

Math 5605

Algebraic Topology


Book 1



If X, Y are top. spaces, $f: X \rightarrow Y$ is continuous if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \rightarrow Y$ is a homeomorphism if f is bijective and f, f^{-1} are continuous.


$X \cong Y$ are homeomorphic if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

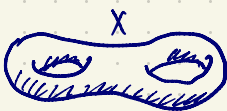
$\mathbb{R}^2 \not\cong S^1$ since S^2 is compact; \mathbb{R}^2 is not.

$S^2 \cong \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} =$ 

$S^2 \not\cong T^2 = \bigcup_{S^1} S^1 =$  $=$ 

S^2, T^2 are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$T^1 = S^1 =$  $=$ circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not. 

In S^1 , every closed path can be "continuously shrink" to a point (homotopic to a point, i.e. null-homotopic)

$X \not\cong T^2$ although both surfaces are compact, connected, not simply connected

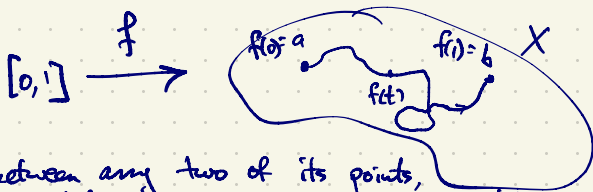
These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T^2) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of homotopy equivalence.

Eg. For every $n \geq 0$, \mathbb{R}^n is homotopy equivalent to $\mathbb{R}^0 = \{0\}$.

Given points $a, b \in X$ (a topological space), a path from a to b is a function $f: [0, 1] \rightarrow X$ such that $f(0) = a, f(1) = b$. (continuous)



All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X .) If $\varphi: [0, 1] \rightarrow [0, 1]$ (recall: continuous) such that $\varphi(0) = 0, \varphi(1) = 1$ then $f \circ \varphi: [0, 1] \rightarrow X$ is just a reparameterization of the same path and we don't distinguish it from f .

If $f, g: [0, 1] \rightarrow X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:



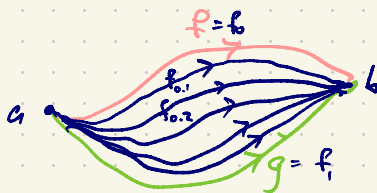
$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t) & t \in [0, \frac{1}{4}] \\ g(4t-1) & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(4t-2) & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3) & t \in [\frac{3}{4}, 1] \end{cases}$$



f, g are homotopic but h is not homotopic to f, g



More precisely, we require a map $[0, 1]^2 \rightarrow X$
 $(s, t) \mapsto f(s, t) = f_s(t)$
 such that $f_0 = f$ i.e. $f_0(t) = f(t)$
 $f_1 = g$ i.e. $f_1(t) = g(t)$
 $f_s(0) = a$
 $f_s(1) = b$ for all $s \in [0, 1]$
 This is a homotopy from f to g .

We say f, g are homotopic if there is a continuous family of paths from a to b in X , f_s ($s \in [0, 1]$) with $f_0 = f, f_1 = g$.

If $\varphi: [0,1] \rightarrow [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f \circ \varphi: [0,1] \rightarrow X$ is homotopic to f . A homotopy from f to $f \circ \varphi$ is

$$[0,1]^2 \rightarrow X$$

$$(s,t) \mapsto f(\underbrace{(1-s)t + s\varphi(t)}_{\uparrow [0,1]}) = f_s(t)$$

$$f_0(t) = f(t)$$

$$f_s(t) = f(\varphi(t))$$

$$f_s(0) = f((1-s) \cdot 0 + s \cdot \varphi(0)) = f(0)$$

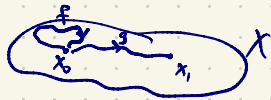
$$f_s(1) = f((1-s) \cdot 1 + s \cdot \varphi(1)) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$$\pi_1(\mathbb{R}^n) = 1 \quad (\text{trivial group}).$$

$$\pi_1(S^1) \cong \mathbb{Z} \quad (\text{free group on one generator})$$



Fix g path in X from x_0 to x_1 .

A isomorphism $\phi: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$

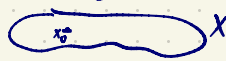
$$f \xrightarrow{\phi} \bar{g}fg$$

$$gf\bar{g} \xleftarrow{\phi^{-1}} h$$

$$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$

$$f_1, f_2 \in \pi_1(X, x_0)$$

γ : Identity in $\pi_1(X, x_0)$



$$\gamma(t) = x_0 \quad \text{for } t \in [0,1]$$

$$\gamma f = f \gamma = f \quad \text{for all } f \in \pi_1(X, x_0)$$

The inverse of $f \in \pi_1(X, x_0)$ is

$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$

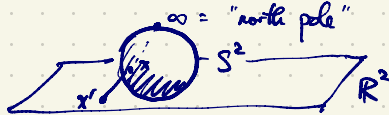
(same path in the reverse direction)



$$f \bar{f} = \bar{f} f = \gamma = \text{null path}$$

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)



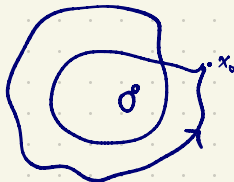
$x \mapsto x'$ is stereographic projection from the north pole ∞



See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
punctured plane



follows from the fact that

$\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S^1$

$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic)

Hatcher writes $X \approx Y$ for homeomorphic

- retraction
- deformation retraction

"def. retraction in the weak sense"

- strong deformation retraction

"def. retraction"

- homotopy
- relative homotopy
- homotopy equivalence

Let $A \subseteq X$ (subspace of a top. space).

A retraction $f: X \rightarrow A$ is a ^(continuous) map such that $f|_A = id_A = 1_A$ i.e. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a retract of X .

Eg. \mathbb{R}^n has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \rightarrow \{a\}$, $x \mapsto a$ is a retraction.

$\mathbb{R}^2 \rightarrow x\text{-axis}$, $(x, y) \mapsto (x, 0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \rightarrow S^1$.
(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to a retraction), $A \subseteq X$.

i.e. $f: [0,1] \times X \rightarrow X$

$$f(t, x) = f_t(x)$$

$$f_0(x) = x$$

i.e. $f_0 = \text{id}_X$

$$f_1(x) \in A$$

f_1 is a retraction $X \rightarrow A$

$$f_t(a) = a \text{ for all } a \in A.$$

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X .

(This is stronger than retract)

Fig. $f: [0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f_t(x) = (1-t)x \text{ is a def. retraction to } \{0\}.$$

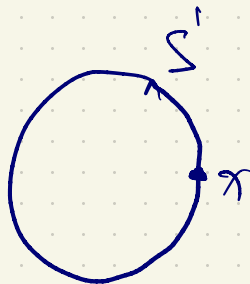
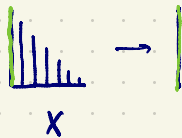
Fig. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1 .

A strong def. retract $f: [0,1] \times X \rightarrow X$:

$$f_0(x) = x \text{ i.e. } f_0 = \text{id}_X$$

f_t is a retraction $X \rightarrow A$

$$f_t|_A = \text{id}_A \text{ for all } t \in [0,1]$$



def. retract
but not strong def. retract.

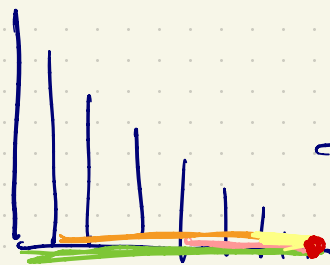
$$X \subseteq \mathbb{R}^2 \quad X = ([0,1] \times \{0\}) \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} (\{r\} \times [0,1-r])$$



This is a def. retract, but not a strong def. retract.

There is no strong def. retract $X \rightarrow A$.

X has a def. ret. to $A = \{0\} \times [0,1]$.



$$0 \leq t \leq \frac{1}{2}$$

$$\frac{1}{2} \leq t \leq 1$$



Let $f_0, f_1: X \rightarrow Y$ be maps. A homotopy from f_0 to f_1 is a map $f: [0,1] \times X \rightarrow Y$ such that $f(0,x) = f_0(x)$ and $f(1,x) = f_1(x)$. ("continuous deformation")

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 is a homotopy $f: [0,1] \times X \rightarrow Y$ such that $f_t(a)$ is independent of $t \in [0,1]$ for all $a \in A$. (constant)

A path from a to b in X is a homotopy from a to b .

Given two paths f_0, f_1 in X from a to b ($a, b \in X$) is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \xrightarrow{f} Y$ such that $f \circ g : Y \rightarrow Y$ and $g \circ f : X \rightarrow X$ are homotopic to id_Y and id_X respectively.

Eg. \mathbb{R}^n is homotopy equivalent to $\mathbb{R}^0 = \{0\}$ (or $\mathbb{R}^n \simeq \{0\}$)

$$\mathbb{R}^n \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} \{0\}$$

$$f(x) = 0 \text{ for all } x \in \mathbb{R}^n$$

$$g(0) = 0 \in \mathbb{R}^n$$

$$g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$x \mapsto 0$$

$$f \circ g : \{0\} \rightarrow \{0\}$$

$$0 \mapsto 0$$

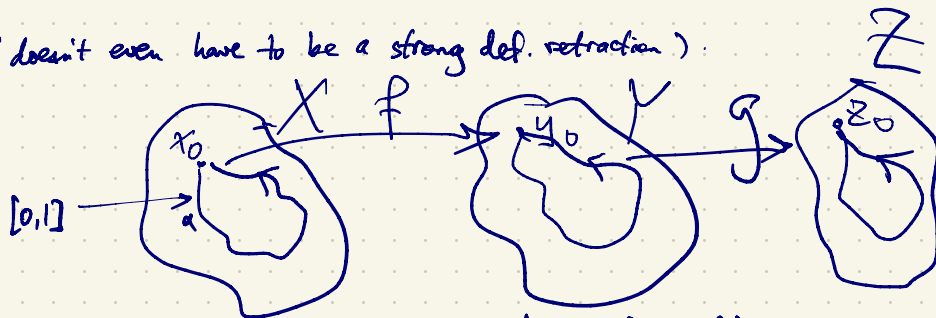
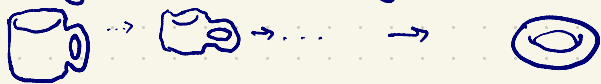
A homotopy from $g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to $\text{id}_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is $h_t(x) = tx$, $0 \leq t \leq 1$, $x \in \mathbb{R}^n$

Not relative to any subspace necessarily.

$$h_0 = g \circ f$$

$$h_1 = \text{id}_{\mathbb{R}^n}$$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).



S^1 is not homotopic to a point.
(S^1 is not null homotopic; not contractible)

If $f : X \rightarrow Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \rightarrow \pi_1(Y)$

$$\alpha : [0,1] \rightarrow X \text{ gives } f_* \alpha = f \circ \alpha : [0,1] \rightarrow Y$$

$$(g \circ f)_* = g_* \circ f_*$$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

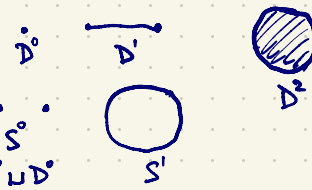
D^n = closed ball in \mathbb{R}^n
 disk

$S^n = \partial D^{n+1}$ = n-sphere

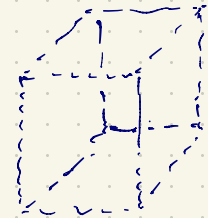
A CW complex is formed

from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$

X^n is a union of copies of D^n with the boundaries of D^n attached to X^{n-1} via attaching maps.



Eg. Torus $T^2 = S^1 \times S^1 =$

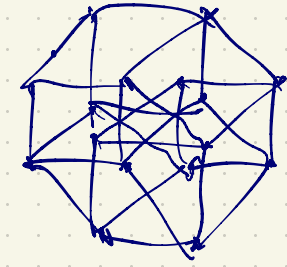


$X^0 = \bullet = D^0$

$X^1 =$  $\cong S^1 \vee S^1$



X^2



$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

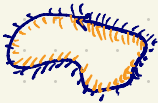
Möbius strip



not homeomorphic to cylinder $S^1 \times [0, 1]$



not orientable



Both are homotopy equivalent to S^1 (def. retract to S^1)
Both have \mathbb{Z} as fund. gp.

orientable