f(2) = 2\big\} \\ &= \bigcup_{k=3}^{\infty} \big\{f \in G : f(1) = 3,\ f^{-1}(1) = k \text{ and } f(2) = 2\big\} \\ &= \big\{f \in G : d(f,(1,3)) < \tfrac{1}{2}\big\} \cup \bigcup_{k=4}^{\infty} \big\{f \in G : d(f,(1,3,k)) < \tfrac{1}{2}\big\}\end{aligned}$$

and vice versa:

$$\begin{aligned}\big\{f \in G : d(f,(1,3,5)) < \tfrac{1}{2}\big\} &= \big\{f \in G : f(1) = 3,\ f(2) = 2 \text{ and } f(5) = 1\big\} \\ &= \bigcup_{k \geqslant 4} \bigcup_{\substack{\ell \geqslant 4 \\ \ell \neq k}} \big\{f \in G : f(1) = 3,\ f(2) = 2,\ f(3) = k,\ f(4) = \ell \text{ and } f(5) = 1\big\} \\ &= \bigcup_{s \in S} \big\{f \in G : d_0(f,s) < \tfrac{1}{32}\big\}\end{aligned}$$

where

$$\begin{aligned}S = {} & \big\{(1,3,5),\ (1,3,4,5)\big\} \cup \big\{(1,3,k,\ell,4,5) : k,\ell \geqslant 6 \text{ and } k \neq \ell\big\} \\ & \cup \big\{(1,3,k,5),\ (1,3,k,4,5) : k \geqslant 6\big\} \\ & \cup \big\{(1,3,5)(4,\ell),\ (1,3,4,\ell,5) : \ell \geqslant 6\big\}.\end{aligned}$$

Similar relations show that sets of the form $\{f \in G : f(k) = \ell\}$ where $k, \ell \in \mathbb{N}$ form a subbasis for the topology of G; and basic open sets are obtained by instead imposing finitely many conditions $f(k_i) = \ell_i$ for $i = 1, 2, \ldots, n$. The group G has the *topology of pointwise convergence*: that is, a sequence $(f_n)_n$ in G converges to $f \in G$ iff for every $k \in \mathbb{N}$, we have $f_n(k) \to f(k)$ as $n \to \infty$. Since $\mathbb{N}$ is discrete, this means that for all $k \in \mathbb{N}$, there exists N such that for all $n > N$, we have $f_n(k) = f(k)$. Thus G is homeomorphic to a subspace of $\mathbb{N}^{\mathbb{N}}$ (a countable product of countable discrete spaces). Each of our subbasic open sets is also closed (i.e. clopen) since

$$\{f \in G : f(k) = \ell\} = G \setminus \bigcup_{m \neq \ell} \{f \in G : f(k) = m\}.$$

So after intersecting finitely many subbasic open sets, we see that every basic open set is also clopen.

5.2 Theorem. G is a topological group for the metric d. □

This says that multiplication in G, i.e. the map $G \times G \to G$, $(f, g) \mapsto f \circ g$, is continuous; also inversion $G \to G$, $f \mapsto f^{-1}$ is continuous. The proof is omitted; but see [C, Sec.2.4].

The subgroup $S_\infty < G$ is not (topologically) closed; in fact it is dense in G: its closure is $\overline{S_\infty} = G$. To see this, let $g \in G$ and $n \geqslant 1$; then the basic open neighbourhood

$$\{f \in G : f(k) = g(k) \text{ for all } k \leqslant n\}$$

of g contains a point of S_∞: simply take $N = \max\{n, g(k) : k \leqslant n\}$ so there exists $f \in S_N < S_\infty$ satisfying $f(k) = g(k)$ for all $k \leqslant n$. However if H is any subgroup of G, then its (topological) closure is also a subgroup. To see this, let $g, g' \in \overline{H}$, so there exist sequences $(h_n)_n, (h'_n)_n$ in H converging to g and g' respectively; then $gg' = \lim h_n h'_n \in \overline{H}$. A similar argument works for inverses in $\overline{H}$.

5.3 Theorem. Let H and K be closed subgroups of $G = \operatorname{Sym}\mathbb{N}$ and $K \leqslant H$. Then $[H : K] \leqslant \aleph_0$ iff K contains the stabilizer of some finite subset $A \subset \mathbb{N}$, i.e. $K \supseteq H_A$ where

$$H_A = \{h \in H : h(A) = A\} \supseteq \{h \in H : h(a) = a \text{ for all } a \in A\}.$$

Proof. Assuming that $K \supseteq H_A$, the H-orbit of A, namely

$$A^H = \{h(A) : h \in H\} \text{ where } h(A) = \{h(a) : a \in A\},$$

has size $|A^H| = [H : H_A] = [H : K][K : H_A]$ (see, e.g. [C2, p.5]). Since $\mathbb{N}$ has only countably many subsets of size $|A|$, this implies that $[H : K] \leqslant |A^H| \leqslant \aleph_0$.

Conversely, suppose we have closed subgroups satisfying $K \leqslant H \leqslant G = \operatorname{Sym}\mathbb{N}$. Since G is a complete metric space, the closed subgroups H and K are also complete. Suppose also that $[H : K] \leqslant \aleph_0$, so that $H = \bigcup_n K h_n$ where the set of coset representatives $h_n \in H$ is countable. Each of the cosets Kh_n is clopen (closed because K itself is closed, and multiplication by h_n is continuous; but also open, because the complement $H \smallsetminus K h_n$ is the union of the other cosets). By the Baire Category Theorem 3.1, some coset Kh_n has nonempty interior. Again, right-multiplying by h_n^{-1} (a continuous operation) means that K itself has nonempty interior; thus K contains an open neighbourhood $H \cap B_\delta(k) \subseteq K$ for some $\delta > 0$ and $k \in K$. We may suppose $k = \iota$; otherwise $\iota \in k^{-1}B_\delta(k) \subseteq K$ so K also contains an open neighbourhood of ι in H:

$$K \supseteq H \cap B_{1/2^n}(\iota) = H_{\{1,2,3,\ldots,n-1\}}.$$ □

The index $[H : K]$ in Theorem 5.3 is conceivably any cardinality $\leqslant |G| = 2^{\aleph_0}$. Now the Continuum Hypothesis is the assertion that there is no set A satisfying $\aleph_0 < |A| < 2^{\aleph_0}$.

However, this statement is independent of the usual axioms ZFC of set theory (the Zermelo-Fraenkel axioms ZF, together with the Axiom of Choice). This means (assuming ZFC is consistent, as we are generally prone to believe) that one can neither prove nor disprove the Continuum Hypothesis using the ZFC axioms—this fact is due to the combined work of Kurt Gödel (1940) and Paul Cohen (1963). Most modern mathematics is founded upon ZFC and has no use for the Continuum Hypothesis. The following, which would be a trivial assertion under the Continuum Hypothesis, is actually proved using only the standard axioms of ZCF, *without* assuming the Continuum Hypothesis. See [C, p.29] and [DNT].

5.4 Theorem. For every subgroup $H \leqslant G = \operatorname{Sym}\mathbb{N}$, we have either $[G:H] \leqslant \aleph_0$ or $[G:H] = 2^{\aleph_0}$.

The key to proving the following (see [C3]) is again the Baire Category Theorem:

5.5 Theorem. Let G be a primitive permutation group on an infinite set X. Then G preserves a nontrivial topology on X iff G preserves a filter on X.

Macpherson and Praeger [MP] make further extensive use of topological methods in the study of infinite permutation groups. A representative result of theirs is:

5.6 Theorem. Let G be a permutation group on $\mathbb{N}$ that does not preserve any nontrivial filter (i.e. the only filters fixed by G are the indiscrete filter $\{\mathbb{N}\}$ and the finite complement filter). Then G is highly transitive (i.e. for every $n \geqslant 1$, G transitively permutes the n-subsets of $\mathbb{N}$).

References

[A] J. Monge-Álvarez, 'Weierstrass cosine function', website accessed Nov. 20, 2015: http://www.mathworks.com/matlabcentral/fileexchange/50292-weierstrass-cosine-function--wcf-

[C] P.J. Cameron, *Oligomorphic Permutation Groups*, Camb. Univ. Press, Cambridge, 1990.

[C2] P.J. Cameron, *Permutation Groups*, Camb. Univ. Press, Cambridge, 1999.

[C3] P.J. Cameron, 'Topology in permutation groups', pp.93–105 in: *Groups: Topological, Combinatorial and Arithmetic Aspects*, ed. T.W. Müller, Camb. Univ. Press, Camidge, 2004.

[DNT] J.D. Dixon, P. Neumann and S. Thomas, 'Subgroups of small index in infinite symmetric groups', *Bull. London Math. Soc.* **18** (1986), 580–586.

[MP] H.D. Macpherson and C.E. Praeger, 'Maximal subgroups of infinite symmetric groups', *J. London Math. Soc.* (2) **42** (1990), 85–92.