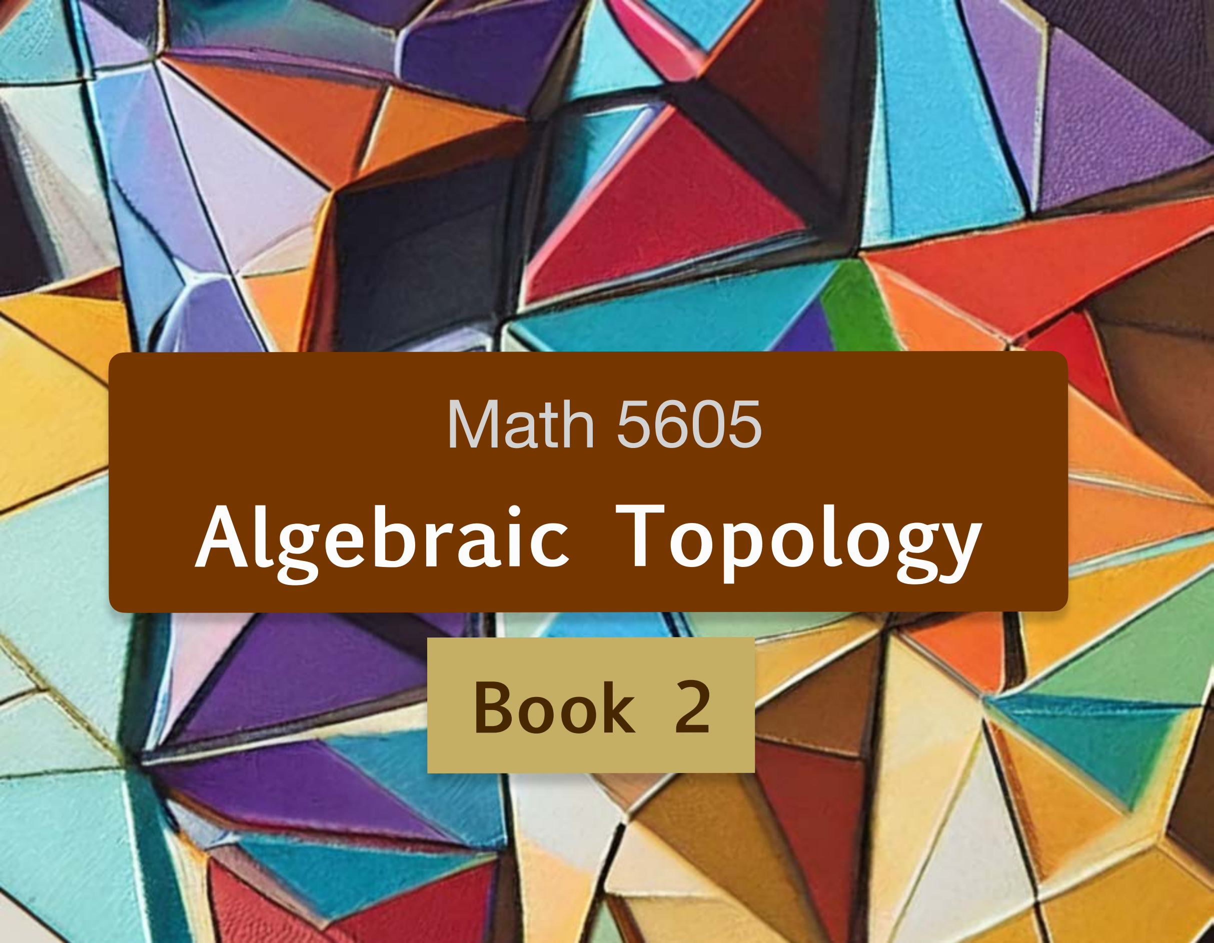
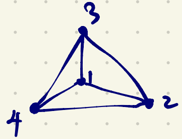


Math 5605

Algebraic Topology

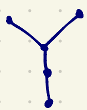
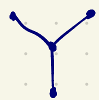
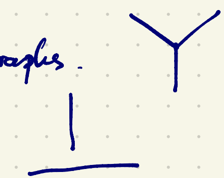
Book 2

When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \dots, \{3,4\}\}$.

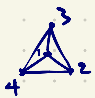
X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

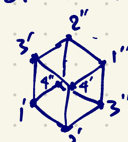
 and  have the same geometric realization although they are different graphs. 

A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency

i.e. $x \sim y$ in Γ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'.
$(x, y \in V(\Gamma)$
$\{x,y\} \in E(\Gamma)\ )$

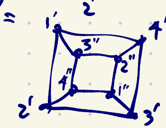
A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

Eg. a double cover of $X =$ 

is the 1-skeleton of the cube $Y =$ 

$Y \longrightarrow X$

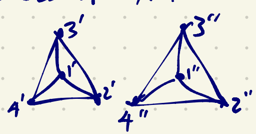
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

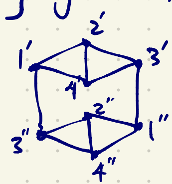
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



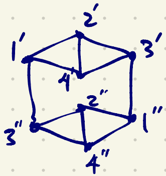
$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

Let $p_1: X_1 \to X$, $p_2: X_2 \to X$ be covering spaces of X. We say $\theta: X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg.



$Z \to X$

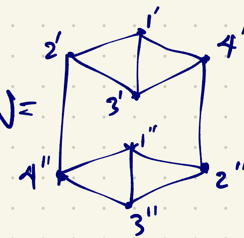
$1', 1'' \mapsto 1$

$\vdots$

$4', 4'' \mapsto 4$

is not equivalent to $Y \to X$. But what about

$W =$



$W \to X$

$1', 1'' \longmapsto 1$

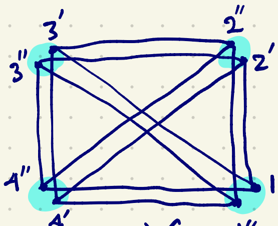
$2', 2'' \longmapsto 2$

$3', 3'' \longmapsto 3$

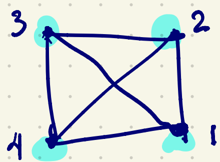
$4', 4'' \longmapsto 4$

Is this equivalent to $Z \to X$? No...

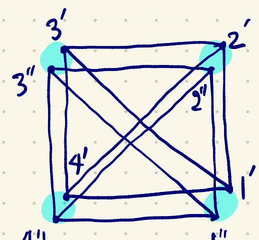
Another picture of these covers:



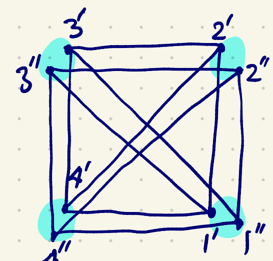
Y



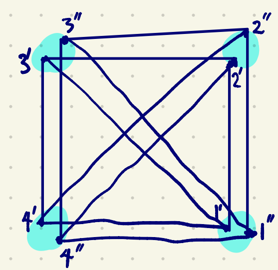
X



$X \sqcup X$

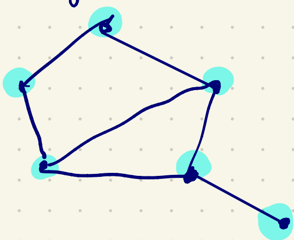


Y



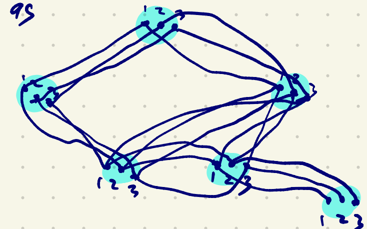
W

To construct an r-fold cover of X, created one copy of $[r] = \{1, 2, \cdots, r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

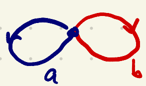
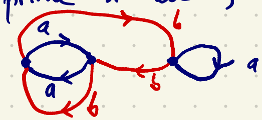


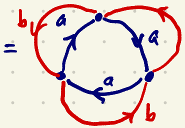
X

A triple cover $Y \to X$ is constructed as



Why is 2 more special than other positive integers (the oddest prime of all)?

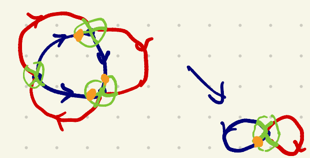
Consider $X =$  has many triple covers including $Y_1 =$ 

$Y_2 =$ 

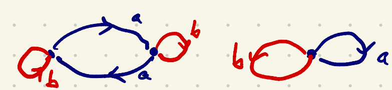
The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

An equivalence between $Y \to X$ and itself (automorphism of the cover) is a <u>deck transformation</u>. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations)
But $Y_1 \to X$ has only one (trivial) deck transformation.



In a connected r-fold cover, there are at most r deck transformations.
If equality holds, the covering space is <u>normal</u> or <u>Galois</u>.
(not the same as normal space in point set topology).
Double covers are always normal.

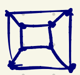
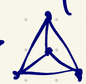
$Y_3 =$ 

In group theory, subgroups of index 2 are normal.

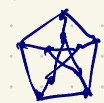
In the case of (separable) extensions of fields, the extension is normal.

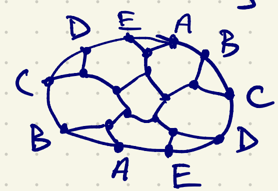
For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$ = dimension of E as a vector space over F. The number of F-automorphisms of E (i.e. $\sigma: E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers: examples

 $\xrightarrow{\text{2-to-1}}$ 

dodecahedral graph $\to$ Petersen

$\to$ 



$S^2 \longrightarrow P^2\mathbb{R}$

real proj. plane

S^n is not a top. group unless $n \in \{1, 3\}$.

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$S^3 = \{z \in \mathbb{H} : |z| = 1\}$ $\qquad \mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ $\qquad i^2 = j^2 = k^2 = ijk = -1$

$\cong SU_2(\mathbb{C}) = \left\{A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C},\ AA^* = A^*A = I,\ \det A = 1\right\}$

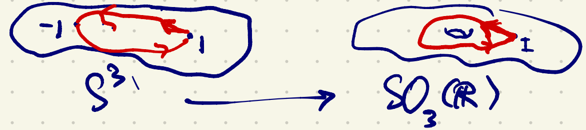
$SO_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I,\ \det A = 1\}$

$O_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I\}$ has two connected components

Fact: $S^3 \cong SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R})$ is a double cover.

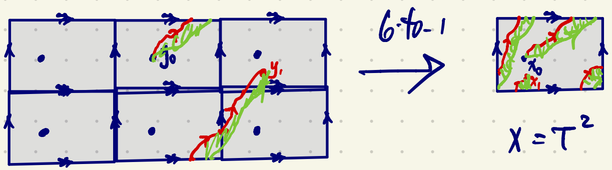
$Z(S^3) = \{\pm 1\}$

$PSU_2(\mathbb{C}) = S^3/Z(S^3) \cong SO_3(\mathbb{R}) \overset{\text{homeomorphism}}{\cong} P^3\mathbb{R}$.



In general for $n \geq 3$, $\pi_1(SO_n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$.

$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R})$ is its universal cover; a simply connected double cover constructed from Clifford Algebras (generalizing $\mathbb{H}$).



$Y = T^2$ $\qquad$ $X = T^2$

$f_0 : [0,1] \to X$ path in X from $f(0) = x_0$ to $f(1) = x_1$

$f_1 : [0,1] \to X$ is another path in X from x_0 to x_1 homotopic to f_0

$f_t : [0,1] \to X$

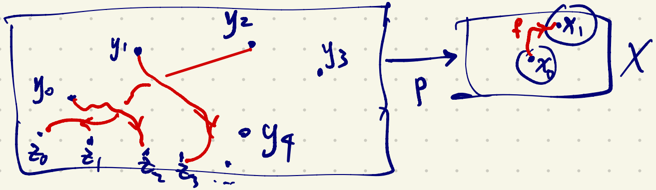
$f : [0,1]^2 \to X$

In any covering space $p: Y \to X$ and given any path $f: [0,1] \to X$ starting at $f(0) = x_0$, the path f can be lifted to Y i.e. there is a path $g: [0,1] \to Y$ such that $f = p \circ g$ i.e.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

and this lift is unique if we say which of the points in $f^{-1}(x_0)$ to take as the starting point for g.

Assuming X is path-connected and $p: Y \to X$ is a path-connected covering space, $X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff $p(y_0) = p(y_1)$.



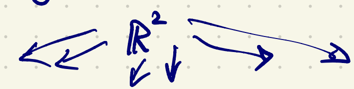
$p^{-1}(x_0) = \{y_0, y_1, y_2, \cdots\}$

$p^{-1}(x_1) = \{z_0, z_1, z_2, \cdots\}$

Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$.

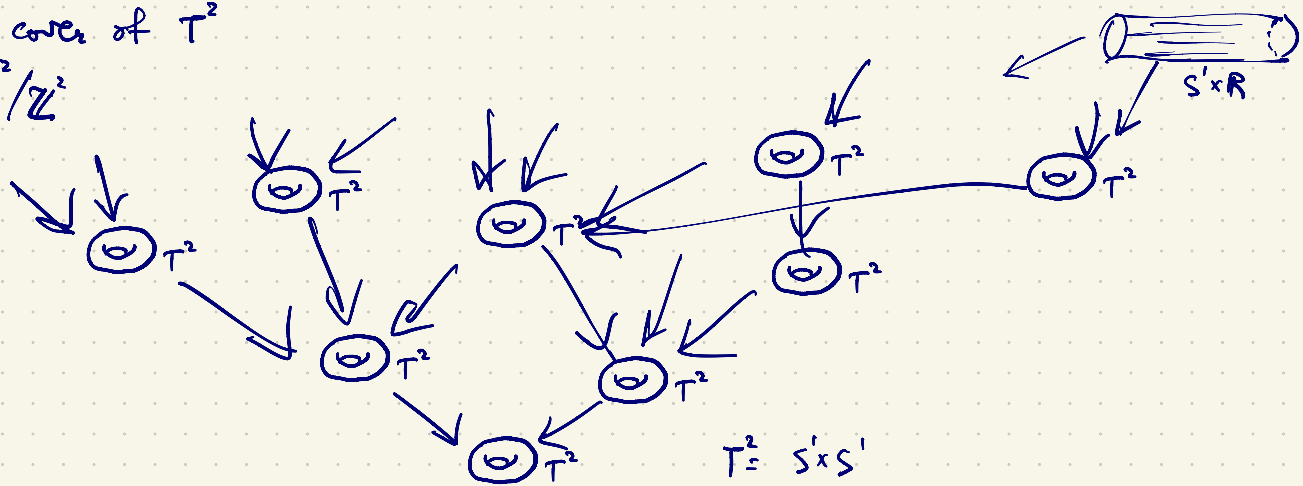
In particular if p is k-to-1 at x_0 i.e. $|p^{-1}(x_0)| = k$ then it is k-to-1 everywhere i.e. $|p^{-1}(x)| = k$ for all $x \in X$.

More generally, if f_t is a homotopy in X and we are given f_0, then every lifting of f_0 to Y extends to a lifting of f_t to Y.



$\mathbb{R}^2$ is the universal cover of T^2

$\mathbb{R}^2 \xrightarrow{p} T^2 = \mathbb{R}^2/\mathbb{Z}^2$

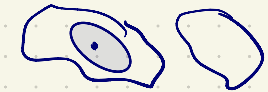


$T^2 = S^1 \times S^1$

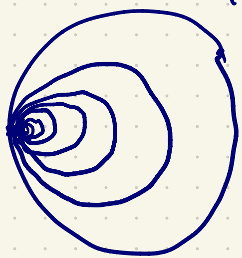
Which top. spaces have a universal cover? (equivalently a path-connected and simply connected cover)

Let X be a path-connected space. Then X has a universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected



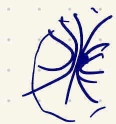
Example of a top. space without a universal cover: Hawaiian earring $\subset \mathbb{R}^2$

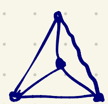


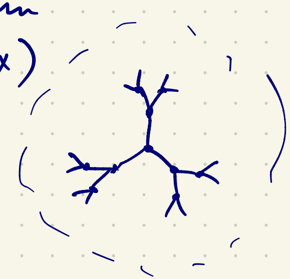
(not a CW complex)

$\not\cong \underbrace{S^1 \vee S^1 \vee S^1 \vee \dots}$

countable wedge sum (CW complex)

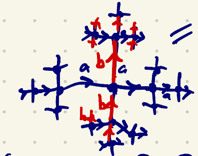


Universal cover of K_4  : trivalent tree



(also the universal cover of any trivalent connected graph)
i.e. regular of degree 3 connected

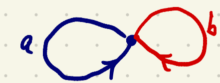
Universal cover of any connected regular graph of degree 4 is



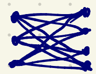
$= \tilde{X} \xrightarrow{p} X$

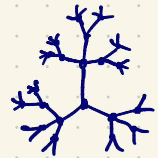
Cayley graph of Free$\{a,b\} = G$
Vertices correspond to elements of G.

Every vertex $w \in G$ has edges to wa, wa^{-1}, wb, wb^{-1}.

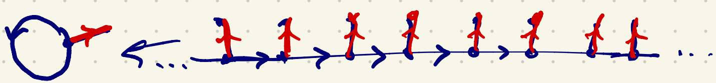


Universal cover of $K_{3,4}$





$\tilde{X} = X/G$



$P^2\mathbb{R}$ has S^2 as its universal cover.

$S^2 \longrightarrow P^2\mathbb{R}$
$= S^2/G$
quotient of S^2 by the antipodal relation.

$G = \{1, -1\}$ acts on S^2

$1x = x$

$(-1)x = -x$ (antipode of x)

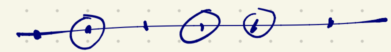
$X/_\sim$ = partition of X into equivalence classes of the equiv. relation "$\sim$"

X/G = partition of X into the orbits of G

$(x \sim xg$ or $g(x))$ for all $g \in G$.

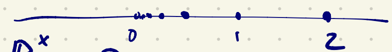
$X \longrightarrow X/_\sim$

$\mathbb{R}/\mathbb{Z} \cong S^1$



$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 = S^1 \times S^1$

A non-discrete action of $\mathbb{Z}$ on $\mathbb{R}$ eg. $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\} < \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$



G acts <u>discretely</u> on X if for every $x \in X$ there is an open nbhd U of x such that the only $g \in G$ mapping $x \longmapsto x^g \in U$ is $g = 1$.

$G = \{ x \longmapsto 2^k x + l \ : \ k, l \in \mathbb{Z} \}$ is non-discrete

---

If X is "nice" (path-connected, locally path-connected, SLSC) then X has a simply connected (and path-connected) cover which is a universal cover. It is unique up to isomorphism of covering spaces.

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[dd, "p"'] & Y \arrow[ddl] \\ & \\ X = \tilde{X}/G & \end{tikzcd}$$

$\tilde{X}$ universal cover. Fix $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$.

$G \cong \pi_1(X, x_0)$.

Every other covering space $Y \to X$ (path-connected) has the form $Y = \tilde{X}/H$, $H \leqslant G$.

eg. $\mathbb{R} = \tilde{S}^1 \quad t \mapsto e^{2\pi i t/k}$

$$\begin{tikzcd} \mathbb{R} \arrow[r] \arrow[d, "p"'] & \mathbb{R}/H = S^1 \arrow[dl, "p_k"] \\ S^1 & \end{tikzcd}$$

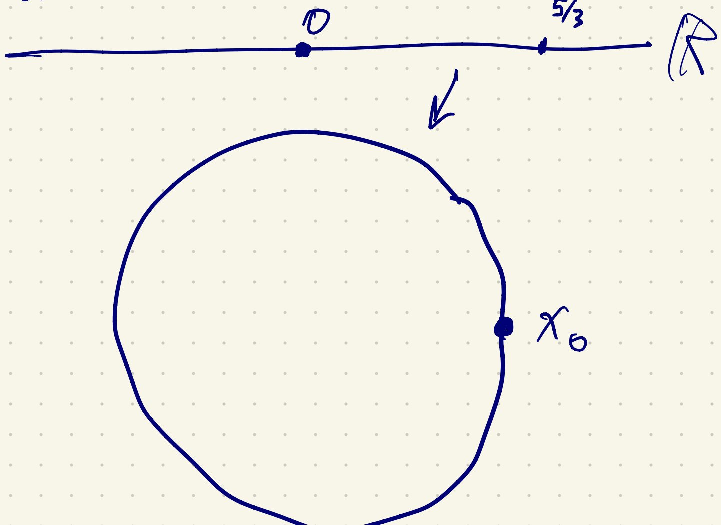
$p : t \longmapsto e^{2\pi i t}$

$p_k : z \longmapsto z^k$

$G = \mathbb{Z}$

$H \leq G$ has the form $H = k\mathbb{Z}$, $k \in \mathbb{Z}$.

Construction of the universal cover $\tilde{X} \to X$?



$\tilde{X} = \{\text{paths in } X \text{ starting at the chosen base point } x_0\}/\sim$

i.e. paths up to homotopy with fixed starting and ending point

If $f \in \tilde{X}$ then

$p(f) = f(1) \in X$ (the endpoint of f)

Cohomology.

Consider a sequence of vector spaces over F given by

$$\cdots \xrightarrow{\partial_4} V_3 \xrightarrow{\partial_3} V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} 0$$

or

$$0 \xrightarrow{0} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} \cdots$$

∂_i, d^i linear transformations (more generally V_i, V^i are modules over a ring R and ∂_i, d^i are R-homomorphisms i.e. $d(av+bw) = a\,dv + b\,dw$, $a,b \in F$; $v,w \in V$)

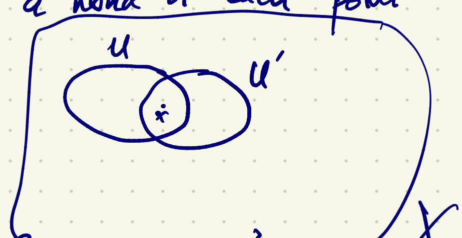
V^i or V_i has i just an index for purposes of reference.

If $\partial_i \circ \partial_{i+1} = 0$ then ∂ is a boundary map and the sequence of V_i's is a complex.

(similarly if $d^{i+1} \circ d^i = 0$, d is a coboundary map.)

Notable example: differential forms

Let X be a real (smooth) n-manifold. In a nbhd of each point $x \in X$, $x \in U \subseteq X$, we have local coordinates $(x_1, \dots, x_n) = x$.



$R = C^\infty(U) = \{$ smooth real-valued functions on $U \}$.

$V^0 = R$. $\quad d: V^0 \to V^1 = \{$ differential 1-forms on $U \} = \{ f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + \cdots + f_n dx_n : f_i \in R \}$

V^1 is a vector space over $\mathbb{R}$ (∞-dimensional) but n-dimensional as module over R

Eg. $X = \mathbb{R}^2 - \{(0,0)\}$.

$$0 \xrightarrow{0} V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} 0$$

$V^0 = \{\text{smooth functions } X \to \mathbb{R}\} = R$ = "0-forms"

$V^1 = \{\text{1-forms on } X\}$ i.e. smooth differential 1-forms

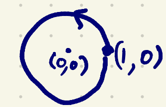
$V^2 = \{\text{2-forms on } X\}$

$V^1 = \{f\,dx + g\,dy : f, g \in R\}$

ω is closed but not exact

Eg. $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ $\quad w = \omega$

Integrate ω over the path $\gamma(t) = (\cos t, \sin t)$ $\quad t \in [0, 2\pi]$



$$\int_\gamma \omega = \int_\gamma \frac{x\,dy - y\,dx}{x^2+y^2} = \int_0^{2\pi} \frac{\cos^2 t\,dt + \sin^2 t\,dt}{1} = \int_0^{2\pi} dt = 2\pi$$

$x = \cos t$

$dx = -\sin t\,dt$

$y = \sin t$

$dy = \cos t\,dt$

x, y : global coordinates in X

r, θ : local coordinates (not global)

$x, y, r, \theta \in R = V^0$

coordinate functions $X \to \mathbb{R}$

or on $U \subset X$

$$\int_\gamma \omega = \int \cdots = 2\pi$$

$x = r\cos\theta$

$y = r\sin\theta$

$\gamma(t)$: $r(t) = 1$

$\theta(t) = t$

$\omega = \cos\theta$

$dx = \cos\theta\,dr - r\sin\theta\,d\theta$

$dy = \sin\theta\,dr + r\cos\theta\,d\theta$

$$V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2$$

$$\begin{aligned} f &\longmapsto df \\ x &\longmapsto dx \\ y &\longmapsto dy \\ r &\longmapsto dr \end{aligned}$$

etc.

If X is an n-manifold then $V^k = \{k\text{-forms on } X\}$ is an R-module of dimension $\binom{n}{k}$.

We need X to be orientable

d is $\mathbb{R}$-linear but not R-linear

$$d(fg) \neq f\,dg$$

$\underline{V^2} = \{ f\, dx \wedge dy \;:\; f \in R \}$

If X has local coordinates $x_1, \dots, x_n$ then $V^1 = \{ f_1 dx_1 + \dots + f_n dx_n \;:\; f_i \in R \}$

$$V^2 = \{ f_{12}\, dx_1 \wedge dx_2 + f_{13}\, dx_1 \wedge dx_3 + \dots \;:\; f_{ij} \in R \}$$

$$dx_i \wedge dx_i = 0$$
$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

Wedge products are R-multilinear eg. $(f\omega + g\omega') \wedge \eta = f\,\omega \wedge \eta + g\,\omega' \wedge \eta$ $\quad f, g \in R$, ω, ω', η diff. forms

$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$$

$$= (-dy \wedge dx) \wedge dz) = -dy \wedge (dx \wedge dz) = -dy \wedge (-dz \wedge dx) = dy \wedge dz \wedge dx$$

$$dx \wedge (dy \wedge dz) = (dy \wedge dz) \wedge dx$$

If ω is an i-form and η is a j-form then $\omega \wedge \eta = (-1)^{ij} \eta \wedge \omega$ is an $i+j$-form.

V^k is spanned by terms like $f\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} =: \omega \in V^k$ WLOG $1 \le i_1 < i_2 < \cdots < i_k \le n$

$V^k \xrightarrow{d} V^{k+1}$

$$d\omega = d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \underset{\sim}{df} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in V^{k+1}$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

In $\mathbb{R}^3$ with (global) coordinates x, y, z

$d: V^k \to V^{k+1}$ is $\mathbb{R}$-linear but not R-linear

$R = V^0 = \{ \text{smooth functions } \mathbb{R}^3 \to \mathbb{R} \}$

Pick $f \in V^0$ i.e. $f: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in V^1$$

This is a very special 1-form because it is <u>exact</u>. $(\in dV^0)$

$$d(df) = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz$$

$$= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial x} dz \right) \wedge dx + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial y} dz \right) \wedge dy$$

$$+ \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} dx + \frac{\partial}{\partial y}\frac{\partial f}{\partial z} dy + \frac{\partial}{\partial z}\frac{\partial f}{\partial z} dz \right) \wedge dz = 0$$

$$\boxed{d^2 = 0}$$

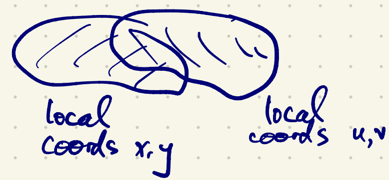
i.e. $d^2\omega := d(d\omega) = 0$ for all ω.

Imagine a surface $S \subset \mathbb{R}^3$. We integrate an arbitrary 2-form $\omega \in V^2$ over S.

If $\omega = f(x,y,z)\,dx\wedge dy + g(x,y,z)\,dx\wedge dz + h(x,y,z)\,dy\wedge dz \in V^2$ then $\int_S \omega$

$$= \int_S f(x,y,z)\,dx\wedge dy + \cdots$$

If $x = x(u,v)$
 $y = y(u,v)$



then $f(x,y)\,dx\wedge dy$

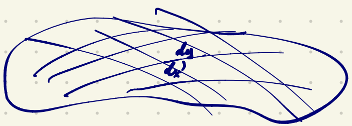
$$= f(x(u,v), y(u,v))\underbrace{\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)}_{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}} du\wedge dv$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$$

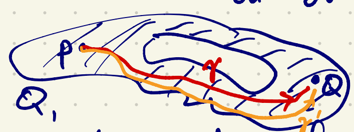
$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv$$

$$dx\wedge dy = \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\wedge\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)du\wedge dv$$



For a region $X \subset \mathbb{R}^2$,
γ path in X from P to Q,
$\omega \in V^1$, we define the path integral $\int_\gamma \omega$.



If $\omega = df$ (an exact 1-form) then $\int_\gamma \omega = \int_\gamma df = f(Q) - f(P)$
by the Fundamental Theorem of calculus
But for

But if $\gamma' \sim \gamma$ in X then $\int_{\gamma'}\omega = \int_\gamma \omega = f(Q) - f(P)$ whenever $\omega = df$.

Stokes' Theorem (general Fundamental Theorem of Calculus)

Let X be an orientable n-manifold with boundary ∂X which is also orientable $(n-1)$-manifold. Let $\omega \in \Lambda^{n-1}$, so that $d\omega \in \Lambda^n$. Then

$$\boxed{\int_{\partial X} \omega = \int_X d\omega}$$

Special case: $X = [a,b] \subset \mathbb{R}$, $\partial X = \{a,b\}$, $\omega = f \in \mathbb{R}$ (smooth function $X \to \mathbb{R}$)

$$\int_a^b f' = \int_a^b f'(t)\,dt = f(b) - f(a) \qquad d\omega = f'(x)\,dx$$

$\gamma \sim \gamma'$ in X $\qquad \partial A = \gamma - \gamma'$

If $\omega \in \Lambda^1(X)$ then $d\omega \in \Lambda^2(X)$

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\partial A} \omega = \int_A d\omega$$

If in particular $d\omega = 0$ (ω a <u>closed</u> 1-form) then RHS $= 0$ giving $\int_\gamma \omega = \int_{\gamma'} \omega$.

Exact forms are automatically closed (if $\omega = df$ then $d\omega = d^2 f = 0$).

Not conversely! unless X is simply connected.

The "gap" between {closed forms} and {exact forms} is measured by cohomology.

$$V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1}$$

$(n-1)$-forms $\quad$ n-forms $\quad$ $(n+1)$-forms

image of $d^{n-1}: V^{n-1} \to V^n$ is $B^n = \{\text{exact } n\text{-forms}\}$

kernel of $d^n: V^n \to V^{n+1}$ is $Z^n = \{\text{closed } n\text{-forms}\}$

$H^n = Z^n / B^n = n^{th}$ cohomology group (or vector space over $\mathbb{R}$)

$\dim H^n$ is the number of "n-dim'l holes" in X.

C stands for cochains; d is the coboundary map

$X = \mathbb{R}^2 \setminus \{0\}$ punctured plane

$$0 \xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} 0$$

$C^k = \{\text{diff. } k\text{-forms on } X\} = C^k(X;\mathbb{R}) = C^k(X)$

$C^0 = \{\text{smooth functions } X \to \mathbb{R}\}$

$C^1 = \{ f(x,y)\,dx + g(x,y)\,dy \;:\; f,g \in C^0 \}$

$C^2 = \{ h(x,y)\,dx \wedge dy \;:\; h \in C^0 \}$

$\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ is closed (i.e. $d\omega = 0$)

but ω is not exact i.e. $\omega \neq df$ for any $f \in C^0$.

$H^1 = \{\text{closed 1-forms}\} / \{\text{exact 1-forms}\} = H^1(X;\mathbb{R})$

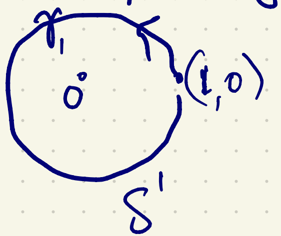
$\dim H^1 = 1$. Proof of this follows.

$df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$d(f\,dx) = df \wedge dx$, $\quad d(g\,dy) = dg \wedge dy$.

Proof that $\dim H^1 = 1$.

Let η be any closed 1-form on X ie. $\eta \in C^1$, $d\eta = 0$.



Let $c = \int_{S^1} \eta = \int_0^{2\pi} (\quad) \, d\theta$ and set $\tilde{\eta} = \eta - \frac{c}{2\pi}\omega$

$d\tilde{\eta} = 0 - 0 = 0$ so $\tilde{\eta}$ is closed

and we claim $\tilde{\eta}$ is exact

i.e. $\eta = \underbrace{\tilde{\eta}}_{\text{exact}} + \frac{c}{2\pi}\omega$

To show $\tilde{\eta}$ is exact use:

For any two paths γ, γ' in X which are homotopic in X, (with common endpoints),

$$\int_\gamma \eta = \int_{\gamma'} \eta$$

$$0 = \int_A d\eta = \int_{\partial A} \eta = \int_\gamma \eta - \int_{\gamma'} \eta$$

Stokes' Theorem

Here is our candidate $f \in C^0$ for which $df = \tilde{\eta}$.

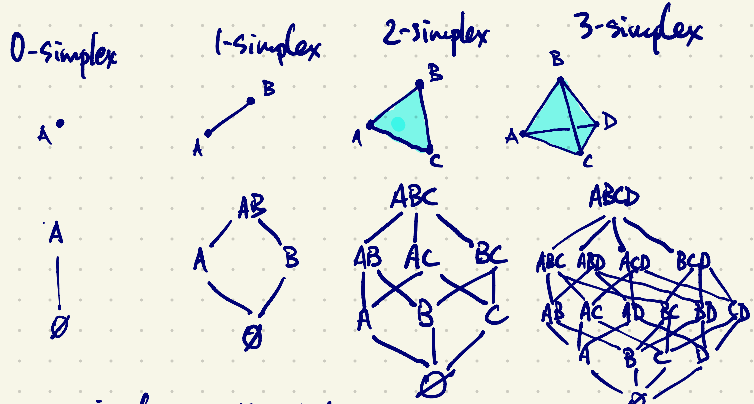
For each $Q \in X$, define $f(Q) = \int_\gamma \tilde{\eta} = \int_{(1,0)}^x \tilde{\eta}$ where γ is any path in X from $(1,0)$ to Q.

To see that this f is well-defined first fix one path γ_0 from $(1,0)$ to Q. Then any path γ in X from $(1,0)$ to Q is homotopic to γ_0 concatenated with γ_1^k so $\int_\gamma \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} + k \int_{\gamma_1} \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} \cancel{+0}$

Finally, $df = \tilde{\eta}$.

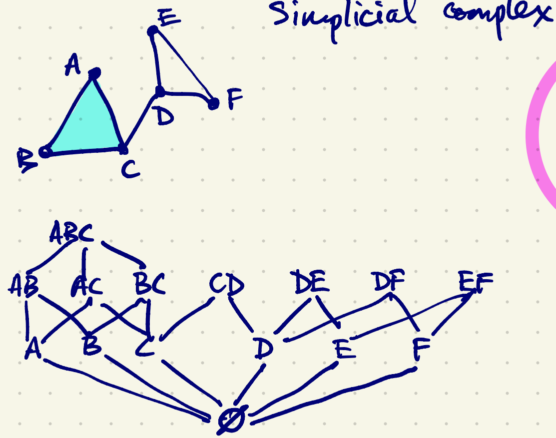
For $X = \mathbb{R}^2 - \{0\}$, (punctured plane), $\pi_1(X) \cong \mathbb{Z}$ since X is is contractible to S^1 and $\dim \underbrace{H^1(X;\mathbb{R})} = 1$. These two facts are related by the theorem of Hurewicz. $\mathbb{R}$ If we define our cohomology groups in a more universal way then $H^1(X) = H^1(X;\mathbb{Z})$ is an additive abelian group ie. $\mathbb{Z}$-module. ($\cong \mathbb{Z}$ in the case of $X = \mathbb{R}^2 - \{0\}$).

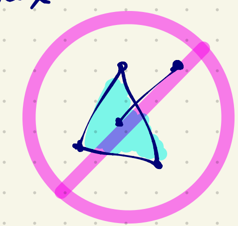
Hurewicz gave a homomorphism $\pi_1(X) \longrightarrow H_1(X) = H_1(X;\mathbb{Z})$ which is surjective ; its kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ so $H_1(X)$ is the abelianization of $\pi_1(X)$.



n-simplex: the lattice of subsets of an (n+1)-set.

Simplicial complex

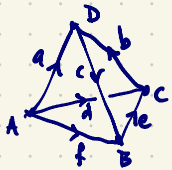




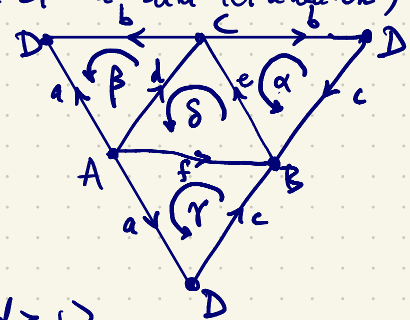
eg. $X =$ 2-skeleton of a 3-simplex

(i.e. surface of a solid tetrahedron)

$X =$



$=$



Every d-simplex $(d \geq 1)$ is orientable. Orient the faces arbitrarily

$X \cong S^2$ (homeomorphic top. spaces)

$\sim \mathbb{R}^3 - \{0\}$ (homotopic equivalent)

So we get the same algebraic invariants including homology groups) as we'll point out later

$$\partial a = D - A \qquad \partial \alpha = -b - c - e$$
$$\partial b = D - C \qquad \partial \beta = -a + b + d$$
$$\partial c = B - D \qquad \partial \gamma = a + c - f$$
$$\partial e = C - B \qquad \partial \delta = -d + e + f$$
$$\partial f = B - A$$
$$\partial d = C - A$$

$$\partial^2 \alpha = \partial(\partial \alpha) = \partial(-b-c-e) = -(\cancel{D}-\cancel{C}) - (\cancel{B}-\cancel{D}) - (\cancel{C}-\cancel{B}) = 0$$

$$\partial_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\partial_1 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We have a chain complex

$$0 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

$\partial =$ boundary map

where $C_k = \{k\text{-chains in } X\}$

A k-chain is a $\mathbb{Z}$-linear combination of k-faces of X (k-simplices)

$$C_2 = \{x_1\alpha + x_2\beta + x_3\gamma + x_4\delta \ : \ x_1, x_2, x_3, x_4 \in \mathbb{Z}\}$$
$$C_1 = \{x_1 a + x_2 b + x_3 c + \cdots + x_6 f \ : \ x_1, \ldots, x_6 \in \mathbb{Z}\}$$
$$C_0 = \{x_1 A + x_2 B + x_3 C + x_4 D \ : \ x_1, \ldots, x_4 \in \mathbb{Z}\}$$

C_k is an additive abelian group i.e. $\mathbb{Z}$-module.

$\partial_k : C_k \longrightarrow C_{k-1}$ is additive i.e.

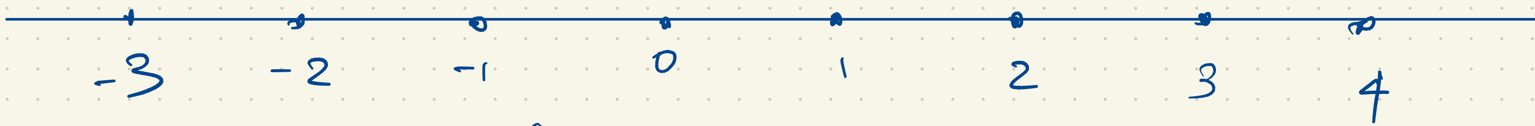
$$\partial(u \pm v) = \partial u \pm \partial v$$

$$\Longrightarrow \partial(ru + sv) = r\partial u + s\partial v \text{ for all } r, s \in \mathbb{Z}$$

$\partial_1 \circ \partial_2 = 0$

i.e. ∂ is a $\mathbb{Z}$-module homomorphism

C_k is a <u>free</u> $\mathbb{Z}$-module



$\mathbb{R}/\sim \qquad x \sim y \iff (x = y \text{ or } x, y \in \mathbb{Z})$

$\cong \bigvee_{n=-\infty}^{\infty} S^1$

homeomorphic

$\mathbb{R}/\mathbb{Z} \cong S^1$ quotient of top. groups.

not the same (one is a group, the other not)

For any chain complex $\cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$ i.e. $\partial_{k-1} \circ \partial_k = 0$

$\partial = (\partial_k)_k$ homomorphisms of additive abelian groups

$Z_k, B_k \leq C_k$ subgroups of $C_k = \{k\text{-chains}\}$

$B_k = \partial_{k+1} C_{k+1} = \text{image of } \partial_{k+1} : C_{k+1} \longrightarrow C_k = \{k\text{-boundaries}\}$

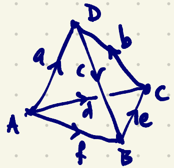
$Z_k = \ker(\partial_k : C_k \to C_{k-1}) = \{k\text{-cycles}\}$

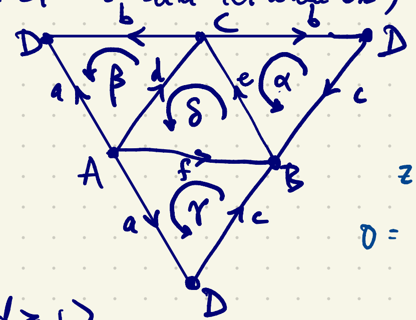
$B_k \leq Z_k \leq C_k$

The k^{th} homology group $H_k = Z_k / B_k$

Two elements $z, z' \in Z_k$ are homologous if $z + B_k = z' + B_k \iff z - z' \in B_k$.

eg. X = 2-skeleton of a 3-simplex (i.e. surface of a solid tetrahedron)

$X =$ 

$=$ 

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$$\begin{aligned}
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\partial b &= D - C & \partial \beta &= -a+b+d \\
\partial c &= B - D & \partial \gamma &= a+c-f \\
\partial e &= C - B & \partial \delta &= -d+e+f \\
\partial f &= B - A & & \\
\partial d &= C - A & &
\end{aligned}$$

$$0 \xrightarrow{0} C_2 \xrightarrow{\partial_2} C_1 \longrightarrow C_0 \longrightarrow 0$$

$$H_1 = Z_1 / B_1 =$$

$$Z_1 = \ker \partial_1 : C_1 \to C_0$$

$$z = z_1 a + z_2 b + z_3 c + z_4 d + z_5 e + z_6 f \in Z^1 \iff$$

$$0 = \partial z = z_1(D-A) + z_2(D-C) + z_3(B-D) + z_4(C-A) + z_5(C-B) + z_6(B-A)$$

$$= (-z_1 - z_4 - z_6)A + (z_3 - z_5 + z_6)B + (-z_2 + z_4 + z_5)C + (z_1 + z_2 - z_3)D$$

"null space" over $\mathbb{Z}$ of

$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$a = -d-f$
$b = d+e$
$c = e-f$

$$\partial_1 = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\partial_2 = \begin{array}{c|c} & \begin{matrix} \alpha & \beta & \gamma & \delta \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -d-f \\ d+e \\ e-f \\ d \\ e \\ f \end{bmatrix} = d \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$B_1 = \langle \text{cols of } \partial_2 \rangle = Z_1$$

$$Z_1 = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} \cong \mathbb{Z}^3$$

$$\boxed{H_1 = Z_1 / B_1 = 0}$$

$$\boxed{H_0 = Z_0/B_0 \cong \mathbb{Z}} \qquad H_0 = Z_0/B_0 = \{ kA + B_0 \ : k \in \mathbb{Z} \}.$$

$$B_0 = \langle A-B, A-C, A-D, B-C, B-D, C-D \rangle = \{ x_1 A + x_2 B + x_3 C + x_4 D : \ x_1, \dots, x_4 \in \mathbb{Z}, \ x_1 + x_2 + x_3 + x_4 = 0 \}$$

$$Z_0 = \langle A, B, C, D \rangle$$

$$x_1 A + x_2 B + x_3 C + x_4 D = \underbrace{(x_1 + x_2 + x_3 + x_4)}_{k} A + \underbrace{(-x_2 - x_3 - x_4) A + x_2 B + x_3 C + x_4 D}_{\in B_0}$$

$$[\vdots]$$

$$H_2 = Z_2/B_2 = \langle \alpha+\beta+\gamma+\delta \rangle_{\mathbb{Z}} / 0 \cong \mathbb{Z} \leftarrow \text{comes from (the abelianization of } \pi_2(X))$$

$$H_1 = 0$$

$$H_0 \cong \mathbb{Z}$$

If we had included the interior of the tetrahedron (X = 3-simplex) then

$$0 \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

and then $H_2(\text{3-simplex}) = 0$ and that's no surprise since the 3-simplex is contractible.

$H_0(X) \cong \mathbb{Z}^k$ where k is the number of path-connected components of X.

Reduced homology groups of X $\tilde{H}_k(X)$ are computed using the modified chain complex

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \xrightarrow{\partial_0} C_{-1} = \mathbb{Z} \to 0$$

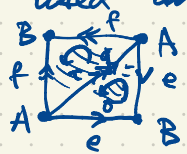
$$C_0 = \{0\text{-chains}\} = \{\sum_i x_i A_i : x_i \in \mathbb{Z}\} = \langle A_i \text{ vertices} \rangle_{\mathbb{Z}}$$

$$\partial_0\left(\sum_i x_i A_i\right) = \sum_i x_i \in \mathbb{Z}$$

$$\tilde{H}_i(X) \cong \begin{cases} H_i(X), & i \geq 1 \\ \mathbb{Z}^{k-1}, & i = 0 \end{cases}$$

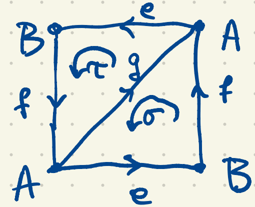
where k = no. of path-connected components in X.

We could have used any triangulation of S^2 and we'd get the same homology groups eg.



$$C_2 = \langle \sigma, \tau \rangle_{\mathbb{Z}}$$
$$C_1 = \langle e, f, g \rangle_{\mathbb{Z}}$$
$$C_0 = \langle A, B \rangle_{\mathbb{Z}}$$

Homology of $X \cong \mathbb{P}^2\mathbb{R}$



$$\partial e = B - A \qquad \partial f = A - B \qquad \partial g = A - A = 0$$

$$\partial_1 = \begin{array}{c|ccc} & e & f & g \\ \hline A & -1 & 1 & 0 \\ B & 1 & -1 & 0 \end{array}$$

$$\partial \sigma = e + f - g \qquad \partial \tau = e + f + g$$

$$\partial_2 = \begin{array}{c|cc} & \sigma & \tau \\ \hline e & 1 & 1 \\ f & 1 & 1 \\ g & -1 & 1 \end{array}$$

$$0 \xrightarrow{\partial_3 = 0} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0 = 0} 0$$

$$C_2 = \langle \sigma, \tau \rangle_{\mathbb{Z}}, \quad C_1 = \langle e, f, g \rangle_{\mathbb{Z}}, \quad C_0 = \langle A, B \rangle_{\mathbb{Z}}$$

$$\partial_1 \circ \partial_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Z_1 = \ker \partial_1 = \langle e+f, g \rangle = \left\langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} = \{\text{1-cycles}\}$$

$$B_1 = \operatorname{im}(\partial_2 : C_2 \to C_1) = \left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} = \langle e+f-g,\ e+f+g \rangle_{\mathbb{Z}} = \langle e+f+g,\ 2g \rangle_{\mathbb{Z}}$$

[Over $\mathbb{R}$, this simplifies to $\langle e+f, g \rangle_{\mathbb{R}}$]

$$H_1 = H_1(X) = Z_1 / B_1 = \langle e+f, g \rangle / \langle e+f+g, 2g \rangle \cong \left(\langle g \rangle + \langle e+f+g, 2g \rangle \right) / \langle e+f+g, 2g \rangle$$

$$\cong \langle g \rangle / \left(\langle g \rangle \cap \langle e+f+g, 2g \rangle \right)$$

$$\cong \langle g \rangle / \langle 2g \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

using 2nd Isomorphism Theorem for Groups / Rings $(R+S)/S \cong R / R \cap S$