Math 5605

Algebraic Topology

Book 1

If X, Y are top. spaces, $f: X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a <u>homeomorphism</u> if f is bijective and f, f^{-1} are continuous.

$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

$\mathbb{R}^2 \not\cong S^2$ Since S^2 is compact; $\mathbb{R}^2$ is not.

2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} = $

$S^2 \not\cong T^2 = S' \times S' = $ $ = $ S^2, T^2 are compact surfaces. They are locally homeomorphic
$\underset{S}{\overset{S''}{}}$ but not globally homeomorphic.

$T' = S' = $ $ = $ circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not.
In S^2, every closed path can be "continuously shrunk" to a point
                    (homotopic to a point, i.e. null-homotopic)

$\overset{X}{}$ $\not\cong$ $\overset{T = T^2}{}$ although both surfaces are compact, connected, not simply connected

These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial)
                                                                                                    abelian group.
If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.
For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of
<u>homotopy</u> equivalence.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$.

Given points $a, b \in X$ (a topological space), a path from a to b is a function (continuous) $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.

$$[0,1] \xrightarrow{f}$$

$f(0) = a$ $f(1) = b$ X

$f(t)$
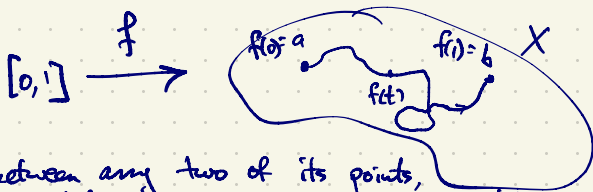
All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:

$f(0) \cdot$ f $f(1) = g(0)$ g $g(1) = h(0)$ h $h(1)$

$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$
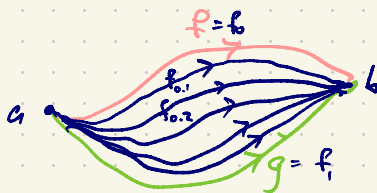
$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$

$(s,t) \mapsto f(s,t) = f_s(t)$

Such that $f_0 = f$ ie. $f_0(t) = f(t)$
$f_1 = g$ ie. $f_1(t) = g(t)$
$f_s(0) = a$
$f_s(1) = b$ for all $s \in [0,1]$

This is a homotopy from f to g.

f, g are homotopic
but h is not homotopic to f, g

$f = f_0$

$f_{0.1}$

$f_{0.2}$

$g = f_1$

a b

We say f, g are homotopic if there is a continuous family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi : [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f \circ \varphi : [0,1] \to X$ is homotopic to f. A homotopy from f to $f \circ \varphi$ is

$$[0,1]^2 \to X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{[0,1]}) = f_s(t)$$

$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
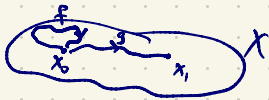$$f_s(0) = f((1-s)\cdot 0 + s\cdot \varphi(0)) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot \varphi(1)) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S') \cong \mathbb{Z}$ (free group on one generator)

Fix g path in X from x_0 to x_1

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f \gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is

$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$

(same path in the reverse direction)

$f\bar{f} = \bar{f}f = \gamma = $ null path

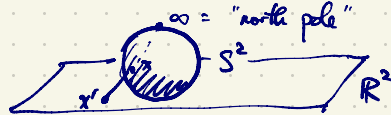An isomorphism $\phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$f \overset{\phi}{\longmapsto} \bar{g}fg$$
$$g f \bar{g} \overset{\phi^{-1}}{\longleftarrow} h$$

$$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$
$$f_1, f_2 \in \pi_1(X, x_0)$$

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

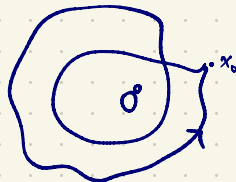$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)

$\infty = $ "north pole"

$x \longmapsto x'$ is stereographic projection from the north pole ∞

See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
   punctured plane

follows from the fact that
$\mathbb{R}^2 - \{0\}$ and S' have the same homotopy type
$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S'$
$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).
      Hatcher writes $X \approx Y$ for homeomorphic.

- retraction
- deformation retraction
- strong deformation retraction
- homotopy
- relative homotopy
- homotopy equivalence

"def. retraction in the weak sense"

"def. retraction"

Let $A \subseteq X$ (subspace of a top. space).
A retraction $f: X \longrightarrow A$ is a $\overset{\text{(continuous)}}{\text{map}}$ such that $f|_A = id_A = 1_A$ ie. $f(a) = a$
                                                                                for all $a \in A$.
If such a map exists then A is a retract of X.
Eg. $\mathbb{R}^n$ has a retraction to any one of its points.   If $a \in \mathbb{R}^n$ then
    the constant map $\mathbb{R}^n \longrightarrow \{a\}$, $x \longmapsto a$ is a retraction.
$\mathbb{R}^2 \longrightarrow x\text{-axis}$ $(x,y) \longmapsto (x,0)$.
If $S' \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \longrightarrow S'$.
(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to $\underset{\underset{X \to A}{\text{refraction}}}{a}$), $A \subseteq X$.

i.e. $f : [0,1] \times X \longrightarrow X$

$$f(t, x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
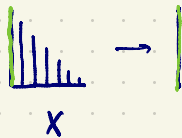
$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.

Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f : [0,1] \times X \longrightarrow X$:

$f_0(x) = x$ i.e. $f_0 = id_X$

f_1 is a retraction $X \to A$

$f_t |_A = id_A$ for all $t \in [0,1]$

X

S^1

x
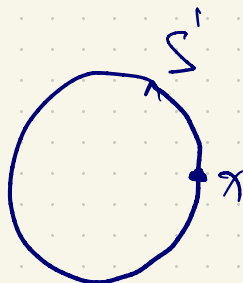
def. retract
but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $\qquad X = \left([0,1] \times \{0\}\right) \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} \left(\{r\} \times [0, 1-r]\right)$ $\qquad$ X has a def. ret. to A $= \{0\} \times [0,1]$.

$0 \leq t \leq \frac{1}{2}$ $\qquad\qquad \frac{1}{2} \leq t \leq 1$

This is a def. retract but not a strong def. retract.
There is no strong def. retract $X \to A$.
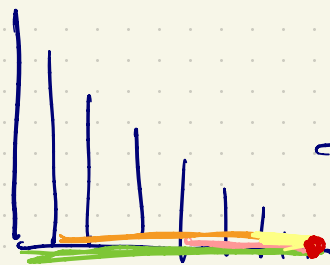
Let $f_0, f_1 : X \to Y$ be maps. A **homotopy** from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0, x) = f_0(x)$, $f(1, x) = f_1(x)$

$f(b, x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 $\quad (f_0, f_1 : X \to Y)$ is a homotopy $f: [0,1] \times X \to Y$ such that $f_t(a)$ is independent of $t \in [0,1]$ for all $a \in A$.
(constant)

A path from a to b in X is a homotopy from a to b.
Given two paths f_0, f_1 in X from a to b $\quad (a, b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{}{\overset{f}{\rightleftarrows}} Y$ such that $f \circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$ $f(x) = 0$ for all $x \in \mathbb{R}^n$
                $g(0) = 0 \in \mathbb{R}^n$

$g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ $f \circ g : \{0\} \to \{0\}$
$\qquad x \mapsto 0$ $0 \mapsto 0$
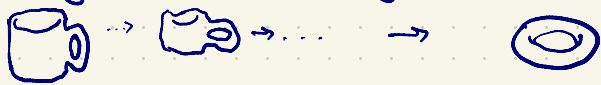
A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ to $\text{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$
$\qquad\qquad\qquad x \mapsto 0$ $x \mapsto x$

$\qquad$ Not relative
$\qquad$ to any subspace
$\qquad$ necessarily.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_0 = g \circ f$
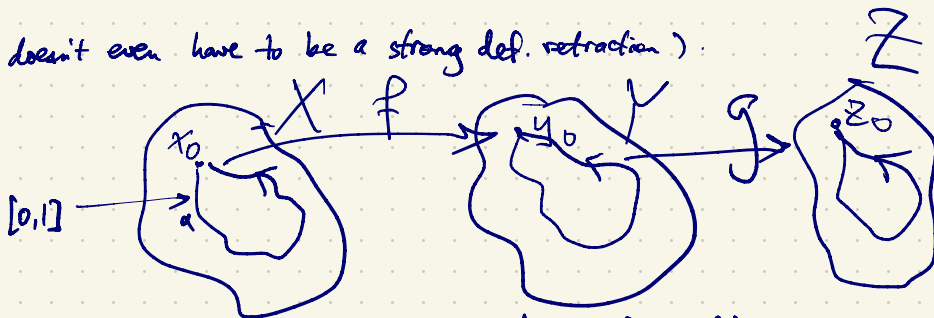$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_1 = \text{id}_{\mathbb{R}^n}$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).

S^1 is not homotopic to a point.
(S^1 is not null-homotopic ; not contractible)

If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

$\quad \alpha : [0,1] \to X$ gives $f_* \alpha = f \circ \alpha : [0,1] \to Y$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (g \circ f)_* = g_* \circ f_*$

D^n = closed ball in $\mathbb{R}^n$ "disk"

$\overset{\bullet}{D^0}$ $\overline{}^{\bullet}$ D^1

D^2

$S^n = \partial D^{n+1}$ = n-sphere

$\overset{\bullet}{\underset{\bullet}{S^0}}$
$= D^0 \sqcup D^0$

S^1

A CW complex is formed from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$

X^n is a union of copies of D^n with the boundaries of D_α^n attached to X^{n-1} via attaching maps.

$\underbrace{}_{D_\alpha^n}$
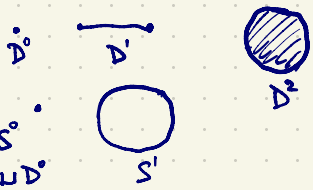
Eg. Torus $T^2 = S^1 \times S^1 = $ $=$ $=$

$X^0 = \cdot = D^0$

$X^1 = $ $\cong S^1 \vee S^1$

$\underset{D_0^1}{\underline{}}_{\bullet}$ ω $_{\bullet}\underset{D_1^1}{\underline{}}$

X^2
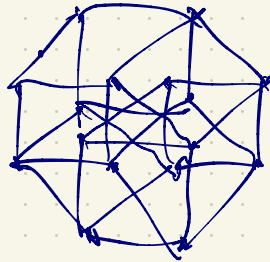
$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

Möbius strip not homeomorphic to cylinder $S^1 \times [0,1] =$ $=$

$X^0 = $ • •

Cylinder: $X^0 = $ • • $\mathring{D} \cup D^0$

$X^1 = $ (A B)

$X^2 = $

$X^1 = $

$X^2 = $

orientable

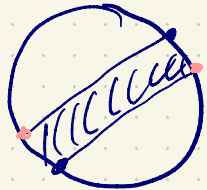Both are homotopy equivalent to S^1 (def. retract to S^1)

Both have $\mathbb{Z}$ as fund. gp.
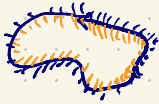
not orientable

$P^2\mathbb{R}$ (or $\mathbb{R}P^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified
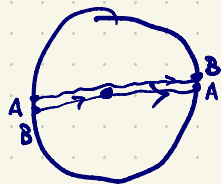

(non-orientable surface)

D^2

$P^2\mathbb{R} = D^2$ glued to a Möbius strip

$\pi_1(\mathbb{P}\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ a is not homotopic to the null path γ. a^2 is homotopic to γ

a^2
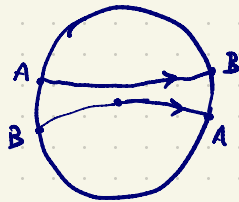
$\mathbb{R}^2 - \{0\} \simeq S^1$
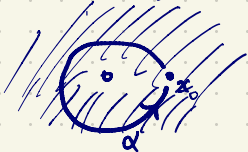
A homotopy equivalence

$\mathbb{R}^2 - \{0\} \xrightarrow{f_1} S^1$

$f_1((x,y)) = \left(\dfrac{x}{\sqrt{x^2+y^2}} , \dfrac{y}{\sqrt{x^2+y^2}} \right)$

$f_t(v) = (1-t)v + t\dfrac{v}{|v|}$ $f_1(v) = \dfrac{v}{|v|}$

$f_t : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$

$f_0 = \mathrm{id}_{\mathbb{R}^2 - \{0\}}$

f_1 is a retraction to S^1

strong def. retraction since

$f_t |_{S^1} = \mathrm{id}_{S^1}$ for all $t \in [0,1]$.

$\{\alpha^n : n \in \mathbb{Z}\}$
$\parallel$
$\langle \alpha \rangle = \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$ free group on one generator

$$\pi_1(S') \cong \mathbb{Z}$$

Given a closed path in S' with base point $1 \in S' = \{ z \in \mathbb{C} : |z| = 1 \}$

define $w(\beta) = \dfrac{1}{2\pi i} \underset{\beta}{\int} \dfrac{dz}{z} = \dfrac{1}{2\pi i} \displaystyle\int_{\beta(0)}^{\beta(1)} \dfrac{dz}{z}$

$\beta : [0,1] \longrightarrow S'$

$\beta(0) = 1$

$\beta(e^{2\pi i \theta}) = \theta + n$

$w(\alpha) = 1$

$w(\alpha^n) = n$

w is an isomorphism from $\pi_1(S')$ to $\mathbb{Z}$.

In $\mathbb{C} - \{0\}$ the same argument works

$$\int \dfrac{dz}{z} = \ln|z| + 2\pi i \operatorname{Arg} z$$