

Math 5605

Algebraic Topology

Book 3

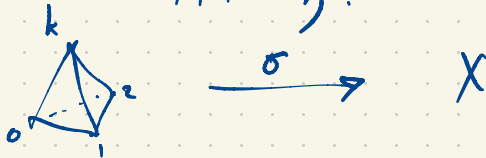
Cup product for simplicial cohomology $H^k \times H^l \xrightarrow{\cup} H^{k+l}$
 makes $H^*(X; \mathbb{Z})$ or $H^*(X; \mathbb{R})$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k -chains: ($k = 0, 1, 2, 3, \dots$) ways of mapping k -simplices
 into X , not necessarily embeddings.

Take an abstract k -simplex {all subsets of $\{0, 1, 2, \dots, k\}$ }.

This has a geometric realization



$$\Delta = \Delta^n = \left\{ \underbrace{(v_0, v_1, \dots, v_n)}_{\text{barycentric coordinates}} : v_i \geq 0, \sum v_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

(convex combinations of $e_0 = (1, 0, \dots, 0)$, $e_1, \dots, e_n = (0, \dots, 0, 1)$)

An n -chain is a formal linear combination of maps $\sigma: \Delta^n \rightarrow X$.

$$C_n = \{n\text{-chains in } X\} = C_n(X; \mathbb{R}), \quad \mathbb{R} \text{ any commutative ring with } 1 \quad \text{eg. } \mathbb{R}, \mathbb{Z}, \mathbb{F}_2$$

$$C^n = C_n^* = \{n\text{-cochains in } X\} = \text{Hom}(C_n, \mathbb{R}) = \{\mathbb{R}\text{-homomorphisms } C_n \rightarrow \mathbb{R}\}$$

$$d: C_n \rightarrow C_{n-1}, \quad d\sigma = \sum_{i=0}^n \sigma \circ \partial_i \quad d^2 = 0, \quad (d^*)^2 = 0$$

$$d^*: C^{n-1} \rightarrow C^n$$

If $\phi \in C^k$ k -cochain then $\phi \cup \psi \in C^{k+l}$ cochain; for any $(k+l)$ -chain $\sigma: \Delta^{k+l} \rightarrow X$
 $\psi \in C^l$ l -cochain $(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_k, \dots, v_{k+l}]})$ $[v_0, \dots, v_{k+l}] \mapsto \sigma(v_0, \dots, v_{k+l})$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
 inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)

making $H^*(X; \mathbb{R})$ into a graded ring

$$\bigoplus_{i \geq 0} H^i(X; \mathbb{R}).$$

Eg. $X = \mathbb{P}^n \mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, $H^i(X; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & 0 \leq i \leq n \\ 0 & \text{else} \end{cases}$

$\mathbb{P}^n \mathbb{R} = \{ \text{1-dim'd subspaces of } \mathbb{R}^{n+1} \} = S^n / \text{antipodality}$

$\mathbb{P}^1 \mathbb{R} \cong S^1 / \text{antipodality} \cong S^1$

$\mathbb{O} \cong \mathbb{O}$

$\mathbb{P}^n \mathbb{R}$ is orientable iff n is odd.

$\mathbb{P}^2 \mathbb{R} = S^2 / \text{antipodality} =$ 

$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x] / (x^{n+1})$ Additively: $\{ a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2 \}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.

Proof is by contradiction

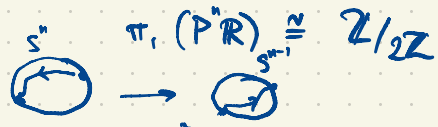
ie. $f(-x) = -f(x)$

Suppose $f: S^n \rightarrow S^{n-1}$ is antipodal. ($f(-x) = -f(x)$)

Then f induces a well-defined map

$$\begin{array}{ccc} P^n \mathbb{R} & \xrightarrow{f} & P^{n-1} \mathbb{R} \\ \downarrow & & \downarrow \\ \pm x & & \pm f(x) \end{array}$$

($x \in S^n$)



f^* maps a generator of $\pi_1(P^{n-1} \mathbb{R})$ to a generator of $\pi_1(P^n \mathbb{R})$

f induces $f^*: H^*(P^{n-1} \mathbb{R}; \mathbb{F}_2) \rightarrow H^*(P^n \mathbb{R}; \mathbb{F}_2)$ mapping $x \mapsto x$

$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{F}_2[x] / (x^n) & & \mathbb{F}_2[x] / (x^{n+1}) \end{array}$$

$x^n \mapsto x^{n+1}$; contradiction.

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A)} \oplus T(A)$ where $T(A) = \text{torsion subgroup of } A = \{\text{elements of } A \text{ of finite order}\}$

$k = \text{rank } A = \dim A$.

For any chain complex $C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots \rightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{0} 0$ (over \mathbb{Q} or \mathbb{R})

we have homology groups $H_n = \ker d_n / \text{im } d_{n+1}$ with well-defined rank $H_n(X; \mathbb{Z}) = \text{rank } H_n(X; \mathbb{Q}) = \text{rank } H_n(X; \mathbb{R})$

and Euler characteristic

$$\chi(X) = \sum_{i=0}^n (-1)^i \text{rank } H_i(X) = \sum_{i=0}^n (-1)^i \text{rank } C_i$$

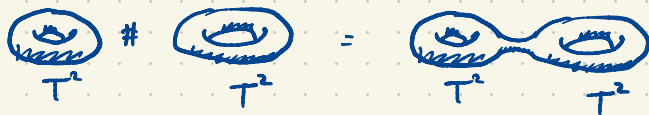
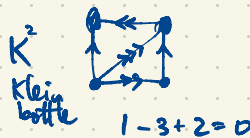
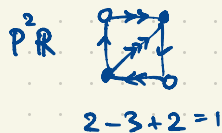
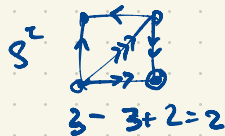
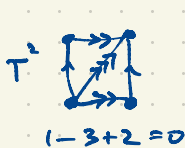
$$\begin{array}{l} C_n \xrightarrow{d_n} C_{n-1} \quad \dim C_n = \dim \ker d_n + \dim \text{im } d_n \\ H_n = \ker d_n / \text{im } d_{n+1} \quad \dim H_n = \dim \ker d_n - \dim \text{im } d_{n+1} \end{array}$$

eg. $\chi(S^2) = 4 - 6 + 4 = 2$



Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

	S^2	T^2	P^2R	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0



$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2 \quad \text{for any two closed surfaces } S_1, S_2$$

$$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0 + 0 - 2 = -2$$

$$\underbrace{T^2 \# \dots \# T^2}_g = \text{(genus } g \text{ surface)}$$

$$\chi(T^2 \# \dots \# T^2) = 2 - 2g$$

g = genus of orientable surface

$$\chi(P^2R \# P^2R) = 1 + 1 - 2 = 0$$

$\underbrace{\hspace{1cm}}_{K^2}$

Exact sequences $\dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$ $\ker d_n = \text{im } d_{n+1}$

$0 \rightarrow C \rightarrow 0$ is exact $\iff C = 0$

$0 \rightarrow A \rightarrow B \rightarrow 0$ is exact $\iff A \cong B$

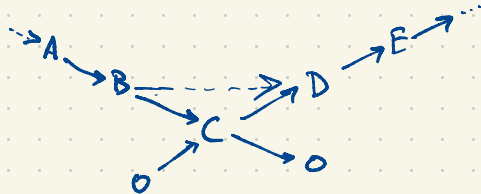
$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact (short exact) $\iff C \cong B/A$

If $f: X \rightarrow X$ is an endomorphism of an abelv. gp. X (or vector space) (at least in an abelian category) some important short exact sequences are

$$0 \rightarrow \ker f \rightarrow X \xrightarrow{f} f(X) \rightarrow 0$$

$$0 \leftarrow \text{coker } f \leftarrow X \xleftarrow{f} f(X) \leftarrow 0$$

If $\dots \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and $0 \rightarrow C \rightarrow D \rightarrow E \rightarrow \dots$



If $f: X \rightarrow Y$ then $\text{coker } f = \varinjlim f(X)$
are exact then we get an exact seq.
 $\rightarrow A \rightarrow B \rightarrow D \rightarrow E \rightarrow \dots$

$0 \rightarrow \ker f \rightarrow X \xrightarrow{f} X \rightarrow \text{coker } f \rightarrow 0$ is exact.

If X is a fin. diml vector space over F then the Euler char. of this sequence is
 $\dim \text{coker } f - \dim X + \dim X - \dim \ker f = 0$

If $T: X \rightarrow X$ is an operator (endomorphism) (don't worry about boundedness)
the index of T is $\text{ind } T = \dim \text{coker } T - \dim \ker T$ when both of these terms are finite
(i.e. T is Fredholm).

Theorem: Let $S, T: X \rightarrow X$ be operators (lin. transf).
 of the three operators S, T, ST , then whenever two are Fredholm then so is the third and
 in this case $\text{ind } ST = \text{ind } S + \text{ind } T$. (or abd. gps)

In general (i.e. for any lin. transf. $S, T: X \rightarrow X$) we have an exact sequence

$$0 \rightarrow \ker T \rightarrow \ker ST \rightarrow \ker S \rightarrow \text{coker } T \rightarrow \text{coker } ST \rightarrow \text{coker } S \rightarrow 0$$

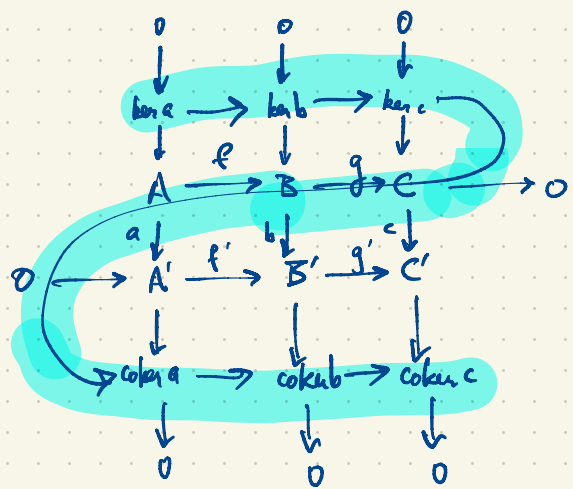
So its Euler characteristic is zero. i.e. $\text{ind } S + \text{ind } T - \text{ind } ST = 0$.

Snake Lemma In an abel. category we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

then we have a six-term exact seq.

$$\ker a \longrightarrow \ker b \longrightarrow \ker c \longrightarrow \text{coker } a \longrightarrow \text{coker } b \longrightarrow \text{coker } c$$



Group Cohomology: used in the study of group extensions

If G and H are groups then an extension of H by G is a group X giving an exact sequence $1 \rightarrow H \rightarrow X \rightarrow G \rightarrow 1$

Note: Groups are not necessarily abelian. We are asking for a new group X having a normal subgp $\cong H$ s.t. $X/H \cong G$. G on top, H on the bottom.

Trivial: $X = G \times H$. (split extension)

G is now an arbitrary group and A is an abelian group (G multiplicative; A additive notation) on which G acts (each $g \in G$ gives $g \in GL(A)$ (automorphisms of A as an abel. gp or \mathbb{Z} -module))

$(g_1 g_2)(a) = g_1(g_2(a))$; $g(a+b) = ga + gb$; $1a = a$. $G \xrightarrow{\text{homo.}} \text{Aut } A = GL(A)$ (fixed)

We construct an extension of A by G i.e. an exact sequence of gps

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \longrightarrow & \hat{G} & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow & & \downarrow \beta & & \downarrow \gamma \\ 1 & \longrightarrow & A & \longrightarrow & \hat{G} & \longrightarrow & G \longrightarrow 1 \end{array}$$

i.e. \hat{G} is a gp with normal subgp iso. to A with $\hat{G}/A \cong G$

Two extensions $\hat{G} \hat{\cong} \hat{G}$ are equivalent if we have a commutative diagram as shown with α, β, γ isomorphisms of groups (with exact rows), Note that the action of G on A is fixed throughout. Cohomology of groups is the tool for this.

... $\xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} D$ is an exact sequence of additive abel. gps where $C^k = C^k(G; A)$ is the set of all maps $G^k \rightarrow A$ as an additive abel. gp. i.e. \mathbb{Z} -module
 $\underbrace{G \times G \times \dots \times G}_k$ i.e. k -tuples of G

$$C^0 = A \quad (\text{maps } \{1\} \rightarrow A)$$

$$C^1 = A^G = \text{maps } G \rightarrow A \quad \text{i.e. } f: G \rightarrow A$$

$$C^2 = A^{G \times G} = \text{maps } G \times G \rightarrow A \quad \text{etc.}$$

Given $a \in C^0$ i.e. $a \in A$, $\delta a \in C^1$ is $\delta a: G \rightarrow A$

$$g \mapsto ga - a$$

Given $f \in C^1$ i.e. $f: G \rightarrow A$

Construct $\delta f \in C^2$ i.e. $(\delta f): G \times G \rightarrow A$

$$(\delta f)(g, h) = \underbrace{g f(h)}_A - \underbrace{f(gh)}_A + \underbrace{f(g)}_A \in A.$$

Given $f \in C^2$ i.e. $f: G \times G \rightarrow A$

Construct $(\delta f): G \times G \times G \rightarrow A$

$$(\delta f)(g, h, k) = g f(h, k) - f(gh, k) + f(g, hk) - f(g, h)$$

See p. 2 bottom of handout for $\delta: C^k \rightarrow C^{k+1}$ in general

$$\text{Check: } C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \quad \delta^2 = 0?$$

Take $a \in C^0 = A$.

$$(\delta a): G \rightarrow A$$

$$(\delta a)(g) = ga - a.$$

$$\delta^2 a: G \times G \rightarrow A$$

$$(\delta^2 a)(f, g) = f(\delta a)(g) - (\delta a)(fg) + \delta a(f)$$

$$= \cancel{f(ga - a)} - (\cancel{f(g)a} - a) + \cancel{f(a - a)} \\ = \cancel{fga - fa} - \cancel{fga} + a + \cancel{fa - a} \\ = 0$$

Classify extensions $1 \rightarrow A \rightarrow \hat{G} \rightarrow G \rightarrow 1$ where G is a group acting on an abelian gp A
 i.e. \hat{G} is a group with normal subgp A with $\hat{G}/A \cong G$, using cohomology. Start with a split extension
 i.e. A has a complementary subgp in \hat{G} . So \hat{G} acts on the subgps complementary to A by conjugation.
 $H^1(G; A)$ classifies the complementary subgps up to conjugacy.

Fix an action of $\underbrace{G}_{\text{mult.}}$ on $\underbrace{A}_{\text{additively}}$

$$a(g, g_2) = (ag_1)g_2$$

here A is a right G -module.

$$a1 = a$$

\uparrow id. of G

$$(a+a')g = ag + a'g$$

for $a, a' \in A; 1, g, g_1, g_2 \in G$.

\hat{G} is isomorphic to the

semidirect product $A \rtimes G = \{ (a, g) : a \in A, g \in G \}$

$$(a_1, g_1)(a_2, g_2) = (a_1g_2 + a_2, g_1g_2)$$

identity $(0, 1)$

Alternative notation: $A \rtimes G = \left\{ \begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} : a \in A, g \in G \right\}$

$G \times A$
for left
action

$$\begin{bmatrix} g_1 & 0 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} g_2 & 0 \\ a_2 & 1 \end{bmatrix} = \begin{bmatrix} g_1g_2 & 0 \\ a_1g_2 + a_2 & 1 \end{bmatrix}$$

$$C^2 \xleftarrow{\delta^1} C^1 \xleftarrow{\delta^0} C^0 \xleftarrow{\quad} 0$$

$\begin{matrix} \text{"} \\ A \end{matrix}$

for $a \in A \stackrel{C^0}{=} C^0$ $(\delta^1)(g) = ag - a$

Complements of A
in \hat{G} are given
by 1-cocycles.

How do we construct a subgrp of $A \rtimes G$ complementary to A ?

Any such subgrp $H \leq A \rtimes G$ has the form $\{ (t_g, g) : g \in G \} = \left\{ \begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix} : g \in G \right\}$

Here $g \mapsto t_g, G \rightarrow A$. This will automatically be a complement to A as long as it is a subgrp. eg. $t_1 = 0$ but most importantly, closure.

$$\begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix} \begin{bmatrix} g' & 0 \\ t_{g'} & 1 \end{bmatrix} = \begin{bmatrix} gg' & 0 \\ t_{gg'} & 1 \end{bmatrix} \text{ i.e. } \underbrace{t_{gg'}}_A = \underbrace{t_g}_{A} \underbrace{g'}_G + \underbrace{t_{g'}}_A \text{ so } (\delta^1)(g, g') = -f(gg') + f(g)g' + f(g') = 0.$$

When are two complements of A conjugate in \hat{G} ? $\hat{G} = A \rtimes G$

If A has complementary subgps $H_1, H_2 \leq \hat{G}$ given by $H_i = \{ (f_i(g), g) : g \in G \}$, $f_i \in Z^1(G; A)$
 when are H_1, H_2 conjugate in \hat{G} ?

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}^{-1} = \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix}$$

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use $\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \in \hat{G}$ (a, g fixed) to conjugate H_1 :

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ f_1(x) & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gx & 0 \\ ax + f_1(x) & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ axg^{-1} + f_1(x)g^{-1} - ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ f_2(gxg^{-1}) & 1 \end{bmatrix}$$

The 1-cycle defining this conjugate subgroup would have to be f_2 : so

$$f_2(gxg^{-1}) = axg^{-1} + f_1(x)g^{-1} - ag^{-1} = f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1}$$

$$ax + f_1(x) - a = f_2(g)x + f_2(x) - f_2(g)$$

$$\begin{aligned} f_2(x) - f_1(x) &= (a - f_2(g))x - (a - f_2(g)) \\ &= \delta(a - f_2(g))(x) \end{aligned}$$

$$(\delta f_i)(g, g^{-1}) = f_i(g^{-1}) - f_i(gg^{-1}) + f_i(g)g^{-1} = 0$$

$$\begin{aligned} f_2(gxg^{-1}) &= f_2(gxg^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} + f_2(g^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1} \end{aligned}$$

$f \in C^1$ i.e. $f: G \rightarrow A$

$$f(xy) = f(x)y + f(y)$$

$$\begin{aligned} f(1) &= f(1 \cdot 1) = f(1) + f(1) \\ &\Rightarrow f(1) = 0 \end{aligned}$$

$$\begin{aligned} 0 &= f(1) = f(gg^{-1}) = f(g)g^{-1} + f(g^{-1}) \\ &\Rightarrow f(g^{-1}) = -f(g)g^{-1} \end{aligned}$$

f is a 1-cocycle: $f \in Z^1$

$$(\delta f)(x, y) = f(xy) - f(x)y - f(y) = 0$$

f is a crossed homomorphism or derivation

(If G acts trivially on A)

i.e. $ag = a$ for all $a \in A, g \in G$

then f is a homo. $G \rightarrow A$.

Extensions of A by G correspond to elements of $H^1 = Z^1/B^1$.

$f \in B^1$ (1-boundary) iff

$$f(x) = ax - a = (\delta a)(x), \quad a \in A.$$

(principal crossed homomorphisms)

(inner derivations)

Ex. Classify extensions of $C_4 = \langle x : x^4 = 1 \rangle$ by $C_2 = \langle y : y^2 = 1 \rangle$

$$1 \longrightarrow C_4 \longrightarrow \hat{G} \longrightarrow C_2 \longrightarrow 1$$

Two cases depending on the action of C_2 on C_4

Case I: y inverts x i.e. $yxy^{-1} = x^{-1} = x^3$

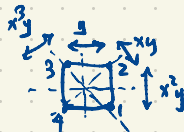
$|H'| = 2$. = how many complementary subgps of G up to conjugacy.

C_4 has four complementary subgps in $\hat{G} \cong$ dihedral gp of order 8.

$\langle y \rangle, \langle x^2y \rangle$ are conjugate to each other in \hat{G} ↗ Not conjugate.

$\langle xy \rangle, \langle x^3y \rangle$ are conjugate to each other in \hat{G} ↖

$$\begin{aligned} x &= (1234) \\ y &= (14)(23) \\ x^2y &= (12)(34) \\ xy &= (13) \\ x^3y &= (24) \end{aligned}$$



Case II: y commutes with x . $xy = yx$ $\hat{G} = C_4 \times C_2$

$\langle x \rangle$ has two complements in \hat{G} , namely $\langle y \rangle, \langle x^2y \rangle$. They are not conjugate. $|H'| = 2$.

How many extensions $1 \longrightarrow C_4 \longrightarrow \hat{G} \longrightarrow C_2 \longrightarrow 1$ are there up to equivalence, if we don't require the extension to be split? (Split \iff there is a complementary subgp for C_4)

eg. C_8 is a nonsplit extension of C_4 by C_2 .

Case I: C_2 acts trivially on C_4 . Here there are two extensions: C_8 (nonsplit) and $C_4 \times C_2$ (split).

Case II: C_2 acts nontrivially on C_4 . Here there are two extensions: dihedral of order 8 (split), quaternion gp of order 8 (nonsplit).

The Schur-Zassenhaus Theorem

Given groups G, N with G acting on N (the action of G on N is fixed) we consider exact sequences of groups

$$1 \longrightarrow N \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

i.e. extensions of N by G i.e. groups \hat{G} having a normal subgroup isomorphic to N .

If $|N|, |G|$ are relatively prime then $H^1 = 1$ and $H^2 = 1$.

This says that the extension splits i.e. \hat{G} has a subgroup complementary to N and any two complements of N are conjugate in \hat{G} .

This generalizes Sylow theory; N and its complements are Hall subgroups.

Note: We do not require N to be abelian. If N is abelian then the formulas are simpler. Even simpler if $N \subset Z(G)$.
(central extension of N by G)

A loop is a set L with a binary operation $(x, y) \mapsto xy$

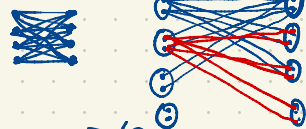
such that any two of x, y, xy uniquely determine the other.

We also assume $\exists 1 \in L$ such that $1x = x1 = x$ for all $x \in L$.

A Bol loop satisfies $((xy)z)y = x((yz)y)$ for all $x, y, z \in L$.

A Hadamard matrix is an $n \times n$ matrix H with entries ± 1 such that $HH^T = nI = H^T H$

eg. $H = \begin{bmatrix} 1 & & & \\ & \oplus & & \\ & & \ddots & \\ & & & \oplus \end{bmatrix}$ gives a (regular) double cover of $K_{n,n}$:



A complex Hadamard matrix is an $n \times n$ matrix H with entries in $S = \{z \in \mathbb{C} : |z| = 1\}$ such that $HH^* = nI = H^* H$

I classified complex Had. matrixes with an automorphism group which is doubly transitive on rows.

A proj. plane has points and lines satisfying



Any $(n^2+n+1) \times (n^2+n+1)$ matrix with $\neq 1$ entries, $n+1$ ones in any row/col, row \cdot row = 1 = col \cdot col

n = The order of the plane

In all known cases, $n = p^k$, p prime, $k \geq 1$.

Some long exact sequences in homology

Take a top. space X having a closed subspace $A \subset X$ which is the deformation retract of an open nbhd in X .



X/A : collapse A to a single point, leaving points outside A untouched.
quotient space

Then we have a long exact sequence

$$\begin{aligned} \dots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X/A) \\ \rightarrow \tilde{H}_{n-1}(A) \rightarrow \tilde{H}_{n-1}(X) \rightarrow \tilde{H}_{n-1}(X/A) \rightarrow \dots \end{aligned}$$

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$$\rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_0 \rightarrow 0$$

has $H_n(X)$

$$\rightarrow C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0$$

has reduced homology $\tilde{H}_n(X)$.

Theorem $H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n \in \{0, k\} \\ 0 & \text{otherwise} \end{cases}$

($k \geq 1$) i.e. $\tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{else} \end{cases}$

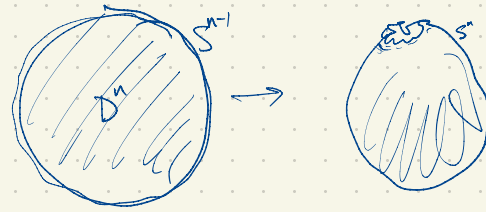
$$A \rightarrow X \rightarrow X/A$$

Proof: Take $X = D^n = \overset{\text{closed}}{n\text{-ball}}$ ($n \geq 1$)
 $A = \partial X = S^{n-1}$
 $X/A \cong S^n = \text{one-point compactification of } \mathbb{R}^n$

$$\tilde{H}_k(D_n) = 0 \text{ for all } k$$

$$\dots \rightarrow \tilde{H}_k\left(\frac{D^n}{X}\right) \rightarrow \tilde{H}_k\left(\frac{S^n}{XA}\right) \rightarrow \tilde{H}_{k-1}\left(\frac{S^{n-1}}{A}\right) \rightarrow \tilde{H}_{k-1}\left(\frac{D^n}{X}\right) \rightarrow \dots$$

$\begin{matrix} \parallel \\ \text{O} \end{matrix} \qquad \qquad \qquad \begin{matrix} \parallel \\ \text{O} \end{matrix}$



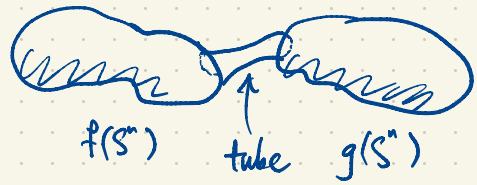
$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ depends only on $k-n$.
 $\tilde{H}_k(S^n) \cong \dots \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$

$$\tilde{H}_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k=1 \\ 0 & \text{else.} \end{cases}$$

Homotopy groups $\pi_n(X)$ "counts" ways to map $S^n \xrightarrow{f, g} X$

X is assumed to be path-connected.

In X :



Cut holes $\cong D^n$ in $f(S^n), g(S^n)$ and join using $\underbrace{S^{n-1} \times [0,1]}_{\text{tube}}$

Identity element: constant.

Elements of $\pi_n(X)$ are up to homotopy,

$f_0, f_1 : S^n \rightarrow X$ are homotopic iff there exists $f : S^n \times [0,1] \rightarrow X, (x,t) \mapsto \frac{f(x)}{t}$

The group operation is associative.

It's commutative for $n \geq 2$.

If X, Y have the same homotopy type then $\pi_n(X) \cong \pi_n(Y)$ for all n . (Not conversely)

If $\pi_k(X) = 1$ for $k < n$ and $\pi_n(X) \neq 1$, then $H_n(X) =$ abelianization of $\pi_n(X)$.
(theorem of Hurewicz)

Homotopy groups are easier to define than homology groups

much harder to compute

$\pi_n(S^k)$ is not known in general

But some is known:

$$\pi_k(S^n) = 0 \text{ for } 1 \leq k \leq n-1$$

$$\pi_n(S^n) \cong \mathbb{Z}$$

eg. $(1, 2) \in \mathbb{P}^1\mathbb{C}$
has preimage (fibre)
 $\left\{ (z, 2z) : \begin{array}{l} |z|^2 + |2z|^2 = 1 \\ |z|^2 = \frac{1}{5} \end{array} \right\}$
 $\left\{ e^{i\theta} \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) : \theta \in [0, 2\pi) \right\}$

$\pi_3(S^2) \cong \mathbb{Z}$ is due to the Hopf fibration

$$1 \rightarrow S^1 \rightarrow S^3 \rightarrow S^2 \rightarrow 1$$

This means we have a map $f: S^3 \rightarrow S^2$ which is surjective and all fibres are circles. In other words we can partition S^3 into circles.

$S^3 =$ unit sphere in $\mathbb{R}^4 =$ unit sphere in \mathbb{H} .

$$= \left\{ (z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1 \right\}$$

$$f(z, w) = \begin{cases} (1, \frac{w}{z}), & \text{if } z \neq 0 \\ (0, 1), & \text{if } z = 0 \end{cases} \in S^2 = \mathbb{P}^1\mathbb{C} = \text{Riemann sphere}$$

$$\begin{array}{l}
 1 \rightarrow S^0 \rightarrow S^1 \rightarrow S^1 \rightarrow 1 \quad \mathbb{R} \subset \mathbb{C} \\
 1 \rightarrow S^1 \rightarrow S^3 \rightarrow S^2 \rightarrow 1 \quad \mathbb{C} \subset \mathbb{H} \\
 1 \rightarrow S^3 \rightarrow S^7 \rightarrow S^4 \rightarrow 1 \quad \mathbb{H} \subset \mathbb{O} \\
 1 \rightarrow S^7 \rightarrow S^{15} \rightarrow S^8 \rightarrow 1 \quad \mathbb{O} \subset \mathbb{S}
 \end{array}$$

$1 \rightarrow S^1 \rightarrow S^1 \times S^2 \rightarrow S^2 \rightarrow 1$ trivial bundle
 has (global) nonvanishing section

The Hopf fibration does not have such a section; it is a nontrivial fibre bundle. (B = base space)

A fibre bundle over B with fibres F is a way of continuously attaching a copy of F at each point of B.

Trivial bundle: $E = F \times B$

Another way: eg. S^1 has more than one kind of line bundle (of dimension 1) having fibre space $F = \mathbb{R}$

(vector bundle over \mathbb{R})

$$\mathbb{R} \rightarrow E \rightarrow S^1$$

\uparrow
 "total space"

Trivial bundle

$$\mathbb{R} \rightarrow S^1 \times \mathbb{R} \rightarrow S^1$$

"infinite cylinder"



A section of a fibre bundle

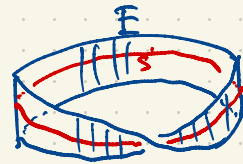
$$F \rightarrow E \xrightarrow{p} B$$

is a right inverse for p

i.e. $g: B \rightarrow E$

Such that $pg = id_B$

Nontrivial:



$$\mathbb{R} \rightarrow E \rightarrow S^1$$

Möbius strip (infinite)

This fibre bundle is nontrivial because it has no nonvanishing global sections.

For every fibre bundle $F \rightarrow E \rightarrow B$ we have a long exact sequence of homotopy groups

$$\dots \rightarrow \pi_k(F) \rightarrow \pi_k(E) \rightarrow \pi_k(B) \rightarrow \pi_{k-1}(F) \rightarrow \dots$$

A line bundle is a bundle with fibre space given by some field (usually \mathbb{R} or \mathbb{C})

A vector bundle has fibre space F^n (eg. $\mathbb{R}^n, \mathbb{C}^n, \dots$)

Sections of a vector bundle are the same thing as vector fields.

A circle bundle has fibre space $\cong S^1$.

A tangent bundle for a ^(hypersurface) surface $M \subset \mathbb{R}^n$ (M an $n-1$ -manifold) has fibres $\cong \mathbb{R}^{n-1}$.

