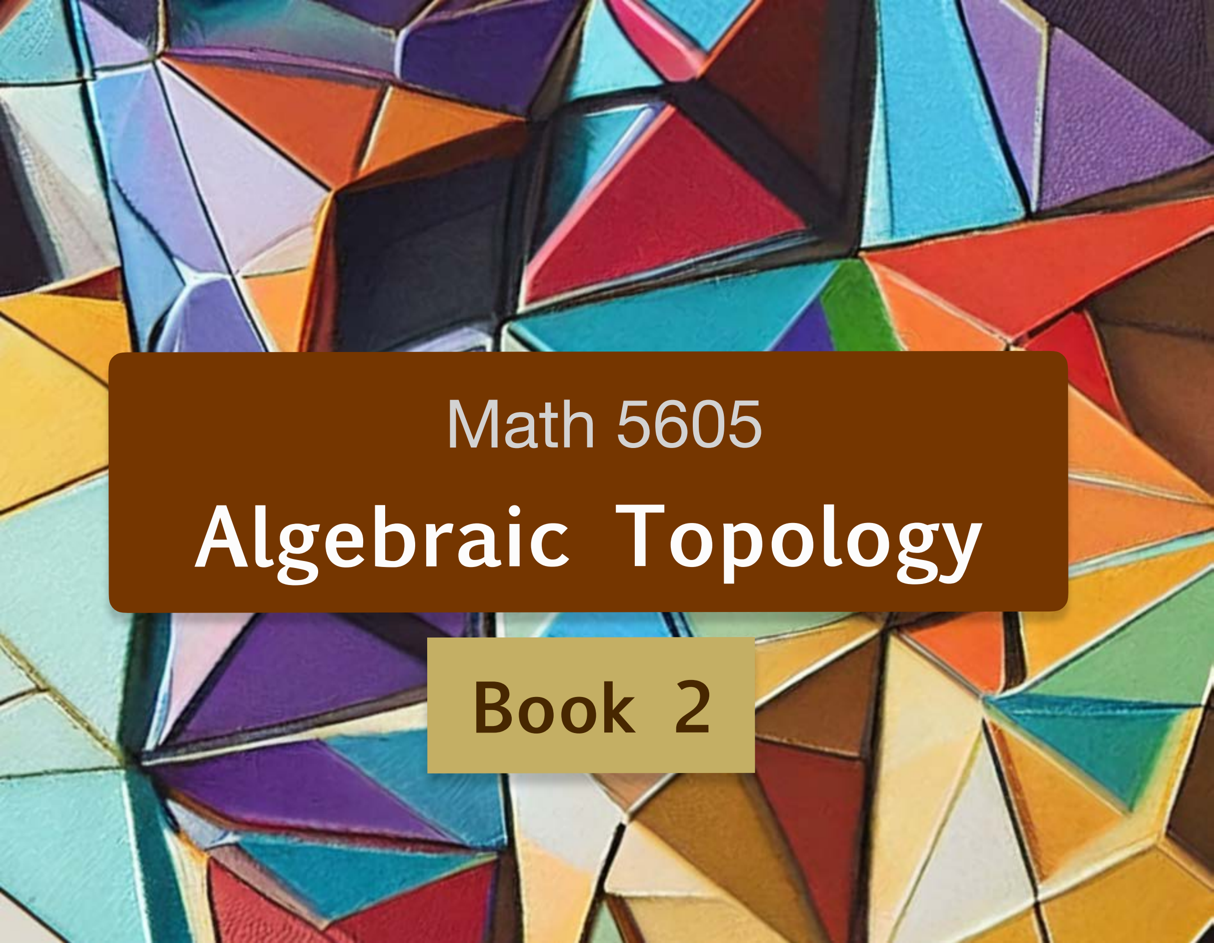
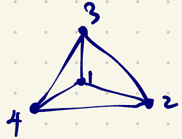


Math 5605

Algebraic Topology

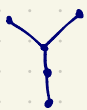
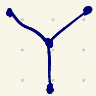
Book 2

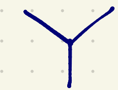
When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

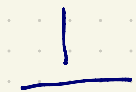
eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \dots, \{3,4\}\}$.

X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

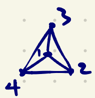
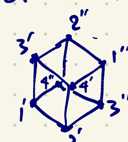
 and  have the same geometric realization although they are different graphs.





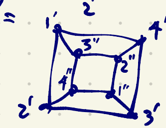
A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in Γ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'.
$(x, y \in V(\Gamma)$
$\{x,y\} \in E(\Gamma)$)

A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

Eg. a double cover of $X =$  is the 1-skeleton of the cube $Y =$ 

$Y \to X$

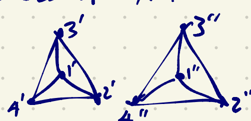
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

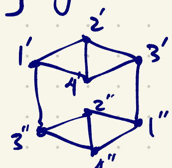
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



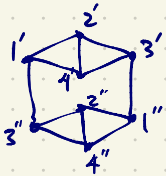
$Z \to X$

$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

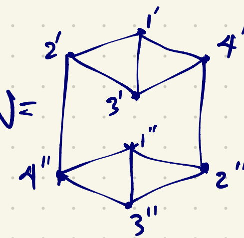
Let $p_1: X_1 \to X$, $p_2: X_2 \to X$ be covering spaces of X. We say $\theta: X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg. 

$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

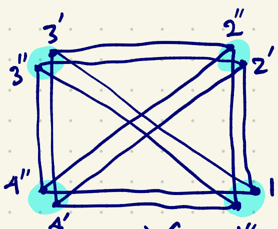
is not equivalent to $Y \to X$. But what about

$W =$ 

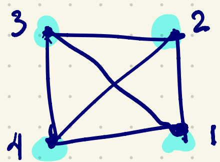
$W \to X$
$1', 1'' \mapsto 1$
$2', 2'' \mapsto 2$
$3', 3'' \mapsto 3$
$4', 4'' \mapsto 4$

Is this equivalent to $Z \to X$? No...

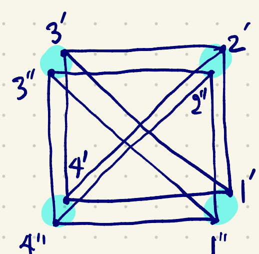
Another picture of these covers:



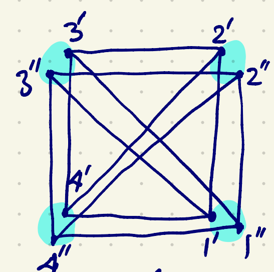
Y



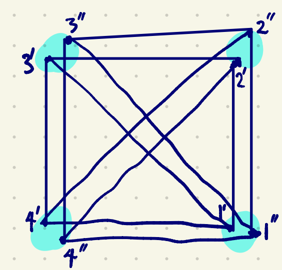
X



$X \sqcup X$



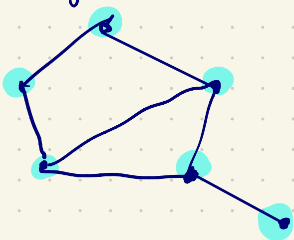
Y



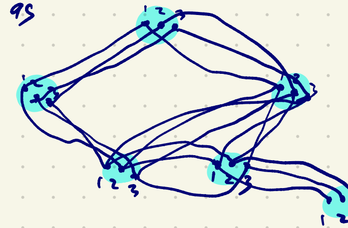
W

To construct an r-fold cover of X, create one copy of $[r] = \{1,2,\cdots,r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

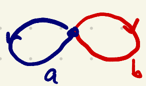
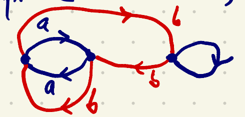
A triple cover $Y \to X$ is constructed as

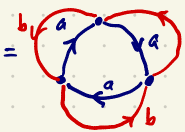


X



Why is 2 more special than other positive integers (the oddest prime of all)?

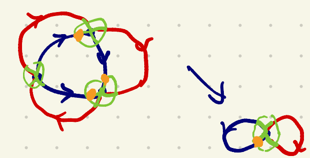
Consider $X =$  has many triple covers including $Y_1 =$ 

$Y_2 =$ 

The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

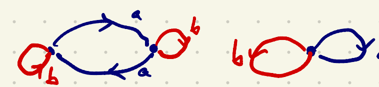
An equivalence between $Y \to X$ and itself (automorphism of the cover) is a <u>deck transformation</u>. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations). But $Y_1 \to X$ has only one (trivial) deck transformation.



In a connected r-fold cover, there are at most r deck transformations. If equality holds, the covering space is <u>normal</u> or <u>Galois</u>. (not the same as normal space in point set topology).

Double covers are always normal.

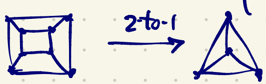
$Y_3 =$ 

In group theory, subgroups of index 2 are normal.

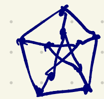
In the case of (separable) extensions of fields, the extension is normal.

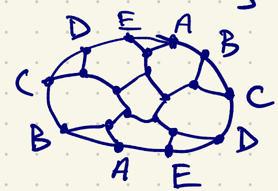
For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$ = dimension of E as a vector space over F. The number of F-automorphisms of E (i.e. $\sigma: E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers: examples



dodecahedral graph $\longrightarrow$ Petersen

$\longrightarrow$ 



$S^2 \longrightarrow P^2\mathbb{R}$ real proj. plane

S^n is not a top. group unless $n \in \{1,3\}$.

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$S^3 = \{z \in \mathbb{H} : |z| = 1\}$ $\quad\quad \mathbb{H} = \{a+bi+cj+dk : a,b,c,d \in \mathbb{R}\}$ $\quad\quad i^2 = j^2 = k^2 = ijk = -1$

$\cong SU_2(\mathbb{C}) = \left\{A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C},\ AA^* = A^*A = I,\ \det A = 1\right\}$

$SO_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I,\ \det A = 1\}$

$O_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I\}$ has two connected components

Fact: $S^3 \cong SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R})$ is a double cover.

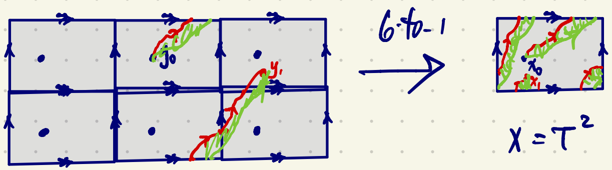
$Z(S^3) = \{\pm 1\}$

$PSU_2(\mathbb{C}) = S^3/Z(S^3) \cong SO_3(\mathbb{R}) \overset{\text{homeomorphism}}{\cong} P^3\mathbb{R}$.

$S^3 \longrightarrow SO_3(\mathbb{R})$

In general for $n \geq 3$, $\pi_1(\underbrace{SO_n(\mathbb{R})}) \cong \mathbb{Z}/2\mathbb{Z}$.

$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R})$ is its universal cover; a simply connected double cover constructed from Clifford Algebras (generalizing $\mathbb{H}$).



$Y = T^2$ $\qquad$ $X = T^2$

$f_0 : [0,1] \to X$ path in X from $f(0) = x_0$ to $f(1) = x_1$

$f_1 : [0,1] \to X$ is another path in X from x_0 to x_1 homotopic to f_0

$f_t : [0,1] \to X$

$f : [0,1]^2 \to X$

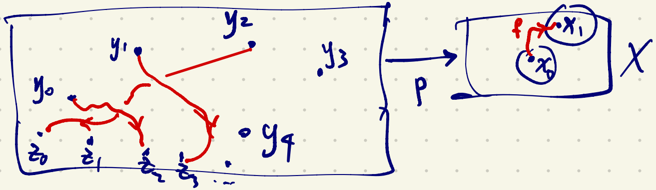
In any covering space $p : Y \to X$ and given any path $f : [0,1] \to X$ starting at $f(0) = x_0$, the path f can be lifted to Y

i.e. there is a path $g : [0,1] \to Y$ such that $f = p \circ g$ i.e.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

and this lift is unique if we say which of the points in $f^{-1}(x_0)$ to take as the starting point for g.

Assuming X is path-connected and $p : Y \to X$ is a path-connected covering space, $X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff $p(y_0) = p(y_1)$.



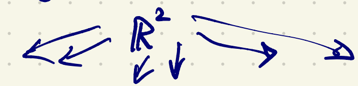
$p^{-1}(x_0) = \{ y_0, y_1, y_2, \cdots \}$

$p^{-1}(x_1) = \{ z_0, z_1, z_2, \cdots \}$

Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$.

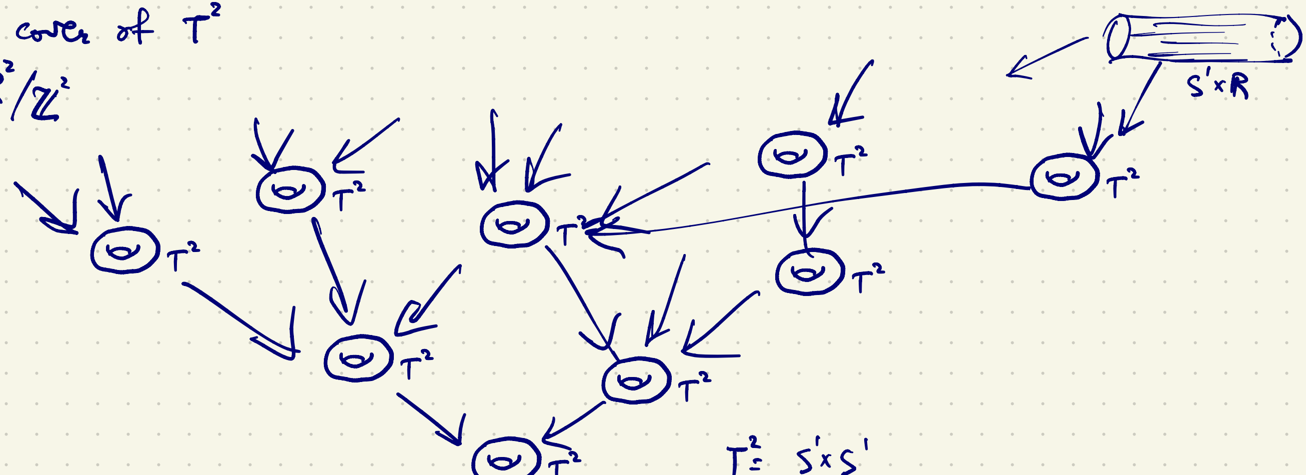
In particular if p is k-to-1 at x_0 i.e. $|p^{-1}(x_0)| = k$ then it is k-to-1 everywhere i.e. $|p^{-1}(x)| = k$ for all $x \in X$.

More generally, if f_t is a homotopy in X and we are given f_0, then every lifting of f_0 to Y extends to a lifting of f_t to Y.



$\mathbb{R}^2$ is the universal cover of T^2

$\mathbb{R}^2 \xrightarrow{p} T^2 = \mathbb{R}^2/\mathbb{Z}^2$



$T^2 = S^1 \times S^1$

Which top. spaces have a universal cover? (equivalently a path-connected and simply connected cover)

Let X be a path-connected space. Then X has a universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected