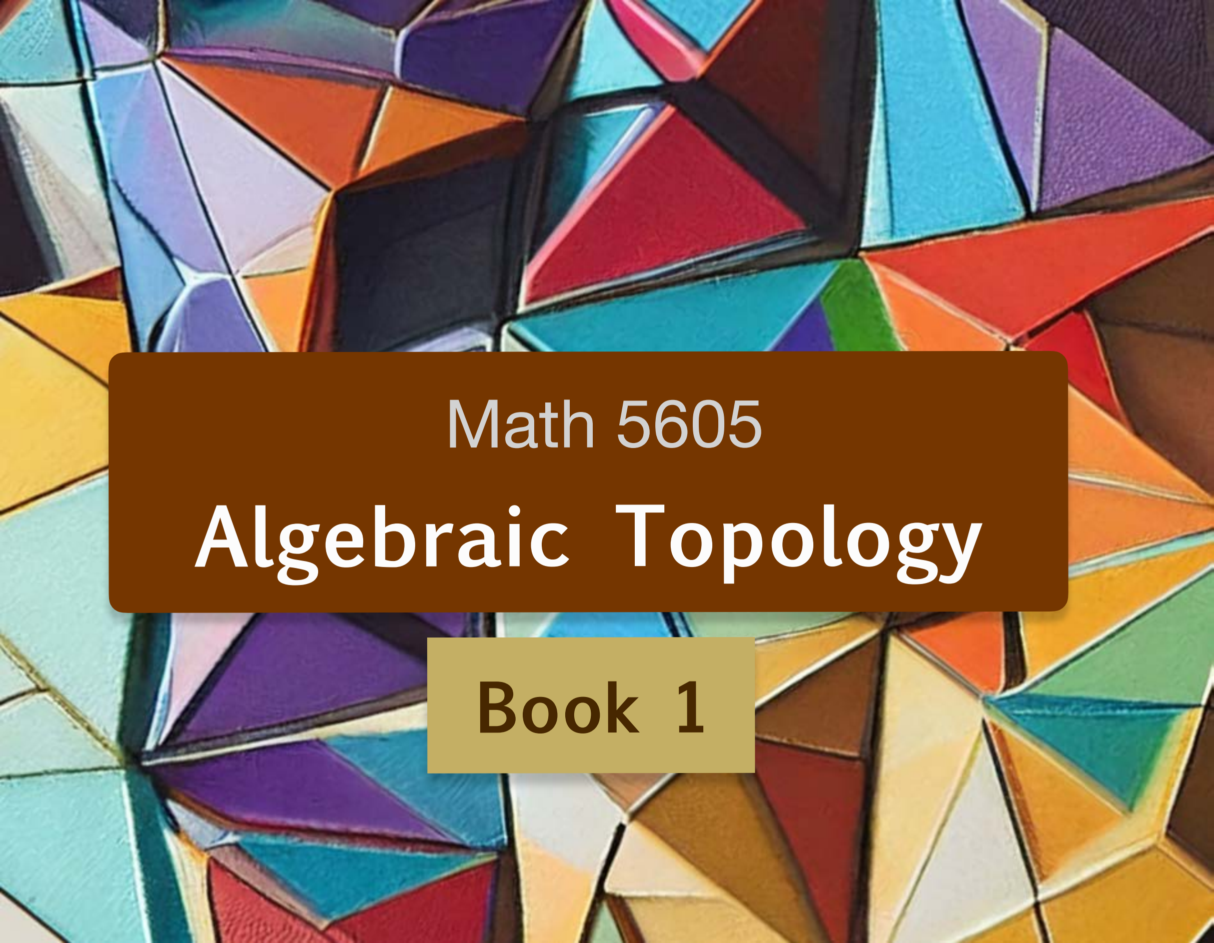


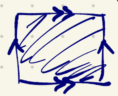
Math 5605

Algebraic Topology

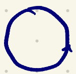
Book 1

If X, Y are top. spaces, $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.
$f: X \to Y$ is a <u>homeomorphism</u> if f is bijectie and f, f^{-1} are continous.

$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \xrightarrow{\cong} Y$.
$\mathbb{R}^2 \ncong S^2$ since S^2 is compact; $\mathbb{R}^2$ is not.
2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ = 

$S^2 \ncong T^2 = S^1 \times S^1 =$  = 
S^1 " S

S^2, T^2 are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$T^1 = S^1 =$  = circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

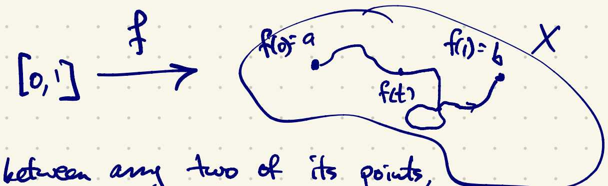
$S^2 \ncong T^2$ because S^2 is simply connected whereas T^2 is not.
In S^2, every closed path can be "continuously shrunk" to a point (homotopic to a point, i.e. null-homotopic)

X  $\ncong$ $T = T^2$  although both surfaces are compact, connected, not simply connected

These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.
If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.
For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of <u>homotopy equivalence</u>.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\bullet\}$.

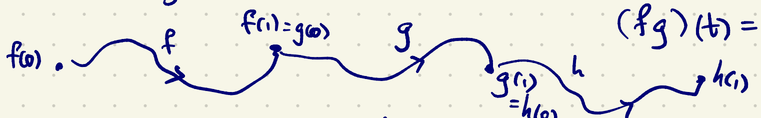
Given points $a, b \in X$ (a topological space), a *path* from a to b is a (continuous) function $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.



All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:



$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$

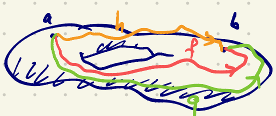
$$(s,t) \mapsto f(s,t) = f_s(t)$$

such that $f_0 = f$ i.e. $f_0(t) = f(t)$

$f_1 = g$ i.e. $f_1(t) = g(t)$

$f_s(0) = a$, $f_s(1) = b$ for all $s \in [0,1]$

This is a homotopy from f to g.



f, g are homotopic but h is not homotopic to f, g

We say f, g are homotopic if there is a continuous family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi : [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f\circ\varphi : [0,1] \to X$ is homotopic to f. A homotopy from f to $f\circ\varphi$ is

$$[0,1]^2 \longrightarrow X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{\in [0,1]}) = f_s(t)$$

$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
$$f_s(0) = f(\cancel{(1-s)\cdot 0} + \cancel{s\cdot\varphi(0)}) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot \underline{\varphi(1)}) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)

Fix g path in X from x_0 to x_1

An isomorphism $\phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$f \overset{\phi}{\longmapsto} \bar{g} f g$$
$$g f \bar{g} \overset{\phi^{-1}}{\longleftarrow} h$$

$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$

$f_1, f_2 \in \pi_1(X, x_0)$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f\gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is

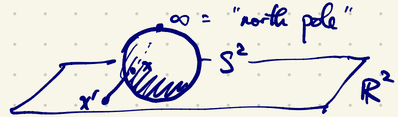
$\bar{f}(t) = f(1-t)$, $t \in [0,1]$

(same path in the reverse direction)

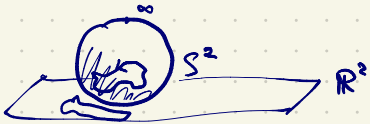
$f\bar{f} = \bar{f} f = \gamma$ = null path

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)



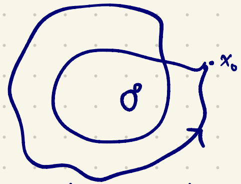
$x \longmapsto x'$ is stereographic projection from the north pole ∞



See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
punctured plane



follows from the fact that

$\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S^1$

$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).

Hatcher writes $X \approx Y$ for homeomorphic.

- retraction
- deformation retraction — "def. retraction in the weak sense"
- strong deformation retraction — "def. retraction".
- homotopy
- relative homotopy
- homotopy equivalence

Let $A \subseteq X$ (subspace of a top. space).

A retraction $f: X \to A$ is a (continuous) map such that $f|_A = \mathrm{id}_A = \mathbb{1}_A$ i.e. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a retract of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \to \{a\}$, $x \mapsto a$ is a retraction.

$\mathbb{R}^2 \to x\text{-axis}$, $(x,y) \mapsto (x,0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \to S^1$.

(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to a retraction $X \to A$), $A \subseteq X$.

i.e. $f : [0,1] \times X \longrightarrow X$

$$f(t,x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

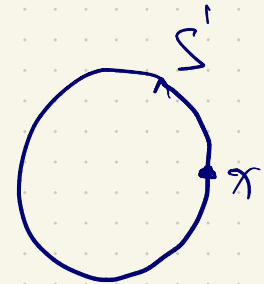
$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$

$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.



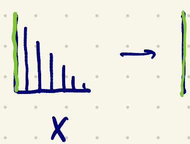
Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f : [0,1] \times X \to X$:

$f_0(x) = x$ i.e. $f_0 = id_X$

f_1 is a retraction $X \to A$

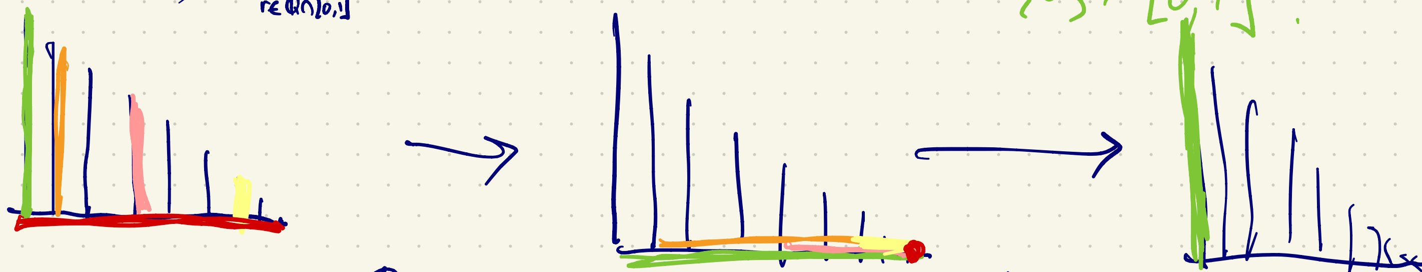
$f_t|_A = id_A$ for all $t \in [0,1]$



def. retract
but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $\quad X = ([0,1] \times \{0\}) \cup \bigcup_{r \in \mathbb{Q}\cap[0,1]} (\{r\} \times [0, 1-r])$

X has a def. ret. to $A := \{0\} \times [0,1]$.



$0 \le t \le \frac{1}{2}$ $\qquad \frac{1}{2} \le t \le 1$

This is a def. retract but not a strong def. retract.

There is no strong def. retract $X \to A$.

Let $f_0, f_1 : X \to Y$ be maps. A <u>homotopy</u> from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0,x) = f_0(x)$, $f(1,x) = f_1(x)$.

$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 $(f_0, f_1 : X \to Y)$ is a homotopy $f : [0,1] \times X \to Y$ such that $f_t(a)$ is independent (constant) of $t \in [0,1]$ for all $a \in A$.

A path from a to b in X is a homotopy from a to b.

Given two paths f_0, f_1 in X from a to b $(a, b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $f\circ g: X \to X$ and $g \circ f: Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$$

$f(x) = 0$ for all $x \in \mathbb{R}^n$
$g(0) = 0 \in \mathbb{R}^n$

$$g\circ f : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto 0$$

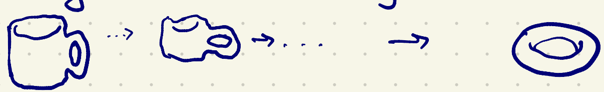
$$f\circ g : \{0\} \to \{0\}, \quad 0 \mapsto 0$$

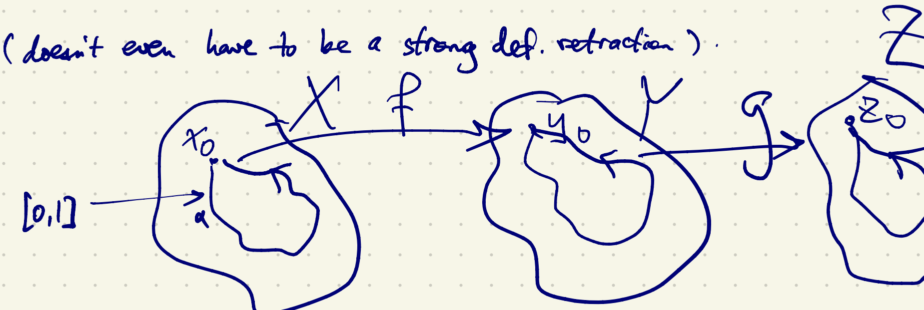
A homotopy from $g\circ f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ to $id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$

$h_0 = g\circ f$
$h_1 = id_{\mathbb{R}^n}$

Not relative to any subspace necessarily.

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).





S^1 is not homotopic to a point.
(S^1 is not null homotopic; not contractible)

If $f: X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_*: \pi_1(X) \to \pi_1(Y)$

$\alpha: [0,1] \to X$ gives $f_*\alpha = f\circ\alpha : [0,1] \to Y$

$(g\circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$