Putting Together all the Fields $\mathbb{Q}_v$:

The Adèles, the Mellin Transform and the Riemann Zeta Function

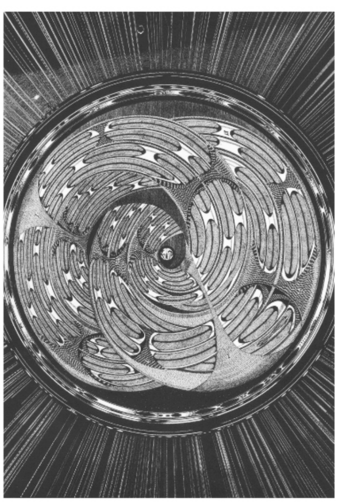
Recall that the *Riemann zeta function* is the meromorphic function defined in the right half-plane $\Re(s) > 1$ by the series

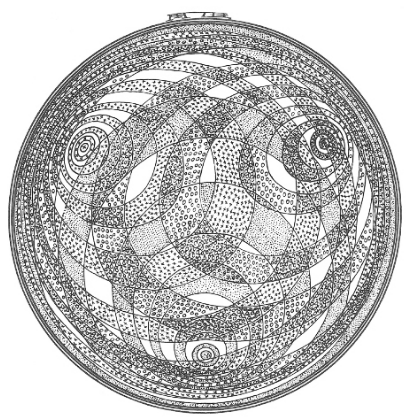
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

The function $\zeta(s)$ has a simple pole at $s = 1$ and elsewhere it is complex analytic. Although the series above converges only for $\Re(s) > 1$, there is a unique function $\zeta(s)$ satisfying the properties above, by general results on analytic continuation. The standard approach to proving this is to first analytically continue $\zeta(s)$ to the half-plane $\Re(s) > 0$, and then to extend to the rest of the plane using the functional identity

$$\zeta(1-s) = \pi^{\frac{1-2s}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \zeta(s).$$

Although we have simply pulled up this identity as a rabbit out of a hat, we will find a perfectly logical explanation for the mysterious factor above. This revelation depends on an understanding of $\mathbb{Q}$ via its embedding in all its completions $\mathbb{Q}_v$ simultaneously (for $v \in \{\infty, 2, 3, 5, 7, 11, \ldots\}$), thus granting to all the p-adic fields $\mathbb{Q}_p$ the same status as the Archimedean field $\mathbb{Q}_\infty = \mathbb{R}$. Given the importance of the Riemann zeta function, as is clear from its role in the foremost open problem in mathematics (the Riemann Hypothesis), we regard this as a compelling reason for studying the p-adic fields $\mathbb{Q}_p$ alongside $\mathbb{R}$.





◀ A 2-adic Solenoid ▲ The 3-adic Unit Disk
artistic conceptions by
Anatoly T. Fomenko (1945–)

One should also keep in mind that the field $\mathbb{Q}$ may be replaced by an arbitrary number field K throughout (meaning that $K \supseteq \mathbb{Q}$ is a finite extension field), while replacing $\zeta(s)$ by the corresponding Dedekind zeta function $\zeta_K(s)$; then v ranges over all valuations of K, i.e. the finitely many Archimedean as well as the infinitely many non-Archimedian 'places' of K). Even more generally, all of this generalizes to Dirichlet L-functions, of which zeta functions are but a special case.

This document is not intended to fully justify the functional equation and other properties of zeta functions, but rather to show enough of the key ideas in its derivation to reveal the parallel roles played by $\mathbb{R}$ and the p-adic fields $\mathbb{Q}_p$ which, when fully implemented, do yield the functional equation of $\zeta(s)$ and much more. For complete explanations, see the famous 1950 thesis of John Tate, which is available as the last chapter in *Algebraic Number Theory*, ed. Cassels and Fröhlich, 1967. Another more recent treatment is Deitmar's *Automorphic Forms*, 2013.

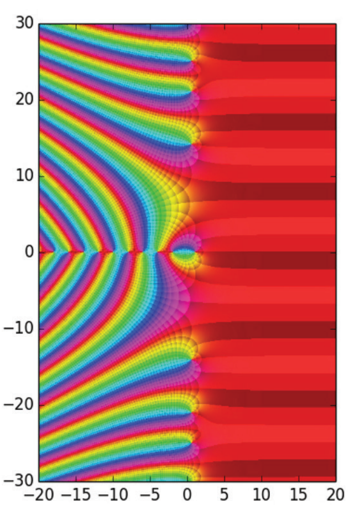
I express my gratitude to Joseph Repka, my official graduate advisor. The main ideas presented in this summary are extracted from notes I took as a graduate student in his course during October, 1984.

Euler Factorization

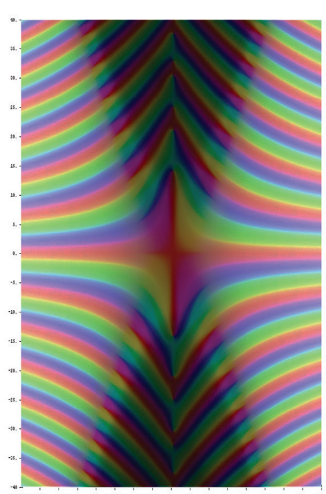
Euler observed the factorization

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

valid for $\Re(s) > 1$, where p ranges over all rational primes. This factorization follows easily from (and is essentially a reformulation of) the Fundamental Theorem of Arithmetic: the fact that every positive integer has a unique factorization as a product of primes. The factor $(1 - p^{-s})^{-1}$ is called the *Euler factor* for $\zeta(s)$ at the prime p. A similar factorization holds for zeta functions, and even for L-functions, defined over general number fields; and once again the factorization is a consequence of unique factorization—not of individual elements, but of ideals in Dedekind domains.



Plot of $\zeta(s)$
(Empetrisor, 2014)



Plot of $\xi(s)$
(J. Homann, 2008)

We obtain a first glimpse of the functional equation of $\zeta(s)$ by recognizing that in addition to Euler factors for each of the finite primes p, one should also include Euler factors for each of the infinite primes v. In the case of $\mathbb{Q}$, the missing Euler factor at $v = \infty$ is

$$\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)$$

which, when included, yields the complete zeta function†. Note that the equation $\xi(1-s) = \xi(s)$ expresses the symmetry of $\xi(s)$ about Riemann's critical line $\Re(s) = \frac{1}{2}$.

Haar Measure

Let G be an abelian topological group with binary operation '$*$'. Thus $x * y = y * x$ for all $x, y \in G$; the map $G \times G \to G$, $(x, y) \mapsto x * y$ is continuous; and the map $G \to G$, $x \mapsto x^{-1}$ is continuous (where x^{-1} denotes the inverse of x with respect to $*$).

Let μ be a measure on G. We say that μ is *translation-invariant* if $\mu(g * X) = \mu(X)$ for every measurable subset $X \subseteq G$ and every $g \in G$. Here we denote a typical translate of X by $g * X = \{g * x : x \in X\}$; and note in particular that translates of measurable subsets are also measurable.

Under certain technical (but very reasonable) assumptions*, Haar's theorem guarantees the existence and essential uniqueness (i.e. up to nonzero scalar multiple) of such a translation-invariant measure on G. Such a measure on G is called *Haar measure.*

Example 1. If G is a finite group, with the discrete topology, then Haar measure is simply counting measure. We usually normalize μ so that $\mu(X) = \frac{|X|}{|G|}$ for all $X \subseteq G$.

Example 2. Let G be the additive group of $\mathbb{R}$. Then Haar measure λ^+ is simply Borel measure, which assigns to each interval $[a, b]$ its length $\lambda^+([a, b]) = b - a$. For measurable subsets $E \subset \mathbb{R}$ we have $\lambda^+(E) = \int_E dx$.

Example 3. Consider the multiplicative group $(0, \infty)$ of positive real numbers. Recall that this is isomorphic to the additive reals of Example 2 via the isomorphism $\ln : (0, \infty) \to \mathbb{R}$.

† Some authors have included the additional factor $s(s-1)/2$. The symmetry $\xi(1-s) = \xi(s)$ holds with or without this additional factor; but $s(s-1)/2$ is not part of any of the Euler factors. The only purpose of this additional factor is to force the function $\xi(s)$ to be entire; and the practice of including it has been decried by several experts.

* We assume that G is locally compact and Hausdorff; and μ is defined on the Borel subsets of G, i.e. members of the σ-algebra generated by the open subsets of G. We further assume that μ is nontrivial; i.e. it is not identically zero, yet every compact subset of G has finite measure.) Haar's Theorem states that there is a nontrivial countably additive measure μ defined on the Borel subsets of G; and that such a measure μ is unique up to positive scalar multiple. If G were non-abelian, we would instead have to consider measures invariant under left-translation and right-translation, leading to left and right Haar measures on G; and these in general do not coincide.

Haar measure $\lambda^\times$ on $(0,\infty)$ is found by pulling back λ^+ to $(0,\infty)$: for measurable subsets $E\subset(0,\infty)$,

$$\lambda^\times(E)=\lambda^+(\ln E)=\int_{\ln E}dx=\int_E\frac{dx}{x}$$

where $\ln E:=\{\ln x: x\in E\}$. The translation invariance of $\lambda^\times$ is expressed by $\frac{d(ax)}{ax}=\frac{dx}{x}$.

Example 4. The full multiplicative group of nonzero reals $\mathbb{R}^\times=\{\pm\infty\}\times(0,\infty)$ has Haar measure found by slightly extending the measure of Example 3: for measurable $E\subset\mathbb{R}^\times$,

$$\lambda^\times(E)=\int_E\frac{dx}{|x|}\,.$$

Example 5. Consider the additive group $\mathbb{Q}_p$, which is locally compact Hausdorff. The Borel subset $\mathbb{Z}_p=\{x\in\mathbb{Q}_p:\|x\|_p\leqslant 1\}$ has a partition

$$\mathbb{Z}_p=A_0\sqcup A_1\sqcup\cdots\sqcup A_{p-1}$$

where $A_j=j+p\mathbb{Z}_p=\big\{j+pa_1+p^2a_2+p^3a_3+\cdots: a_i\in\{0,1,2,\ldots,p-1\}\big\}$. The translation invariance of Haar measure λ^+ on $\mathbb{Q}_p$ means that

$$\lambda^+(A_0)=\lambda^+(A_1)=\cdots=\lambda^+(A_{p-1}).$$

After normalizing so that $\lambda^+(\mathbb{Z}_p)=1$, we have

$$\lambda^+(A_j)=\lambda^+(p\mathbb{Z}_p)=\tfrac{1}{p}\,.$$

More generally, $\lambda^+(aE)=\|a\|_p\lambda^+(E)$ for any $a\in\mathbb{Q}_p$ and measurable $E\subset\mathbb{Q}_p$. In fact by local compactness, every measurable $E\subset\mathbb{Q}_p$ has a partition of the form $E=\bigsqcup_\alpha(j_\alpha+p^{k_\alpha}\mathbb{Z}_p)$ for which

$$\lambda^+(E)=\int_E dx=\sum_\alpha p^{-k_\alpha}.$$

Example 6. Consider the multiplicative group of nonzero p-adic numbers $\mathbb{Q}_p^\times$. Every $a\in\mathbb{Q}_p^\times$ factors uniquely as $a=p^ku$ where $k\in\mathbb{Z}$ and $u=a_k+pa_{k+1}+p^2a_{k+2}+\cdots\in\mathbb{Z}_p^\times$; here each $a_i\in\{0,1,2,\ldots,p-1\}$ and $a_k\neq 0$. This gives $\mathbb{Q}_p^\times=\langle p\rangle\times\mathbb{Z}_p^\times$ where $\langle p\rangle=\{\ldots,p^{-2},p^{-1},1,p,p^2,\ldots\}$ is infinite cyclic (isomorphic to the additive group $\mathbb{Z}$). Note the similarity to Example 4. We want to normalize the Haar measure so that $\lambda^\times(\mathbb{Z}_p^\times)=1$. Using the partition

$$\mathbb{Z}_p^\times=(1+p\mathbb{Z}_p)\sqcup(2+p\mathbb{Z}_p)\sqcup\cdots\sqcup(p{-}1+p\mathbb{Z}_p)$$

together with the fact that

$$\lambda^\times(j+p\mathbb{Z}_p)=\lambda^\times\big(j(1+p\mathbb{Z}_p)\big)=\lambda^\times(1+p\mathbb{Z}_p)$$

for each $j\in\{1,2,\ldots,p{-}1\}\subset\mathbb{Z}_p^\times$, we see that

$$\lambda^\times(j+p\mathbb{Z}_p)=\tfrac{1}{p-1}$$

for each $j\in\{1,2,\ldots,p{-}1\}$. Similarly, for each $k\geqslant 1$ and $j\in\{1,2,\ldots,p{-}1\}$ we have

$$\lambda^\times(j+p^k\mathbb{Z}_p)=\tfrac{p^{1-k}}{p-1}.$$

More generally, each measurable set $E\subset\mathbb{Q}_p^\times$ has a partition $E=\bigsqcup_\alpha(j_\alpha+p^{k_\alpha}\mathbb{Z}_p)$ where $j_\alpha\in\{1,2,\ldots,p{-}1\}$ and $\lambda^\times(j_\alpha+p^{k_\alpha}\mathbb{Z}_p)=\frac{1}{p-1}$, giving

$$\lambda^\times(E)=\sum_\alpha\frac{p^{1-k}}{p-1}=\frac{p}{p-1}\int_E\frac{dx}{\|x\|_p}.$$

Thus the approriate differential for $\lambda^\times$ is $\frac{p}{p-1}\frac{dx}{\|x\|_p}$.

Additive and Multiplicative Characters

For each real number a, the map

$$\chi_a^+:\mathbb{R}\to\mathbb{C}^\times,\quad \chi_a^+(x)=e^{2\pi iax}$$

is an *additive character*: it is a continuous map satisfying $\chi_a^+(x+x')=\chi_a^+(x)\chi_a^+(x')$ and $|\chi_a^+(x)|=1$ for all x. If we drop the 'unitary' requirement $|\chi_a^+(x)|=1$, then we have also *quasicharacters* of the form $x\mapsto e^{sx}$ for arbitrary $s\in\mathbb{C}$. While the term 'character' is understood more broadly in representation theory, in this context we are considering just linear characters (i.e. characters of degree 1) defined on the additive group of $\mathbb{R}$.

The corresponding notion of character on the multiplicative group $\mathbb{R}^\times$ would be a continuous map $\chi^\times:\mathbb{R}^\times\to\mathbb{C}^\times$ satisfying $\chi^\times(xx')=\chi^\times(x)\chi^\times(x')$ and the unitary requirement $|\chi^\times(x)|=1$. Although there are no nontrivial examples of such characters satisfying the unitary condition, we do have nontrivial quasicharacters: for each $a\in\mathbb{C}$ we have $\chi_a^\times(x)=|x|^a=e^{a\ln|x|}$. Moreover, considering that $\mathbb{R}^\times\cong\{\pm1\}\times(0,\infty)$, we can multiply $\chi_a^\times$ by a character of $\{\pm1\}$ to get more quasicharacters of $\mathbb{R}^\times$ having the form

$$x\mapsto\begin{cases}|x|^a=e^{a\ln|x|} & \text{if } x>0;\\ -|x|^a=-e^{a\ln|x|} & \text{if } x<0\end{cases}$$

but these quasicharacters we shall not require here; in fact we shall only use $\chi_a^\times$ for $a \in \mathbb{R}$.

Let us now consider characters on p-adic fields. Every $x \in \mathbb{Q}_p$ has a p-adic expansion

$$x = \underbrace{a_{-k}p^{-k} + \cdots + a_{-1}p^{-1}}_{\text{principal part of } x} + a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots, \quad a_i \in \{0, 1, 2, \ldots, p-1\}$$

in which the leading terms form the *principal part of* x. Note that the principal part of x is a rational number in the interval $[0, 1)$; and it is a distinguished representative of the additive coset $x + \mathbb{Z}_p \in \mathbb{Q}_p/\mathbb{Z}_p$. Moreover, mapping each $x \in \mathbb{Q}_p$ to its principal part is continuous. (This map is in fact locally constant.) When we add two p-adic numbers, we add their principal parts; so it follows that every $a \in \mathbb{Q}_p$ gives an additive character

$$\chi_a^+ : \mathbb{Q}_p \to \mathbb{C}^\times, \quad \chi_a^+(x) = e^{2\pi i (\text{principal part of } ax)}.$$

We also have multiplicative quasicharacters of the p-adics: for each $s \in \mathbb{C}$, define

$$\chi_s^\times : \mathbb{Q}_p^\times \to \mathbb{C}^\times, \quad \chi_s^\times(t) = \|t\|_p^s = e^{s \ln \|t\|_p}.$$

These maps are continuous but not unitary for $s \neq 0$. Once again, there are more general quasicharacters of $\mathbb{Q}_p^\times \cong \langle p \rangle \times \mathbb{Z}_p^\times$ obtained by multiplying $\chi_s^\times$ by an arbitrary multiplicative character $\psi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ to obtain quasicharacters of the form

$$\mathbb{Q}_p^\times \to \mathbb{C}^\times, \quad p^k t \mapsto \|p^k t\|_p^s \psi(t) = p^{-ks}\psi(t)$$

for $t \in \mathbb{Z}_p^\times$; but we will not need these here. (Such quasicharacters would however be used if our goal was to construct Dirichlet L-functions rather than just zeta functions.)

The Fourier Transform

The Fourier transform of a function $f : \mathbb{R} \to \mathbb{C}$ is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{2\pi i x y} f(x)\, dx.$$

(We assume that the integral converges; this will be the case if f is a *Schwartz function* satisfying $x^k f^{(\ell)}(x) \to 0$ as $x \to \pm\infty$, for all k, ℓ.) The factor $\chi_y^+(x) = e^{2\pi i x y}$ here serves as an additive character of $\mathbb{R}$. This transform enjoys several important properties, including

Theorem (Poisson Summation Formula). If $f \in C^\infty(\mathbb{R})$ is a Schwartz function, then

$$\sum_{n \in \mathbb{Z}} \widehat{f}(n) = \sum_{n \in \mathbb{Z}} f(n).$$

This formula in fact gives rise to the functional identity of $\zeta(s)$. We should regard the functional equation for $\zeta(s)$ as arising from an algeraic duality, since this is the source of the Poisson summation formula. A standard example of a smooth Schwartz function is $f(x) = e^{-\pi x^2}$; and it is a standard exercise to check that this function has Fourier transform $\widehat{f} = f$, a fact that we will invoke later. (To be honest, this is not the only function equal to its own Fourier transform; nevertheless it is about the simplest example and so we might regard it as still rather special.)

Next we proceed by analogy to define the Fourier transform of a function $f : \mathbb{Q}_p \to \mathbb{C}$. We replace the additive character $x \mapsto e^{2\pi i x y}$ by the additive character

$$\chi_y^+ : \mathbb{Q}_p \to \mathbb{C}^\times, \quad \chi_y^+(x) = e^{2\pi i (\text{principal part of } xy)}$$

introduced in the previous section. Integration is with respect to Haar measure on $\mathbb{Q}_p$, and the differential is written as dx just as in the real case. Thus given an integrable function $f : \mathbb{Q}_p \to \mathbb{C}$ we define the Fourier transform $\widehat{f} : \mathbb{Q}_p \to \mathbb{C}$ by

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} \chi_y^+(x) f(x)\, dx = \int_{\mathbb{Q}_p} e^{2\pi i (\text{principal part of } xy)} f(x)\, dx.$$

As an example, consider the characteristic function of $\mathbb{Z}_p$ given by

$$f : \mathbb{Q}_p \to \{0, 1\}, \quad f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z}_p, \text{ i.e. } \|x\|_p \leqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

We compute its Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{Z}_p} e^{2\pi i (\text{principal part of } xy)} dx.$$

If $y \in \mathbb{Z}_p$ then $xy \in \mathbb{Z}_p$ has principal part 0 so

$$\widehat{f}(y) = \int_{\mathbb{Z}_p} dx = \lambda^+(\mathbb{Z}_p) = 1.$$

On the other hand, if $\|y\|_p = p^k > 1$ then the principal part of y is a rational number of the form $\frac{a}{p^k}$ where $\gcd(a, p) = 1$. We partition $\mathbb{Z}_p = \bigsqcup_{b=0}^{p^k-1} (b + p^k \mathbb{Z}_p)$ where each of the parts has size $\lambda^+(b + p^k \mathbb{Z}_p) = \frac{1}{p^k}$; also the integrand has constant value ζ^b on $b + p^k \mathbb{Z}_p$ where $\zeta = e^{2\pi i a / p^k}$ is a complex primitive p^k-th root of unity. This gives

$$\widehat{f}(y) = \int_{\mathbb{Z}_p} e^{2\pi i (\text{principal part of } xy)} dx = \frac{1}{p^k} \sum_{b=0}^{p^k-1} \zeta^b = \frac{1}{p^k} \frac{\zeta^{p^k} - 1}{\zeta - 1} = 0.$$

Thus $\widehat{f} = f$ and we may regard f, the characteristic function of $\mathbb{Z}_p$, as a kind of p-adic analogue of the function $x \mapsto e^{-\pi x^2}$ on the reals.

The Mellin Transform over the Reals and the p-Adics

The *Mellin transform* is designed for functions defined on the multiplicative groups of the reals or p-adics, analogous to the Fourier transform of functions on the corresponding additive groups. From the definition of the Fourier transform over $\mathbb{R}$, we first replace the additive character $\chi_y^+(x) = e^{2\pi ixy}$ by the multiplicative quasicharacter

$$\chi_s^\times : (0,\infty) \to \mathbb{C}, \quad \chi_s(t) = t^s = e^{s \ln t}$$

for $s \in \mathbb{C}$. Secondly, we need to replace the differential dx by $\frac{dx}{x}$ in order that we integrate with respect to Haar measure on $(0,\infty)$. With these two changes, we obtain the Mellin transform $\mathcal{M}f$ of a function $f : (0,\infty) \to \mathbb{C}$:

$$(\mathcal{M}f)(s) = \int_{(0,\infty)} \chi_s^\times(t) f(t)\, \frac{dt}{t} = \int_0^\infty t^{s-1} f(t)\, dt.$$

For example, the Mellin transform of $f(t) = e^{-t}$ on $(0,\infty)$ is the Gamma function:

$$(\mathcal{M}f)(s) = \int_0^\infty t^{s-1} e^{-t}\, dt = \Gamma(s).$$

For functions $f : \mathbb{R}^\times \to \mathbb{C}$, one may instead take

$$(\mathcal{M}f)(s) = \int_{\mathbb{R}^\times} \chi_s^\times(t) f(t)\, \frac{dt}{|t|} = \int_{\mathbb{R}^\times} |t|^{s-1} f(t)\, dt.$$

Next we look for the appropriate transform for functions $\mathbb{Q}_p^\times \to \mathbb{C}$. Proceeding as before, we first replace the quasicharacter by $\chi_s(t) = \|t\|_p^s$. Secondly, we replace the differential dx by Haar measure $\frac{p}{p-1} \frac{dt}{\|t\|_p}$ on $\mathbb{Q}_p^\times$. Thus we arrive at the appropriate Mellin transform on $\mathbb{Q}_p^\times$:

$$(\mathcal{M}f)(s) \;=\; \frac{p}{p-1} \int_{\mathbb{Q}_p^\times} \|t\|_p^s f(t)\, \frac{dt}{\|t\|_p} \;=\; \frac{p}{p-1} \int_{\mathbb{Q}_p^\times} \|t\|_p^{s-1} f(t)\, dt.$$

The Euler Factors

We obtain the Euler factors of $\zeta(s)$ by taking the corresponding Mellin transforms of the functions $x \mapsto e^{-\pi x^2}$ (on the reals) and the characteristic function of $\mathbb{Z}_p$ (on the p-adics). We do not justify why this is the 'right' thing to do (for that, see the references we have cited); we simply point out that these two functions play analogous roles in that each is its own Fourier transform for functions on the corresponding domain.

For each prime p, take $f : \mathbb{Q}_p \to \{0,1\}$ to be the characteristic function of $\mathbb{Z}_p$. The Mellin transform of f is

$$(\mathcal{M}f)(s) \;=\; \frac{p}{p-1}\int_{\mathbb{Q}_p^\times} \|t\|_p^{s-1} f(t)\,dt \;=\; \frac{p}{p-1}\int_{\mathbb{Q}_p^\times\cap\mathbb{Z}_p} \|t\|_p^{s-1}\,dt.$$

Now partition $\mathbb{Q}_p^\times \cap \mathbb{Z}_p = \bigsqcup_{k=0}^\infty p^k\mathbb{Z}_p^\times$. The integrand has constant value $\|t\|_p^{s-1} = p^{-(s-1)k}$ on $p^k\mathbb{Z}_p$; and this part of the domain has size $\lambda^+(p^k\mathbb{Z}_p^\times) = p^{-k}\lambda^+(\mathbb{Z}_p^\times) = \frac{p-1}{p^{k+1}}$ so

$$\begin{aligned}(\mathcal{M}f)(s) \;&=\; \frac{p}{p-1}\sum_{k=0}^\infty \int_{p^k\mathbb{Z}_p^\times} \|t\|_p^{s-1}\,dt \;=\; \frac{p}{p-1}\sum_{k=0}^\infty p^{-(s-1)k}\frac{p-1}{p^{k+1}} \\ &=\; \sum_{k=0}^\infty p^{-sk} \;=\; \frac{1}{1-p^{-s}}\end{aligned}$$

which is exactly the desired Euler factor for the prime p.

By analogy, we compute the Mellin transform of the function $f(x) = e^{-\pi x^2}$ on $\mathbb{R}^\times$:

$$(\mathcal{M}f)(s) \;=\; \int_{\mathbb{R}^\times} |t|^{s-1}e^{-\pi t^2}\,dt \;=\; 2\int_0^\infty t^{s-1}e^{-\pi t^2}\,dt.$$

Substituting $u = \pi t^2$,

$$(\mathcal{M}f)(s) \;=\; \pi^{-s/2}\int_0^\infty u^{\frac{s}{2}-1}e^{-u}\,du \;=\; \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)$$

which is also the desired Euler factor at $v = \infty$.

The Functional Equation

Having thus arrived at $\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, let us verify the functional equation $\xi(1-s) = \xi(s)$. As usual, we will leave it to the reader to check the details of convergence at each step. For $\Re(s) > 1$,

$$\begin{aligned}\xi(s) \;&= \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s) \\ &= \sum_{n=1}^\infty \frac{\pi^{-s/2}}{n^s}\int_0^\infty t^{s/2}e^{-t}\,\frac{dt}{t} \\ &= \sum_{n=1}^\infty \int_0^\infty e^{-t}\left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}}\frac{dt}{t} \\ &= \sum_{n=1}^\infty t^{s/2}\int_0^\infty e^{-\pi n^2 t}\,\frac{dt}{t} \\ &= \int_0^\infty t^{s/2}\omega(t)\,\frac{dt}{t}\end{aligned}$$

where $\omega(t) = \sum_{n=1}^{\infty} e^{-n\pi^2 t}$. This says that $\xi(-2s)$ is the Mellin transform of $\omega(t)$. The identity in fact holds whenever $\Re(s) > 0$ since both sides are analytic due to the rapid decay of $\omega(t)$ as $t \to \infty$. It is easy to check that the Fourier transform of $f(x) = e^{-\pi t x^2}$ is $\widehat{f}(s) = \frac{1}{\sqrt{t}} e^{-\pi s^2/t}$ so the Poisson summation formula gives

$$1 + 2\omega(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}}\Big(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2/t}\Big) = \frac{1 + 2\omega(1/t)}{\sqrt{t}}.$$

Now in the vertical strip $0 < \Re(s) < 1$,

$$\begin{aligned}
\frac{1}{s} + \int_0^1 u^{s/2}\omega(u)\,\frac{du}{u} &= \frac{1}{2}\int_0^1 u^{s/2}\big(1 + 2\omega(u)\big)\,\frac{du}{u} \\
&= \frac{1}{2}\int_1^{\infty} t^{-s/2}\big(1 + 2\omega\big(\tfrac{1}{t}\big)\big)\,\frac{dt}{t} \\
&= \frac{1}{2}\int_1^{\infty} t^{\frac{1-s}{2}}\big(1 + 2\omega(t)\big)\,\frac{dt}{t} \\
&= \frac{1}{s-1} + \int_1^{\infty} t^{\frac{1-s}{2}}\omega(t)\,\frac{dt}{t}
\end{aligned}$$

using the substitution $u = t^{-1}$, $\frac{du}{u} = -\frac{dt}{t}$; also using the identity obtained above from Poisson's formula. So again for $0 < \Re(s) < 1$,

$$\begin{aligned}
\xi(s) + \frac{1}{s} + \frac{1}{1-s} &= \frac{1}{s} + \frac{1}{1-s} + \int_0^{\infty} u^{s/2}\omega(u)\,\frac{du}{u} \\
&= \frac{1}{s} + \frac{1}{1-s} + \int_0^1 u^{s/2}\omega(u)\,\frac{du}{u} + \int_1^{\infty} t^{s/2}\omega(t)\,\frac{dt}{t} \\
&= \int_1^{\infty} t^{\frac{1-s}{2}}\omega(t)\,\frac{dt}{t} + \int_1^{\infty} t^{\frac{s}{2}}\omega(t)\,\frac{dt}{t} \\
&= \int_1^{\infty} \big(t^{\frac{1-s}{2}} + t^{\frac{s}{2}}\big)\omega(t)\,\frac{dt}{t}
\end{aligned}$$

which is entire due to the rapid decay $\omega(t) \to 0$ as $t \to \infty$. Analytic continuation to the remaining s-values gives the functional equation $\xi(1-s) = \xi(s)$, while also showing that $\xi(s)$ is analytic outside the simple poles at $0, 1$.

The Adèles

The ring of *adèles* is the set $\mathbb{A}$ consisting of all sequences

$$a = (a_\infty, a_2, a_3, a_5, a_7, a_{11}, \ldots) \in \prod_v \mathbb{Q}_v$$

such that $\|a_v\|_v \leqslant 1$ for almost all v, i.e. $a_p \in \mathbb{Z}_p$ for all but finitely many primes p. (Convention: v ranges over $\{\infty\} \cup \{\text{primes}\}$ whereas p ranges over $\{\text{primes}\}$; so 'for all but finitely many v' means the same as 'for all but finitely many p'.) Note that $\mathbb{A}$ is a ring with componentwise addition and multiplication; and $\mathbb{Q} \subset \mathbb{A}$ is realized as a subring via the diagonal embedding

$$a \mapsto (a, a, a, a, \ldots)$$

since every nonzero $a \in \mathbb{Q}$ satisfies $\|a\|_p = 1$ for almost all p. We take a topology on $\mathbb{A}$ defined by the collection of basic open sets of the form

$$U = U_\infty \times U_2 \times U_3 \times U_5 \times U_7 \times U_{11} \times \cdots$$

where each $U_v \subseteq \mathbb{Q}_v$ is open, and $U_p = \mathbb{Z}_p$ for all p sufficiently large. (This is different from the product topology. The problem is that a product of locally compact spaces is not locally compact, unless most of the factors are actually compact.) One checks that $\mathbb{A}$ is a locally compact Hausdorff space; thus it has an additive Haar measure λ^+. This measure is normalized so that $\lambda^+(\mathbb{Z}_p) = 1$ for every p (here each $\mathbb{Z}_p$, and more generally every $\mathbb{Q}_v$, is realized as a subring of $\mathbb{A}$). More generally for finite products $E = \prod_v E_v \subset \mathbb{A}$ having measurable factors $E_v \subset \mathbb{Q}_v$, we have $\lambda^+(E) = \prod_v \lambda_v^+(E_v)$ where λ_v^+ is additive Haar measure on $\mathbb{Q}_v$.

Exercise. $\mathbb{Q} \subset \mathbb{A}$ is discrete and co-compact. (Co-compact means that the quotient space $\mathbb{A}/\mathbb{Q}$ is compact.)

The multiplicative group of units $\mathbb{A}^\times$ is the group of *idèles*. Thus an idèle is a sequence of the form $a = (a_\infty, a_2, a_3, a_5, a_7, a_{11}, \ldots) \in \prod_v \mathbb{Q}_v$ such that $a_v \neq 0$ for all v; and $\|a_p\|_p = 1$ for all p sufficiently large. If we take the subspace topology for $\mathbb{A}^\times \subset \mathbb{A}$, then multiplication is continuous in $\mathbb{A}^\times$; but the inverse map $a \mapsto a^{-1}$ is not continuous. Instead we take a refinement of the subspace topology, defined as follows. Basic open sets $U \subseteq \mathbb{A}^\times$ have the form $U = \prod_v U_v$ where $U_v \subseteq \mathbb{Q}^v$ is open for every v; and $U_p = \mathbb{Z}_p^\times$ for all but finitely many p. The resulting topology on the idèles is the coarsest possible refinement of the subspace topology for $\mathbb{A}^\times \subset \mathbb{A}$, for which $\mathbb{A}^\times$ is a topological group. The *absolute value* of an idèle $a = (a_v)_v$ is the product

$$|a| = \prod_v \|a_v\|_v$$

which is well-defined since almost all factors are 1. This definition extends to a multiplicative function on the adèles by taking $|a| = 0$ if $a \in \mathbb{A} \smallsetminus \mathbb{A}^\times$. Thus $|a| = \prod_v \|a_v\|_v$ for all $a \in \mathbb{A}$. The set of idèles $a \in \mathbb{A}^\times$ of absolute value 1 is a subgroup containing $\mathbb{Q}^\times$. The idèle group is locally compact Hausdorff; its Haar measure extends the multiplicative Haar measure on the $\mathbb{Q}_v$'s.

In his thesis (and following ideas of his advisor Emil Artin), Tate gave an interpretation of zeta functions based on the ring of adèles, and on the subring of idèles, which clarified the role of the functional identity. Given $x = (x_\infty, x_2, x_3, x_5, \ldots) \in \mathbb{A}$, define $f(x) = (f_v(x_v))_v \in \mathbb{A}$ where $f_\infty(x_\infty) = e^{-\pi x_\infty^2}$ and $f_p(x_p) = 1$ or 0 according as $x_p \in \mathbb{Z}_p$ or not. The Mellin transform of f is the product of the Mellin transforms of the individual coordinates, which therefore yields $\xi(s)$.