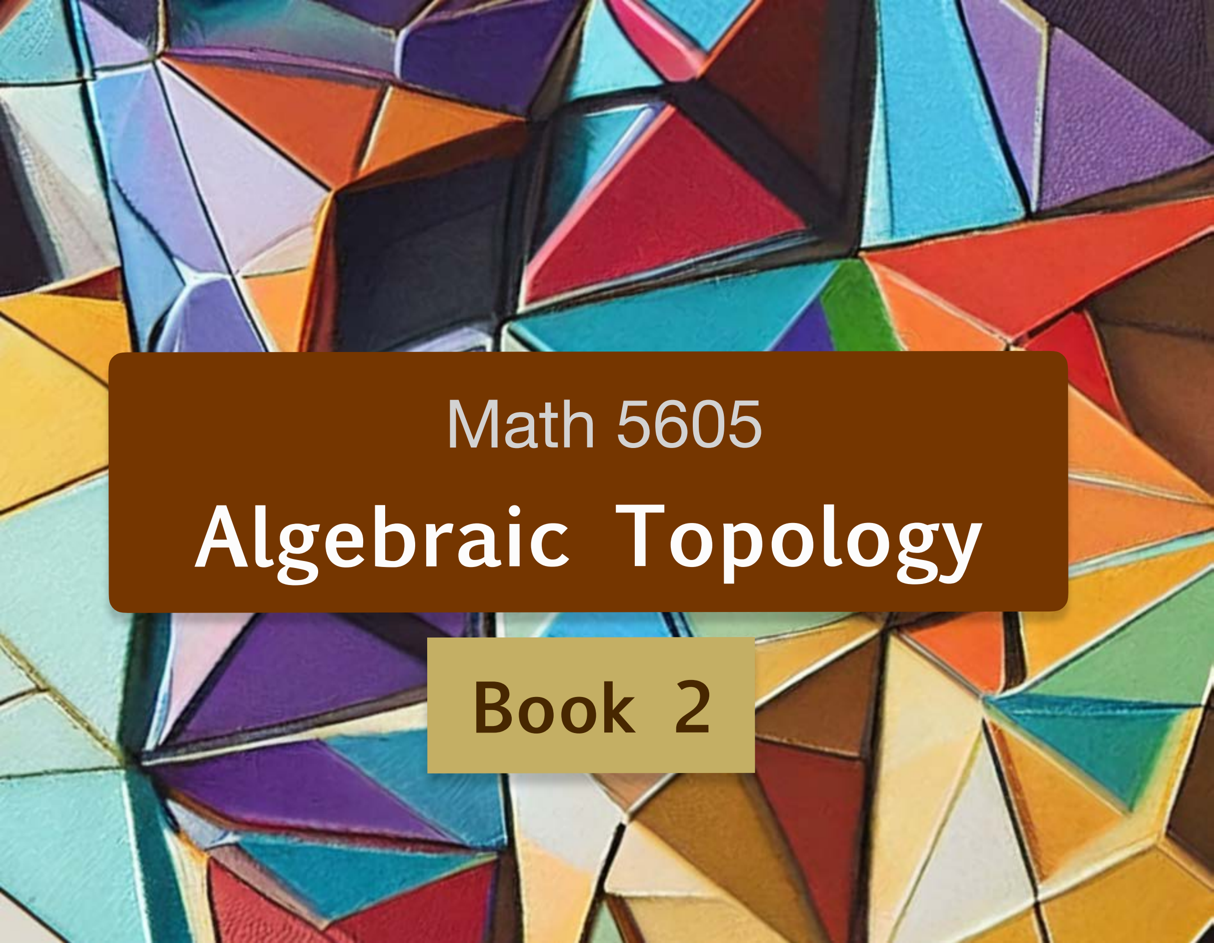
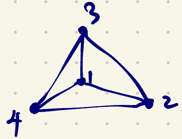


Math 5605

Algebraic Topology

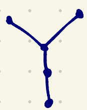
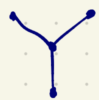
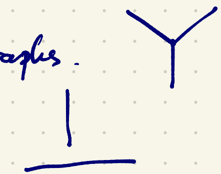
Book 2

When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \ldots, \{3,4\}\}$.

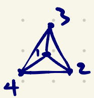
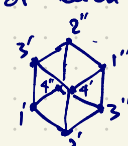
X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

 and  have the same geometric realization although they are different graphs. 

A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency

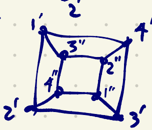
i.e. $x \sim y$ in Γ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'. A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

$(x, y \in V(\Gamma)$
$\{x,y\} \in E(\Gamma)\ )$

Eg. a double cover of $X =$  is the 1-skeleton of the cube $Y =$ 

$Y \longrightarrow X$

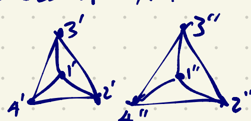
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

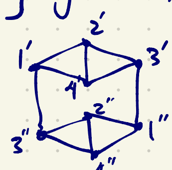
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



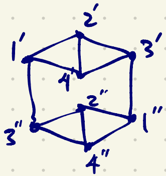
$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

Let $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ be covering spaces of X. We say $\theta : X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

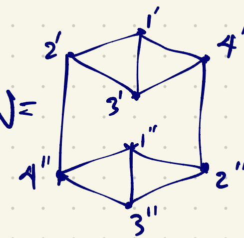
$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg.



$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

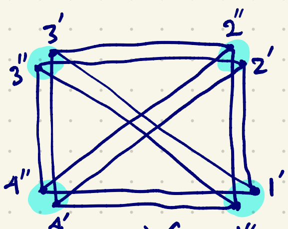
is not equivalent to $Y \to X$. But what about

$W =$ 

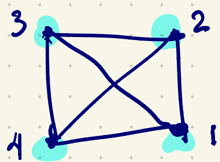
$W \to X$
$1', 1'' \mapsto 1$
$2', 2'' \mapsto 2$
$3', 3'' \mapsto 3$
$4', 4'' \mapsto 4$

Is this equivalent to $Z \to X$? No...

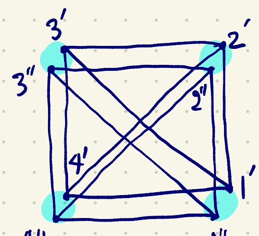
Another picture of these covers:



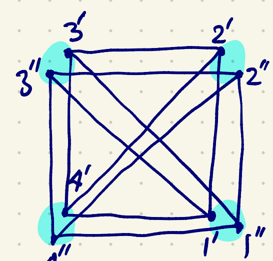
Y



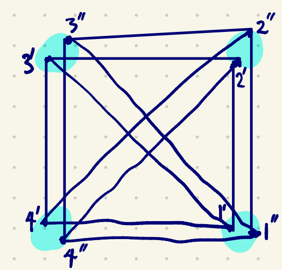
X



$X \sqcup X$

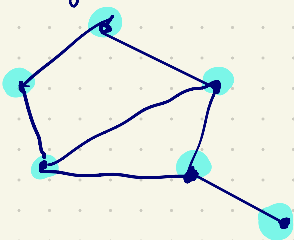


Y



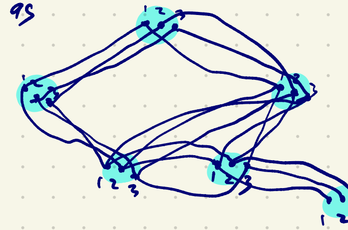
W

To construct an r-fold cover of X, created one copy of $[r] = \{1,2,\cdots,r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

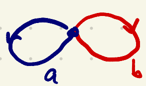


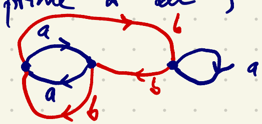
X

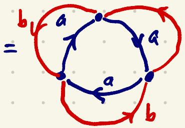
A triple cover $Y \to X$ is constructed as



Why is 2 more special than other positive integers (the oddest prime of all)?

Consider $X =$ 

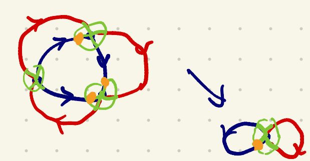
has many triple covers including $Y_1 =$ 

$Y_2 =$ 

The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

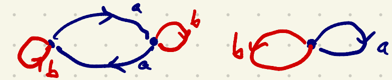
An equivalence between $Y \to X$ and itself (automorphism of the cover) is a <u>deck transformation</u>. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations). But $Y_1 \to X$ has only one (trivial) deck transformation.



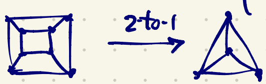
In a connected r-fold cover, there are at most r deck transformations. If equality holds, the covering space is <u>normal</u> or <u>Galois</u>. (not the same as normal space in point set topology).

Double covers are always normal.

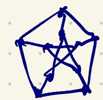
$Y_3 =$ 

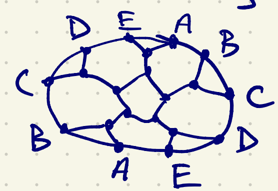
In group theory, subgroups of index 2 are normal.
In the case of (separable) extensions of fields, the extension is normal.
For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$: dimension of E as a vector space over F. The number of F-automorphisms of E (i.e. $\sigma : E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers : examples



dodecahedral graph $\longrightarrow$ Petersen

$\longrightarrow$ 



$S^2 \longrightarrow P^2\mathbb{R}$
real proj. plane

S^n is not a top. group unless $n \in \{1, 3\}$.

$$S^1 = \{ z \in \mathbb{C} : |z| = 1 \}.$$

$$S^3 = \{ z \in \mathbb{H} : |z| = 1 \} \qquad \mathbb{H} = \{ a + bi + cj + dk : a, b, c, d \in \mathbb{R} \} \qquad i^2 = j^2 = k^2 = ijk = -1$$

$$\cong SU_2(\mathbb{C}) = \left\{ A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C},\ AA^* = A^*A = I,\ \det A = 1 \right\}$$

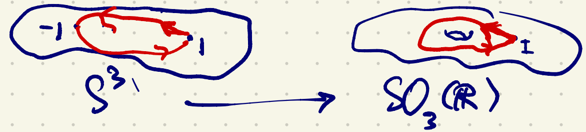
$$SO_3(\mathbb{R}) = \{ A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I,\ \det A = 1 \}$$

$$O_3(\mathbb{R}) = \{ A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I \}$$ has two connected components

$$Z(S^3) = \{ \pm 1 \}$$

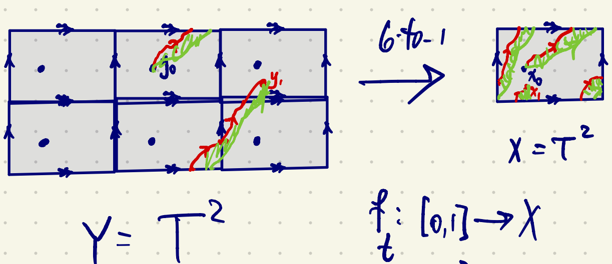
Fact : $S^3 \cong SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R})$ is a double cover.

$$PSU_2(\mathbb{C}) = S^3/Z(S^3) \cong SO_3(\mathbb{R}) \overset{\text{homeomorphism}}{\cong} P^3\mathbb{R}.$$



In general for $n \geq 3$, $\pi_1(\underbrace{SO_n(\mathbb{R})}) \cong \mathbb{Z}/2\mathbb{Z}$.

$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R})$ is its universal cover; a simply connected double cover constructed from Clifford Algebras (generalizing $\mathbb{H}$).



$f_0 : [0,1] \to X$ path in X from $f(0) = x_0$ to $f(1) = x_1$

$f_1 : [0,1] \to X$ is another path in X from x_0 to x_1 homotopic to f_0

$f_t : [0,1] \to X$

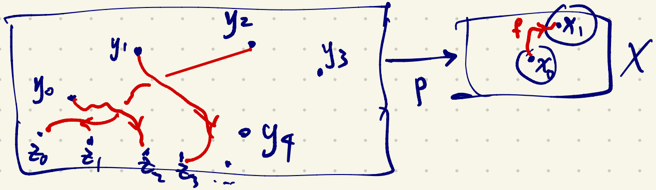
$f : [0,1]^2 \to X$

In any covering space $p: Y \to X$ and given any path $f: [0,1] \to X$ starting at $f(0) = x_0$, the path f can be lifted to Y i.e. there is a path $g: [0,1] \to Y$ such that $f = p \circ g$ i.e.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

and this lift is unique if we say which of the points in $f^{-1}(x_0)$ to take as the starting point for g.

Assuming X is path-connected and $p: Y \to X$ is a path-connected covering space, $X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff $p(y_0) = p(y_1)$.



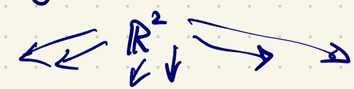
$p^{-1}(x_0) = \{y_0, y_1, y_2, \cdots\}$

$p^{-1}(x_1) = \{z_0, z_1, z_2, \cdots\}$

Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$.

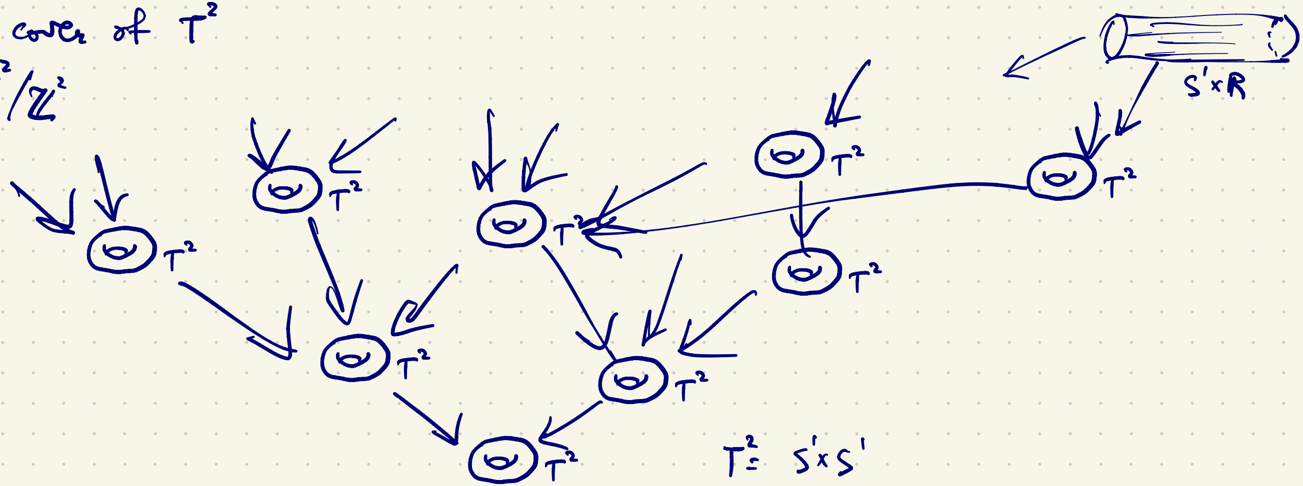
In particular if p is k-to-1 at x_0 i.e. $|p^{-1}(x_0)| = k$ then it is k-to-1 everywhere i.e. $|p^{-1}(x)| = k$ for all $x \in X$.

More generally, if f_t is a homotopy in X and we are given f_0, then every lifting of f_0 to Y extends to a lifting of f_t to Y.



$\mathbb{R}^2$ is the universal cover of T^2

$$\mathbb{R}^2 \xrightarrow{p} T^2 = \mathbb{R}^2/\mathbb{Z}^2$$

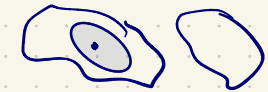


$T^2 = S^1 \times S^1$

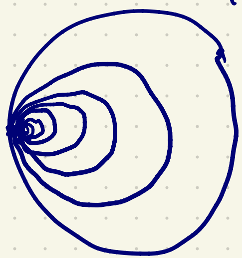
Which top. spaces have a universal cover? (equivalently a path-connected and simply connected cover)

Let X be a path-connected space. Then X has a universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected



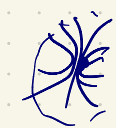
Example of a top. space without a universal cover: Hawaiian earring $\subset \mathbb{R}^2$



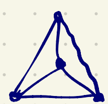
(not a CW complex)

$\not\cong \underbrace{S^1 \vee S^1 \vee S^1 \vee \dots}$

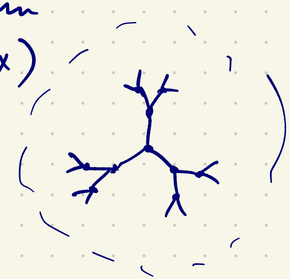
countable wedge sum (CW complex)



Universal cover of K_4

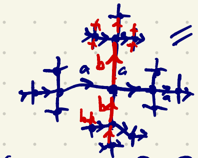
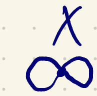


: trivalent tree



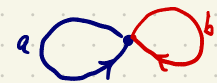
(also the universal cover of any trivalent connected graph)
i.e. regular of degree 3 connected

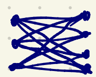
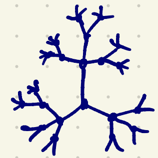
Universal cover of any connected regular graph of degree 4 is

 $= \tilde{X} \xrightarrow{p} X$ 

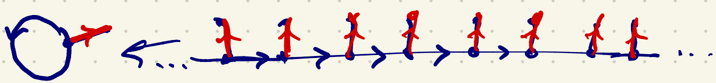
Cayley graph of $\mathrm{Free}\{a,b\} = G$
Vertices correspond to elements of G.

Every vertex $w \in G$ has
edges to wa, wa^{-1}, wb, wb^{-1}.



Universal cover of $K_{3,4}$  

$\tilde{X} = X/G$



$P^2\mathbb{R}$ has S^2 as its universal cover.

$S^2 \longrightarrow P^2\mathbb{R}$
$= S^2/G$

quotient of S^2
by the antipodal relation.

$G = \{1, -1\}$ acts on S^2

$1x = x$
$(-1)x = -x$ (antipode of x)

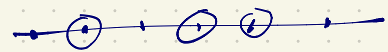
$X/\sim$ = partition of X into equivalence classes of the equiv. relation "$\sim$"

X/G = partition of X into the orbits of G

($x \sim xg$ or $g(x)$ for all $g \in G$.)

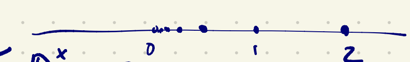
$X \longrightarrow X/\sim$

$\mathbb{R}/\mathbb{Z} \cong S^1$



$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 = S^1 \times S^1$

A non-discrete action of $\mathbb{Z}$ on $\mathbb{R}$ eg. $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\} < \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$



G acts <u>discretely</u> on X if for every $x \in X$ there is an open nbhd U of x such that the only $g \in G$ mapping $x \mapsto x^g \in U$ is $g = 1$.

$G = \{x \mapsto 2^k x + l \ : \ k, l \in \mathbb{Z}\}$ is non-discrete

---

If X is "nice" (path-connected, locally path-connected, SLSC) then X has a simply connected (and path-connected) cover which is a universal cover. It is unique up to isomorphism of covering spaces.

$$\begin{tikzcd} \tilde{X} \arrow[dd, "p"'] \arrow[dr] & \\ & Y \arrow[dl] \\ X = \tilde{X}/G & \end{tikzcd}$$

$\tilde{X}$ universal cover. Fix $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$.

$G \cong \pi_1(X, x_0)$.

Every other covering space (path-connected) $Y \to X$

has the form $Y = \tilde{X}/H$, $H \leq G$.

eg.

$$\begin{tikzcd} \mathbb{R} = \tilde{S}^1 \arrow[d, "p"'] \arrow[r] & \mathbb{R}/H = S^1 \arrow[dl, "p_k"] \\ S^1 & \end{tikzcd}$$

$t \mapsto e^{2\pi i t/k}$

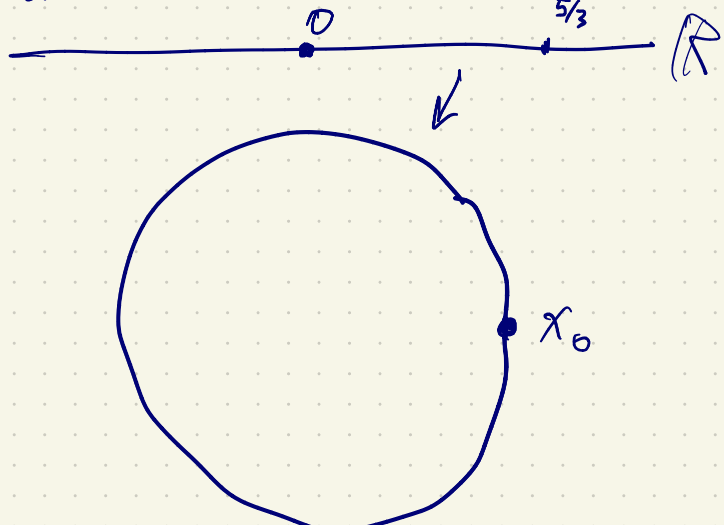
$p : t \longmapsto e^{2\pi i t}$

$p_k : z \longmapsto z^k$

$G = \mathbb{Z}$

$H \leq G$ has the form $H = k\mathbb{Z}$, $k \in \mathbb{Z}$.

Construction of the universal cover $\tilde{X} \to X$?



$\tilde{X} = \{\text{paths in } X \text{ starting at the chosen base point } x_0\}/\sim$

i.e. paths up to homotopy with fixed starting and ending point

If $f \in \tilde{X}$ then

$p(f) = f(1) \in X$ (the endpoint of f)

Cohomology. Consider a sequence of vector spaces over F given by

$$\cdots \xrightarrow{\partial_4} V_3 \xrightarrow{\partial_3} V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} 0$$

or

$$0 \xrightarrow{0} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} \cdots$$

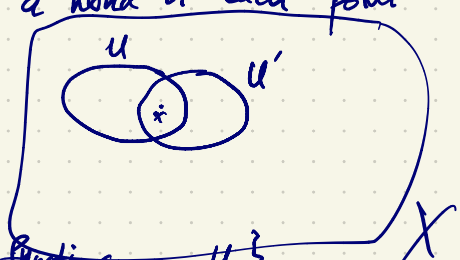
∂_i, d^i linear transformations (more generally V_i, V^i are modules over a ring R and ∂_i, d^i are R-homomorphisms i.e. $d(av+bw) = a\,dv + b\,dw$, $a,b \in F$; $v,w \in V$)

V^i or V_i has i just an index for purposes of reference.

If $\partial_i \circ \partial_{i+1} = 0$ then ∂ is a boundary map and the sequence of V_i's is a complex.
(similarly if $d^{i+1} \circ d^i = 0$, d is a coboundary map.)

Notable example: differential forms

Let X be a real (smooth) n-manifold. In a nbhd of each point $x \in X$, $x \in U \subseteq X$, we have local coordinates $(x_1, \dots, x_n) = x$.



$R = C^\infty(U) = \{$ smooth real-valued functions on $U \}$.

$V^0 = R$. $\quad d : V^0 \to V^1 = \{$ differential 1-forms on $U \} = \{ f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + \cdots + f_n dx_n : f_i \in R \}$

V^1 is a vector space over $\mathbb{R}$ (∞-dimensional) but n-dimensional as module over R

Eg. $X = \mathbb{R}^2 - \{(0,0)\}$.

$$0 \xrightarrow{0} V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} 0$$

$V^0 = \{\text{smooth functions } X \to \mathbb{R}\} = R = \text{" 0-forms "}$

$V^1 = \{\text{1-forms on } X\}$ i.e. smooth differential 1-forms

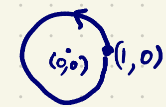
$V^2 = \{\text{2-forms on } X\}$

$V^1 = \{f\,dx + g\,dy : f, g \in R\}$

ω is closed but not exact

Eg. $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ $\qquad W = \omega$

Integrate ω over the path $\gamma(t) = (\cos t, \sin t)$ $\quad t \in [0, 2\pi]$



$$\int_\gamma \omega = \int_\gamma \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{\cos^2 t\,dt + \sin^2 t\,dt}{1} = \int_0^{2\pi} dt = 2\pi.$$

$x = \cos t$
$dx = -\sin t\,dt$
$y = \sin t$
$dy = \cos t\,dt$

x, y: global coordinates in X

r, θ: local coordinates (not global)

$x, y, r, \theta \in R = V^0$
coordinate functions $X \to \mathbb{R}$
or on $U \subset X$

$$\int_\gamma \omega = \int \cdots = 2\pi$$

$x = r\cos\theta$
$y = r\sin\theta$

$\omega = \cos\theta$

$\gamma(t)$: $r(t) = 1$
$\theta(t) = t$

$dx = \cos\theta\,dr - r\sin\theta\,d\theta$
$dy = \sin\theta\,dr + r\cos\theta\,d\theta$

$$V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2$$

$$f \longmapsto df$$
$$x \longmapsto dx$$
$$y \longmapsto dy$$
$$r \longmapsto dr$$

etc.

If X is an n-manifold then $V^k = \{k\text{-forms on } X\}$ is an R-module of dimension $\binom{n}{k}$.

We need X to be orientable

d is $\mathbb{R}$-linear but not R-linear

$d(fg) \neq f\,dg$

$V^2 = \{f\,dx \wedge dy : f \in R\}$

If X has local coordinates $x_1, \ldots, x_n$ then $V^1 = \{f_1 dx_1 + \cdots + f_n dx_n : f_i \in R\}$

$V^2 = \{f_{12}\, dx_1 \wedge dx_2 + f_{13}\, dx_1 \wedge dx_3 + \cdots : f_{ij} \in R\}$

$dx_i \wedge dx_i = 0$

$dx_i \wedge dx_j = -dx_j \wedge dx_i$

Wedge products are R-multilinear eg. $(f\omega + g\omega') \wedge \eta = f\,\omega \wedge \eta + g\,\omega' \wedge \eta$ $\quad f, g \in R$, ω, ω', η diff. forms

$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$

$= (-dy \wedge dx) \wedge dz) = -dy \wedge (dx \wedge dz) = -dy \wedge (-dz \wedge dx) = dy \wedge dz \wedge dx$

$dx \wedge (dy \wedge dz) = (dy \wedge dz) \wedge dx$

If ω is an i-form and η is a j-form then $\omega \wedge \eta = (-1)^{ij}\, \eta \wedge \omega$ is an $i{+}j$-form.

V^k is spanned by terms like $f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} =: \omega \in V^k$ WLOG $1 \le i_1 < i_2 < \cdots < i_k \le n$

$$V^k \xrightarrow{d} V^{k+1} \qquad d\omega = d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \underset{\sim}{df} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in V^{k+1}$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

In $\mathbb{R}^3$ with (global) coordinates x, y, z

$d: V^k \to V^{k+1}$ is $\mathbb{R}$-linear but not R-linear

$R = V^0 = \{ \text{smooth functions } \mathbb{R}^3 \to \mathbb{R} \}$

Pick $f \in V^0$ i.e. $f: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in V^1$$

This is a very special 1-form because it is <u>exact</u>. $(\in dV^0)$

$$d(df) = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz$$

$$= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial x} dz \right) \wedge dx + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial y} dz \right) \wedge dy$$

$$+ \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial z} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial z} dz \right) \wedge dz = 0$$

$$\boxed{d^2 = 0}$$

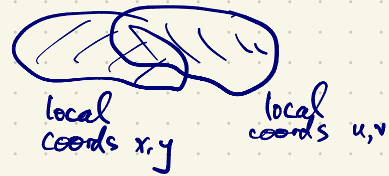
i.e. $d^2\omega := d(d\omega) = 0$ for all ω.

Imagine a surface $S \subset \mathbb{R}^3$. We integrate an arbitrary 2-form $\omega \in V^2$ over S.

If $\omega = f(x,y,z)\, dx \wedge dy + g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dy \wedge dz \in V^2$ then $\int_S \omega$

$$= \int_S f(x,y,z)\, dx \wedge dy + \cdots$$

If $x = x(u,v)$, $y = y(u,v)$



$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

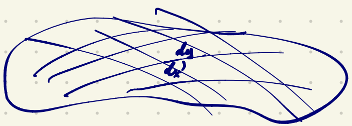
$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right)$$

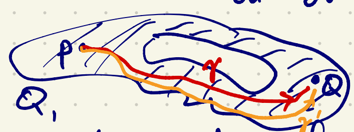
$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) du \wedge dv$$

then $f(x,y)\, dx \wedge dy$

$$= f(x(u,v), y(u,v)) \underbrace{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)}_{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}} du \wedge dv$$



For a region $X \subset \mathbb{R}^2$,
γ path in X from P to Q,
$\omega \in V^1$, we define the path integral $\int_\gamma \omega$.



If $\omega = df$ (an exact 1-form) then $\int_\gamma \omega = \int_\gamma df = f(Q) - f(P)$
by the Fundamental Theorem of calculus
But for

But if $\gamma' \sim \gamma$ in X then $\int_{\gamma'} \omega = \int_\gamma \omega = f(Q) - f(P)$ whenever $\omega = df$.

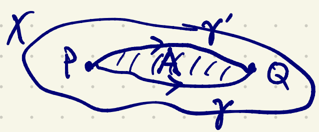
Stokes' Theorem (general Fundamental Theorem of Calculus)

Let X be an orientable n-manifold with ~~boundary~~ ∂X which is also orientable $(n-1)$-manifold. Let $\omega \in \Lambda^{n-1}$, so that $d\omega \in \Lambda^n$. Then

$$\boxed{\int_{\partial X} \omega = \int_X d\omega}$$

Special case: $X = [a,b] \subset \mathbb{R}$, $\partial X = \{a, b\}$, $\omega = f \in R$ (smooth function $X \to \mathbb{R}$)

$$\int_a^b f' = \int_a^b f'(t)\,dt = f(b) - f(a) \qquad d\omega = f'(x)\,dx$$



$\gamma \sim \gamma'$ in X $\qquad \partial A = \gamma - \gamma'$

If $\omega \in \Lambda^1(X)$ then $d\omega \in \Lambda^2(X)$

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\partial A} \omega = \int_A d\omega$$

If in particular $d\omega = 0$ (ω a <u>closed</u> 1-form) then RHS $= 0$ giving $\int_\gamma \omega = \int_{\gamma'} \omega$.

Exact forms are automatically closed (if $\omega = df$ then $d\omega = d^2 f = 0$).

Not conversely! unless X is simply connected.

The "gap" between {closed forms} and {exact forms} is measured by cohomology.

$$V^{n-1} \xrightarrow{d^{n-1}} V^{n} \xrightarrow{d^{n}} V^{n+1}$$
$(n-1)$-forms, n-forms, $(n+1)$-forms

image of $d^{n-1}: V^{n-1} \to V^{n}$ is $B^{n} = \{\text{exact } n\text{-forms}\}$
kernel of $d^{n}: V^{n} \to V^{n+1}$ is $Z^{n} = \{\text{closed } n\text{-forms}\}$
$H^{n} = Z^{n}/B^{n}$ = n^{th} cohomology group (or vector space over $\mathbb{R}$)

$\dim H^{n}$ is the number of "n-dim'l holes" in X.

$X = \mathbb{R}^2 \setminus \{0\}$ punctured plane

C stands for cochains; d is the coboundary map

$$0 \xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} 0$$

$C^k = \{\text{diff. } k\text{-forms on } X\} = C^k(X;\mathbb{R}) = C^k(X)$
$C^0 = \{\text{smooth functions } X \to \mathbb{R}\}$
$C^1 = \{ f(x,y)\,dx + g(x,y)\,dy : f,g \in C^0\}$
$C^2 = \{ h(x,y)\,dx \wedge dy : h \in C^0\}$

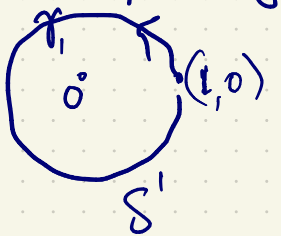
$\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ is closed (i.e. $d\omega = 0$)
but ω is not exact i.e. $\omega \neq df$ for any $f \in C^0$.

$H^1 = \{\text{closed 1-forms}\}/\{\text{exact 1-forms}\} = H^1(X;\mathbb{R})$
$\dim H^1 = 1$. Proof of this follows.

$df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$
$d(f\,dx) = df \wedge dx$, $d(g\,dy) = dg \wedge dy$.

Proof that $\dim H^1 = 1$.

Let η be any closed 1-form on X ie. $\eta \in C^1$, $d\eta = 0$.



Let $c = \int_{S^1} \eta = \int_0^{2\pi} (\quad) \, d\theta$ and set $\tilde{\eta} = \eta - \frac{c}{2\pi} \omega$

$d\tilde{\eta} = 0 - 0 = 0$ so $\tilde{\eta}$ is closed

and we claim $\tilde{\eta}$ is exact

i.e. $\eta = \underbrace{\tilde{\eta}}_{\text{exact}} + \frac{c}{2\pi}\omega$

To show $\tilde{\eta}$ is exact use:

For any two paths γ, γ' in X which are homotopic in X, (with common endpoints),

$$\int_\gamma \eta = \int_{\gamma'} \eta$$

$$0 = \int_A d\eta = \underbrace{\int_{\partial A} \eta}_{\text{Stokes' Theorem}} = \int_\gamma \eta - \int_{\gamma'} \eta$$

Here is our candidate $f \in C^0$ for which $df = \tilde{\eta}$.

For each $Q \in X$, define $f(Q) = \int_\gamma \tilde{\eta} = \int_{(1,0)}^x \tilde{\eta}$ where γ is any path in X from $(1,0)$ to Q.

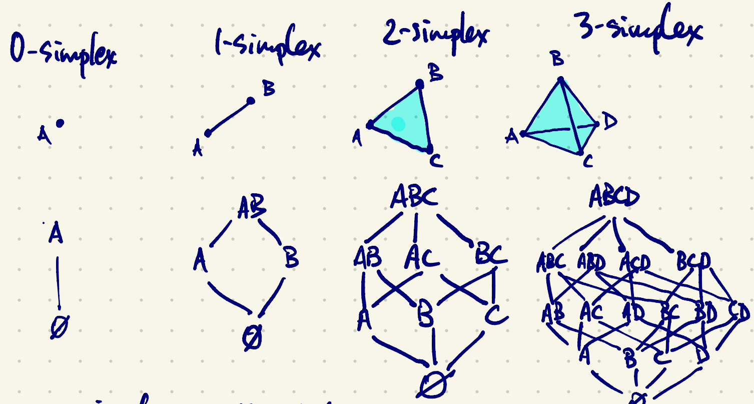
To see that this f is well-defined first fix one path γ_0 from $(1,0)$ to Q. Then any path γ in X from $(1,0)$ to Q is homotopic to γ_0 concatenated with γ_1^k so $\int_\gamma \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} + k \int_{\gamma_1} \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} \cancel{+ 0}$

Finally, $df = \tilde{\eta}$.

For $X = \mathbb{R}^2 - \{0\}$, (punctured plane), $\pi_1(X) \cong \mathbb{Z}$ since X is is contractible to S^1 and $\dim \underbrace{H^1(X;\mathbb{R})}_{\mathbb{R}} = 1$. These two facts are related by the theorem of Hurewicz.

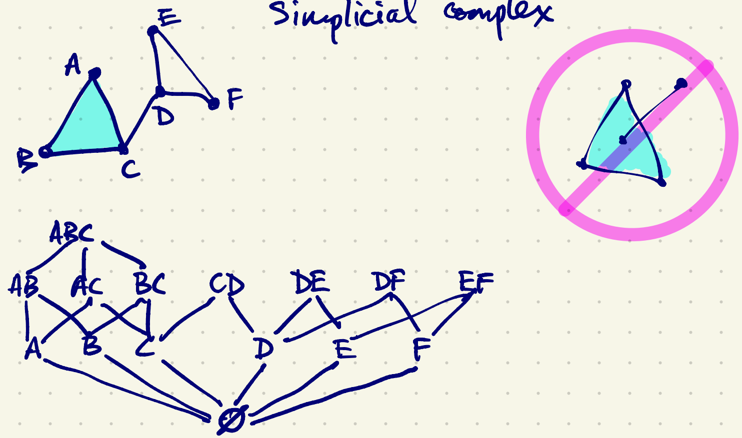
If we define our cohomology groups in a more universal way then $H^1(X) = H^1(X;\mathbb{Z})$ is an additive abelian group ie. $\mathbb{Z}$-module. ($\cong \mathbb{Z}$ in the case of $X = \mathbb{R}^2 - \{0\}$).

Hurewicz gave a homomorphism $\pi_1(X) \longrightarrow H_1(X) = H_1(X;\mathbb{Z})$ which is surjective; its kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ so $H_1(X)$ is the abelianization of $\pi_1(X)$.

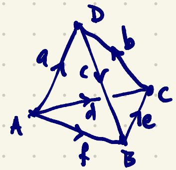


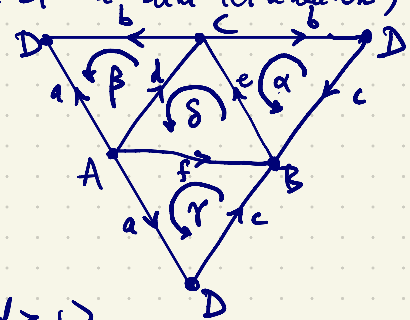
n-simplex: the lattice of subsets of an (n+1)-set.

Simplicial complex



eg. X = 2-skeleton of a 3-simplex
(i.e. surface of a solid tetrahedron)

X = 

= 

Every d-simplex $(d \geq 1)$ is orientable. Orient the faces arbitrarily

$X \cong S^2$ (homeomorphic top. spaces)
$\sim \mathbb{R}^3 - \{0\}$ (homotopic equivalent)

So we get the same algebraic invariants including homology groups)
as we'll point out later

$\partial a = D - A$
$\partial b = D - C$
$\partial c = B - D$
$\partial e = C - B$
$\partial f = B - A$
$\partial d = C - A$

$\partial \alpha = -b - c - e$
$\partial \beta = -a + b + d$
$\partial \gamma = a + c - f$
$\partial \delta = -d + e + f$

$\partial^2 \alpha = \partial(\partial \alpha) = \partial(-b-c-e)$
$= -(\cancel{D}-\cancel{C}) - (\cancel{B}-\cancel{D}) - (\cancel{C}-\cancel{B}) = 0$

$$\partial_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\partial_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We have a chain complex

$$0 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

where $C_k = \{k\text{-chains in } X\}$ $\qquad \partial$ = boundary map

A k-chain is a $\mathbb{Z}$-linear combination of k-faces of X (k-simplices)

$C_2 = \{x_1\alpha + x_2\beta + x_3\gamma + x_4\delta : x_1, x_2, x_3, x_4 \in \mathbb{Z}\}$
$C_1 = \{x_1 a + x_2 b + x_3 c + \cdots + x_6 f : x_1, \ldots, x_6 \in \mathbb{Z}\}$
$C_0 = \{x_1 A + x_2 B + x_3 C + x_4 D : x_1, \ldots, x_4 \in \mathbb{Z}\}$

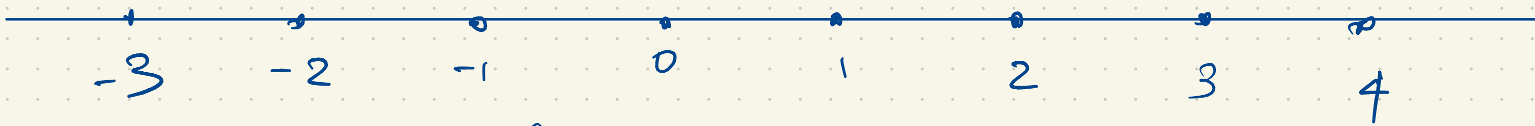
C_k is an additive abelian group i.e. $\mathbb{Z}$-module.

$\partial_k : C_k \longrightarrow C_{k-1}$ is additive i.e.
$\partial(u \pm v) = \partial u \pm \partial v$
$\Longrightarrow \partial(ru + sv) = r\partial u + s\partial v$ for all $r, s \in \mathbb{Z}$

$\partial_1 \circ \partial_2 = 0$

i.e. ∂ is a $\mathbb{Z}$-module homomorphism

C_k is a <u>free</u> $\mathbb{Z}$-module



$\mathbb{R}/\sim \qquad x \sim y \iff (x = y \text{ or } x, y \in \mathbb{Z})$

$\underset{\text{homeomorphic}}{\cong} \bigvee_{n=-\infty}^{\infty} S^1$

$\mathbb{R}/\mathbb{Z} \cong S^1$ quotient of top. groups.

not the same (one is a group, the other not)

For any chain complex $\cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$ i.e. $\partial_{k-1} \circ \partial_k = 0$

$\partial = (\partial_k)_k$ homomorphisms of additive abelian groups

$Z_k, B_k \leq C_k$ subgroups of $C_k = \{k\text{-chains}\}$

$B_k = \partial_{k+1} C_{k+1} =$ image of $\partial_{k+1} : C_{k+1} \longrightarrow C_k = \{k\text{-boundaries}\}$

$Z_k = \ker(\partial_k : C_k \to C_{k-1}) = \{k\text{-cycles}\}$

$B_k \leq Z_k \leq C_k$

The k^{th} homology group $H_k = Z_k / B_k$

Two elements $z, z' \in Z_k$ are homologous if $z + B_k = z' + B_k \iff z - z' \in B_k$.

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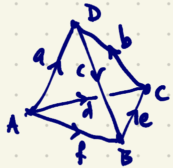
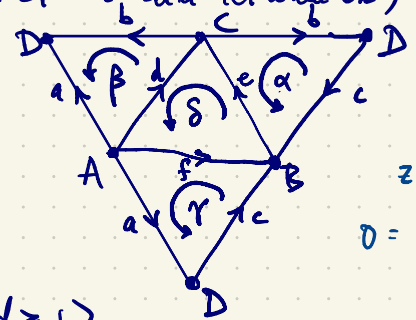
$X =$  $=$ 

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so we get the same algebraic invariants including homology groups) as we'll point out later

$$\begin{aligned} \partial a &= D - A \\ \partial b &= D - C \\ \partial c &= B - D \\ \partial e &= C - B \\ \partial f &= B - A \\ \partial d &= C - A \end{aligned} \qquad \begin{aligned} \partial\alpha &= -b-c-e \\ \partial\beta &= -a+b+d \\ \partial\gamma &= a+c-f \\ \partial\delta &= -d+e+f \end{aligned}$$

$$0 \xrightarrow{0} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 \longrightarrow 0$$

$$H_1 = Z_1/B_1 =$$

$$Z_1 = \ker \partial_1 : C_1 \to C_0$$

$$z = z_1 a + z_2 b + z_3 c + z_4 d + z_5 e + z_6 f \in Z' \iff$$

$$0 = \partial z = z_1(D-A) + z_2(D-C) + z_3(B-D) + z_4(C-A) + z_5(C-B) + z_6(B-A)$$

$$= (-z_1 - z_4 - z_6)A + (z_3 - z_5 + z_6)B + (-z_2 + z_4 + z_5)C + (z_1 + z_2 - z_3)D$$

"null space" of over $\mathbb{Z}$

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad \begin{aligned} a &= -d-f \\ b &= d+e \\ c &= e-f \end{aligned}$$

$$\partial_1 = \begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline A & -1 & 0 & 0 & -1 & 0 & -1 \\ B & 0 & 0 & 1 & 0 & -1 & 1 \\ C & 0 & -1 & 0 & 1 & 1 & 0 \\ D & 1 & 1 & -1 & 0 & 0 & 0 \end{array}$$

$$\partial_2 = \begin{array}{c|cccc} & \alpha & \beta & \gamma & \delta \\ \hline a & 0 & -1 & 1 & 0 \\ b & -1 & 1 & 0 & 0 \\ c & -1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & -1 \\ e & -1 & 0 & 0 & 1 \\ f & 0 & 0 & -1 & 1 \end{array}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -d-f \\ d+e \\ e-f \\ d \\ e \\ f \end{bmatrix} = d\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B_1 = \langle \text{cols of } \partial_2 \rangle = Z_1$$

$$Z_1 = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} \cong \mathbb{Z}^3$$

$$\boxed{H_1 = Z_1/B_1 = 0}$$

$$\boxed{H_0 = Z_0 / B_0 \cong \mathbb{Z}} \qquad H_0 = Z_0/B_0 = \{ kA + B_0 \;:\; k \in \mathbb{Z} \}.$$

$$B_0 = \langle A-B, A-C, A-D, B-C, B-D, C-D \rangle = \{ x_1 A + x_2 B + x_3 C + x_4 D : \; x_1, \dots, x_4 \in \mathbb{Z}, \; x_1+x_2+x_3+x_4 = 0 \}$$

$$Z_0 = \langle A, B, C, D \rangle$$

$$x_1 A + x_2 B + x_3 C + x_4 D = \underbrace{(x_1+x_2+x_3+x_4)}_{k} A + \underbrace{(-x_2-x_3-x_4)A + x_2 B + x_3 C + x_4 D}_{\in B_0}$$

$H_2 = Z_2/B_2 = \langle \alpha+\beta+\gamma+\delta \rangle_{\mathbb{Z}} / 0 \cong \mathbb{Z} \leftarrow$ comes from (the abelianization of $\pi_2(X)$)

$H_1 = 0$

$H_0 \cong \mathbb{Z}$

If we had included the interior of the tetrahedron (X = 3-simplex) then

$$0 \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

and then $H_2(\text{3-simplex}) = 0$ and that's no surprise since the 3-simplex is contractible.

$H_0(X) \cong \mathbb{Z}^k$ where k is the number of path-connected components of X.

Reduced homology groups of X $\tilde{H}_k(X)$ are computed using the modified chain complex

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \xrightarrow{\partial_0} C_{-1} = \mathbb{Z} \to 0$$

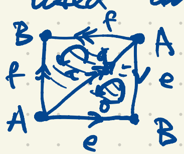
$$C_0 = \{0\text{-chains}\} = \left\{ \sum_i x_i A_i : x_i \in \mathbb{Z} \right\} = \langle A_i \text{ vertices} \rangle_{\mathbb{Z}}$$

$$\partial_0\left(\sum_i x_i A_i \right) = \sum_i x_i \in \mathbb{Z}$$

$$\tilde{H}_i(X) \cong \begin{cases} H_i(X), & i \geq 1 \\ \mathbb{Z}^{k-1}, & i = 0 \end{cases}$$

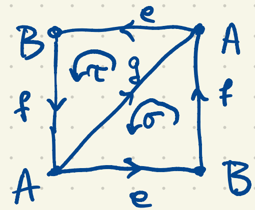
where k = no. of path-connected components in X.

We could have used any triangulation of S^2 and we'd get the same homology groups eg.



$$C_2 = \langle \sigma, \tau \rangle_{\mathbb{Z}}$$
$$C_1 = \langle e, f, g \rangle_{\mathbb{Z}}$$
$$C_0 = \langle A, B \rangle_{\mathbb{Z}}$$

Homology of $X \cong \mathbb{P}^2\mathbb{R}$



$$\partial e = B - A \qquad \partial_1 = \begin{array}{c|ccc} & e & f & g \\ \hline A & -1 & 1 & 0 \\ B & 1 & -1 & 0 \end{array}$$
$$\partial f = A - B$$
$$\partial g = A - A = 0$$

$$\partial\sigma = e + f - g \qquad \partial_2 = \begin{array}{c|cc} & \sigma & \tau \\ \hline e & 1 & 1 \\ f & 1 & 1 \\ g & -1 & 1 \end{array}$$
$$\partial\tau = e + f + g$$

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

$C_2 = \langle \sigma, \tau \rangle_{\mathbb{Z}}$, $C_1 = \langle e, f, g \rangle_{\mathbb{Z}}$, $C_0 = \langle A, B \rangle_{\mathbb{Z}}$

$$\partial_1 \circ \partial_2 = 0$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Z_1 = \ker \partial_1 = \langle e+f, g \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} = \{1\text{-cycles}\}$$

$$B_1 = \operatorname{im}(\partial_2 : C_2 \to C_1) = \left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} = \langle e+f-g, e+f+g \rangle_{\mathbb{Z}} = \langle e+f+g, 2g \rangle_{\mathbb{Z}}$$

Over $\mathbb{R}$, this simplifies to $\langle e+f, g \rangle_{\mathbb{R}}$

$$H_1 = H_1(X) = Z_1 / B_1 = \langle e+f, g \rangle / \langle e+f+g, 2g \rangle \cong \left(\langle g \rangle + \langle e+f+g, 2g \rangle \right) / \langle e+f+g, 2g \rangle$$
$$\cong \langle g \rangle / \left(\langle g \rangle \cap \langle e+f+g, 2g \rangle \right)$$
$$\cong \langle g \rangle / \langle 2g \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

$$H_0 \cong Z_0 / B_0 = \langle A, B \rangle / \langle A - B \rangle \cong \mathbb{Z}$$
$$H_2 \cong Z_2 / B_2 \cong \langle 0 \rangle / \langle 0 \rangle = 0$$

using 2^{nd} Isomorphism Theorem for Groups / Rings $(R+S)/S \cong R / R \cap S$

Tensoring (over $\mathbb{Z}$) with $\mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$ gives $H_k(X; \mathbb{F}_2)$

Tensoring (over $\mathbb{Z}$) with $\mathbb{R}$ gives $H_k(X; \mathbb{R})$

Simplicial cohomology is obtained by dualizing $\underbrace{C_k}_{\substack{\text{additive} \\ \text{abelian group} \\ \text{i.e. } \mathbb{Z}\text{-module}}} \rightsquigarrow C_k^* = \mathrm{Hom}(C_k, \mathbb{Z})$

$= \{\text{homomorphisms } C_k \to \mathbb{Z}\}$ of additive abel. gps / $\mathbb{Z}$-modules

Dualizing the chain complex $0 \to C_2 \to C_1 \to C_0 \to 0$

gives a cochain complex $0 \xleftarrow{0} C_2^* \xleftarrow{\partial_2^*} C_1^* \xleftarrow{\partial_1^*} C_0^* \xleftarrow{0} 0$

$C_2 = \langle \sigma, \tau \rangle$ $\quad C_2^* = \langle \phi_\sigma, \phi_\tau \rangle$

$C_1 = \langle e, f, g \rangle$ $\quad C_1^* = \langle \phi_e, \phi_f, \phi_g \rangle$

$C_0 = \langle A, B \rangle$ $\quad C_0^* = \langle \phi_A, \phi_B \rangle$

A k-cochain is a homomorphism $\phi : C_k \to \mathbb{Z}$

eg. a 0-cochain has the form $\phi(xA + yB) = x\,\phi(A) + y\,\phi(B)$ $\quad x, y \in \mathbb{Z}$

$\phi_A : C_0 \to \mathbb{Z}$

$\phi_A(A) = 1$

$\phi_A(B) = 0$

$(\phi \circ \psi)^* = \psi^* \circ \phi^*$

$$0 \xrightarrow{0} C_2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}} C_1 \xrightarrow{\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}} C_0 \xrightarrow{0} 0$$

$C_2 = \mathbb{Z}^2, \quad C_1 = \mathbb{Z}^3, \quad C_0 = \mathbb{Z}^2$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$0 \leftarrow C_2^* \xleftarrow{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}} C_1^* \xleftarrow{\begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}} C_0^* \leftarrow 0$$

$C_2^* = \mathbb{Z}^2, \quad C_1^* = \mathbb{Z}^3, \quad C_0^* = \mathbb{Z}^2$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$