The Hardin-Taylor Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function (not necessarily continuous). Think of $y = f(t)$ as the value of some interesting quantity at time t, where the actual form of f is unknown. For each time $t_0 \in \mathbb{R}$, we wish to obtain a predictor $\langle f \rangle_{t_0}$ for f such that for each time $t_0 \in \mathbb{R}$,

(a) The function $\langle f \rangle_{t_0} : \mathbb{R} \to \mathbb{R}$ agrees with f (*exactly*!) on $(-\infty, t_0)$.
(b) The predictor $\langle f \rangle_{t_0}$ depends only on the values of f restricted to $(-\infty, t_0)$.
(c) There exists $t_1 > t_0$ such that $\langle f \rangle_{t_0}$ agrees with f on $(-\infty, t_1)$.

Theorem (Hardin, Taylor 2008). *There exists a map $f|_{(-\infty,t_0)} \mapsto \langle f \rangle_{t_0}$ satisfying (a) and (b) for all f and $t_0 \in \mathbb{R}$; and which fails (c) for at most countably many values of $t_0 \in \mathbb{R}$.*

Note that we are *not* assuming f to be continuous. If f were continuous, or even left-continuous, we would be able to predict the present value $f(t)$ from the past values $f|_{(-\infty,t)}$.

For each $t_0 \in \mathbb{R}$ we have an equivalence relation on $\mathbb{R}^{\mathbb{R}}$ defined by

$$f \overset{t_0}{\equiv} g \quad \text{iff} \quad f(t) = g(t) \text{ for all } t \leqslant t_0.$$

Denote the equivalence class of f for this relation by $[f]_{t_0}$. By the Axiom of Choice, we may choose well-ordering relation '$\preceq$' on $\mathbb{R}^{\mathbb{R}}$. We simply define $\langle f \rangle_{t_0}$ to be the $\preceq$-least member of $[f]_{t_0}$. The set of all t-values for which (c) fails is

$$S = \{t \in \mathbb{R} : \langle f \rangle_t \neq \langle f \rangle_{t'} \text{ for all } t' > t\}$$

since $\langle f \rangle_t$ disagrees with $\langle f \rangle_{t'}$ at some point of $(-\infty, t')$, iff $\langle f \rangle_t$ disagrees with f at that point. The main theorem is a corollary of:

Theorem. *The subset $S \subseteq \mathbb{R}$ contains no infinite descending sequence. In particular, S is countable; it has Lebesgue measure zero; and it is nowhere dense.*

Proof. Suppose that S contains an infinite sequence $t_1 > t_2 > t_3 > \cdots$. For each i,

$$\langle f \rangle_{t_{i+1}} \neq \langle f \rangle_{t_i}.$$

Both of these functions are in $[f]_{t_{i+1}}$ so

$$\langle f \rangle_{t_{i+1}} \prec \langle f \rangle_{t_i}.$$

This gives an infinite descending sequence

$$\langle f \rangle_{t_1} \succ \langle f \rangle_{t_2} \succ \langle f \rangle_{t_3} \succ \cdots,$$

a contradiction.

Since $\mathbb{Q} \subset \mathbb{R}$ is dense, there is an injection $\iota : S \to \mathbb{Q}$ satisfying

$$\begin{aligned} t < \iota(t) < t', &\quad \text{if } t' \in S \text{ is the least element of } S \text{ exceeding } t; \\ t < \iota(t), &\quad \text{if } S \text{ has no element exceeding } t. \end{aligned}$$

Thus S is countable. In particular, S has Lebesgue measure zero. If the closure $\overline{S} \subseteq \mathbb{R}$ has nonempty interior, then it contains an open interval (a, b); and from here, it is easy to find an infinite descending sequence in S. This is impossible; so S is nowhere dense. □

A refinement of this procedure yields a predictor for f, based only on the most recent past values of f: given $f|_{(t_0-\varepsilon,t_0)}$ for some $\varepsilon > 0$, we obtain the same conclusions.

This work grew out of the authors' research on generalized hat problems, originated by Gabay and O'Connor, two graduate students at Cornell in 2004.

Let α be an arbitrary ordinal, and consider a set of prisoners indexed by α. After giving the prisoners an opportunity to formulate a stragegy, the warden places a hat (red or blue, chosen at random) on each prisoner's head. Each prisoner can see only the hats on the heads of prisoners with a larger index. Each prisoner must guess the colour of the hat on his/her head, without hearing any of the other guesses. What strategy guarantees that only a finite number of prisoners guess incorrectly? The answer is that prisoners agree beforehand on a well-ordering of the set 2^α of possible hat assignments. Each prisoner bases his guess on the least such assignment that agrees with what he can see. Let S be the set of all indices $\beta \in \alpha$ such that prisoner β guessed incorrectly, and let f_β be the sequence guessed by prisoner $\beta \in S$. The map $S \to 2^\alpha$, $\beta \mapsto f_\beta$ is decreasing, i.e. $f_\alpha > f_\beta$ whenever $\alpha < \beta$. Since $\{f_\alpha\}_{\alpha S}$ is a strictly decreasing sequence in the well-ordered set 2^α, it has only finitely many members; thus $|S| < \infty$.

[HT] C.S. Harding and A.D. Taylor, 'A peculiar connection between the axiom of choice and predicting the future', *Amer. Math. Monthly* **115** no.2 (2008), 91–96.

[HT2] C.S. Harding and A.D. Taylor, *The Mathematics of Coordinated Inference—A Study of Generalized Hat Problems*, Springer-Verlag, 2013.