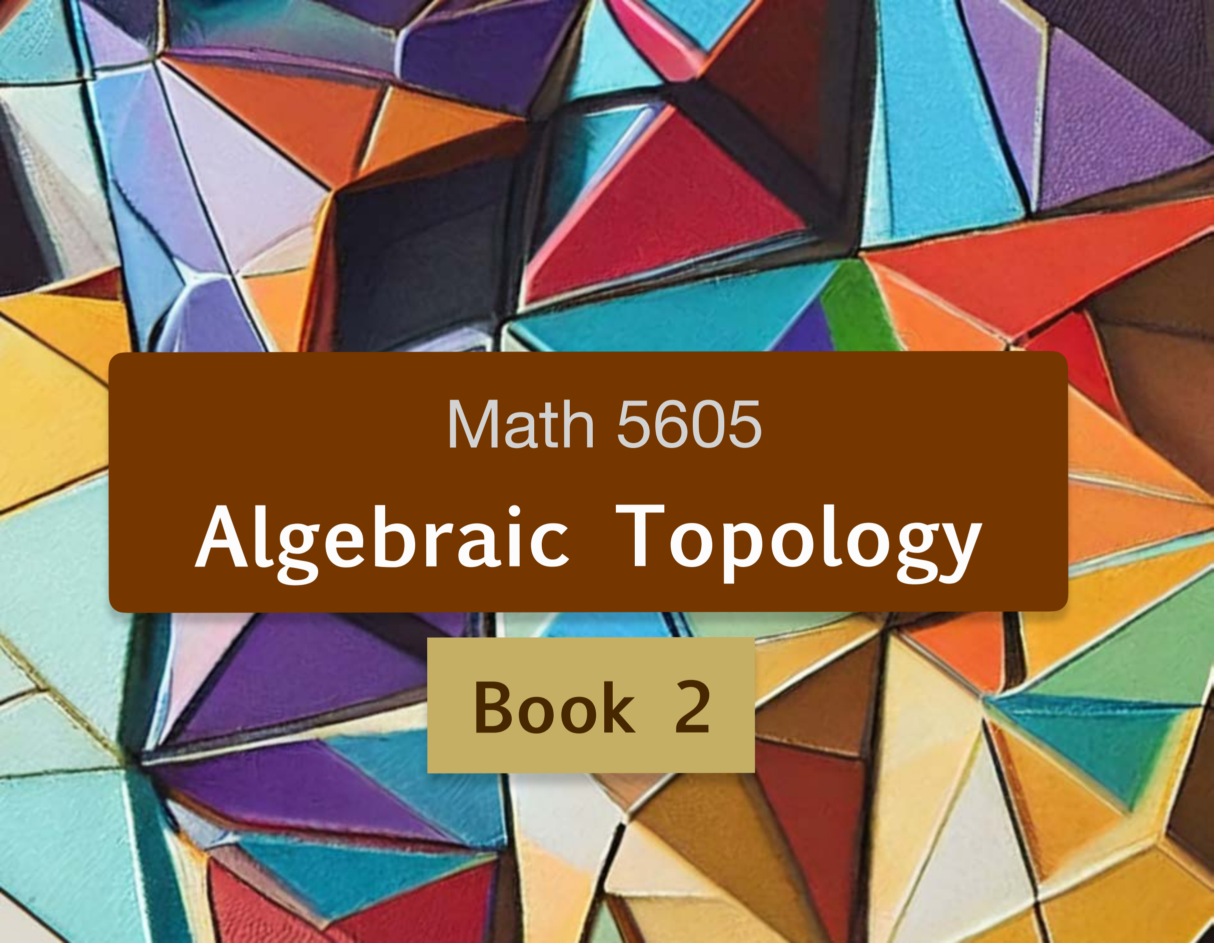
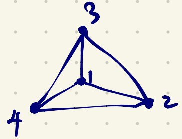


Math 5605

Algebraic Topology

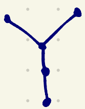
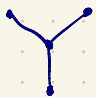
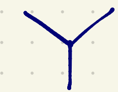
Book 2

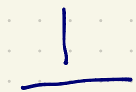
When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \ldots, \{3,4\}\}$.

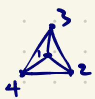
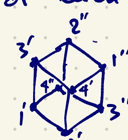
X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

 and  have the same geometric realization although they are different graphs. 



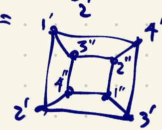
A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in Γ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'.
($x, y \in V(\Gamma)$, $\{x,y\} \in E(\Gamma)$)

A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

Eg. a double cover of $X =$  is the 1-skeleton of the cube $Y =$ 

$Y \longrightarrow X$

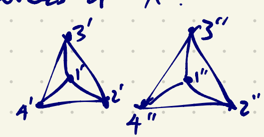
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

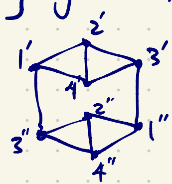
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



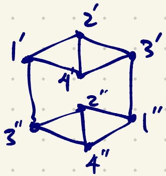
$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

Let $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ be covering spaces of X. We say $\theta : X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

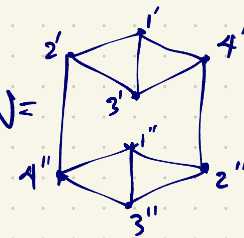
$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg.



$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

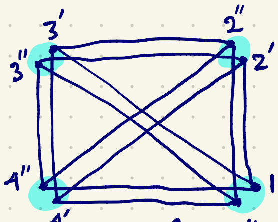
is not equivalent to $Y \to X$. But what about

$W =$ 

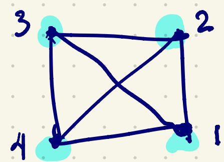
$W \leftrightarrow X$
$1', 1'' \mapsto 1$
$2', 2'' \mapsto 2$
$3', 3'' \mapsto 3$
$4', 4'' \mapsto 4$

Is this equivalent to $Z \to X$? No...

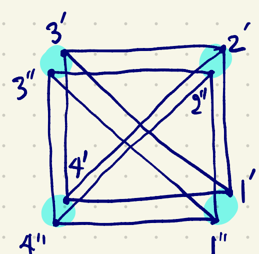
Another picture of these covers:



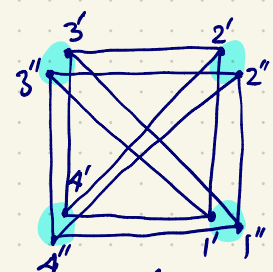
Y



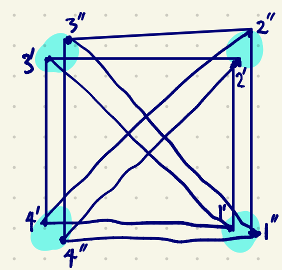
X



$X \sqcup X$

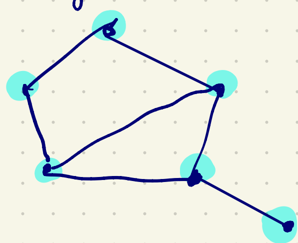


Y



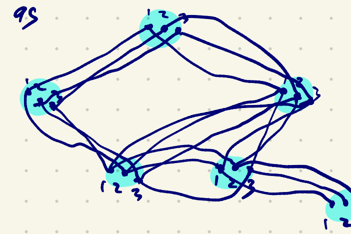
W

To construct an r-fold cover of X, created one copy of $[r] = \{1,2,\cdots,r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

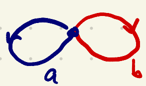


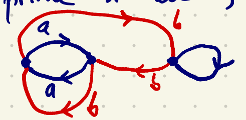
X

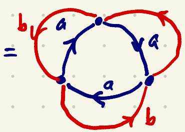
A triple cover $Y \to X$ is constructed as



Why is 2 more special than other positive integers (the oddest prime of all)?

Consider $X =$ 

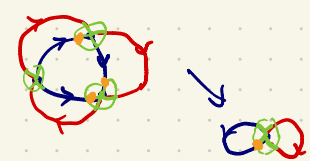
has many triple covers including $Y_1 =$ 

$Y_2 =$ 

The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

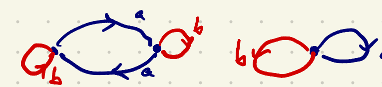
An equivalence between $Y \to X$ and itself (automorphism of the cover) is a <u>deck transformation</u>. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations). But $Y_1 \to X$ has only one (trivial) deck transformation.



In a connected r-fold cover, there are at most r deck transformations. If equality holds, the covering space is <u>normal</u> or <u>Galois</u>. (not the same as normal space in point set topology).

Double covers are always normal.

$Y_3 =$ 

In group theory, subgroups of index 2 are normal.
In the case of (separable) extensions of fields, the extension is normal.
For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$: dimension of E as a vector space over F. The number of F-automorphisms of E (i.e. $\sigma: E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.