Math 5605

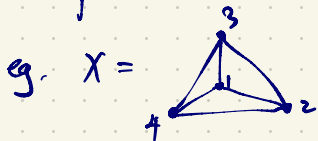# **Algebraic Topology**

Book 2

When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.
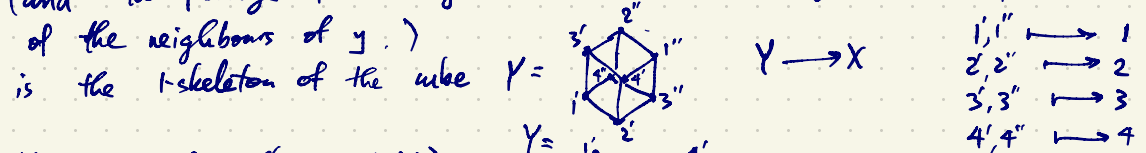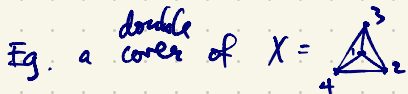
eg. $X =$

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \cdots, \{3,4\}\}$.

X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

 and have the same geometric realization although they are different graphs.
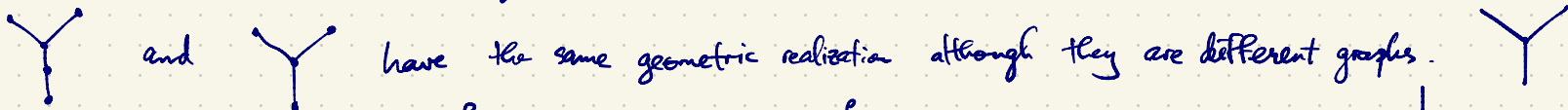
A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in Γ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'. A covering map of graphs is a homomorphism
($x,y \in V(\Gamma)$ inducing a bijection on the neighbours of each vertex of Γ
$\{x,y\} \in E(\Gamma)$) (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbours of y.)

Eg. a double cover of $X =$ is the 1-skeleton of the cube $Y =$ $Y \rightarrow X$

$$1, 1'' \mapsto 1$$
$$2, 2'' \mapsto 2$$
$$3', 3'' \mapsto 3$$
$$4', 4'' \mapsto 4$$

$Y =$

The covering space is $Y \rightarrow X$ (or informally just Y).
Some other double covers of X:
Trivial double cover
$X \sqcup X$

$Z \rightarrow X$
$1, 1' \mapsto 1$
$4', 4'' \mapsto 4$
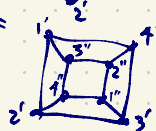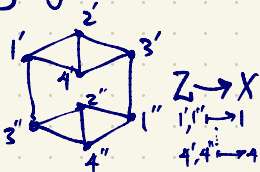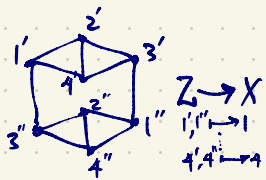
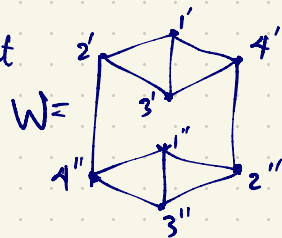When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

Let $p_1: X_1 \longrightarrow X$, $p_2: X_2 \longrightarrow X$ be covering spaces of X. We say $\theta: X_1 \longrightarrow X_2$ is an equivalence or isomorphism of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

$$X_1 \underset{\theta}{\overset{\theta}{\rightleftarrows}} X_2$$

$$p_1 \searrow \quad \swarrow p_2$$

$$X$$

Eg.

$Z \rightarrow X$
$1', 1'' \mapsto 1$
$4', 4'' \mapsto 4$
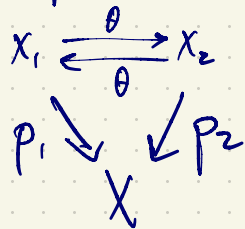
is not equivalent to $Y \rightarrow X$. But what about

$W =$

$$W \longleftrightarrow X$$
$$1', 1'' \longmapsto 1$$
$$2', 2'' \longmapsto 2$$
$$3', 3'' \longmapsto 3$$
$$4', 4'' \longmapsto 4$$

Is this equivalent to $Z \rightarrow X$? No...
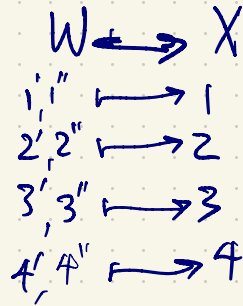
Another picture of these covers:

Y

X

$X \sqcup X$

Y
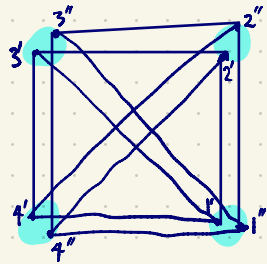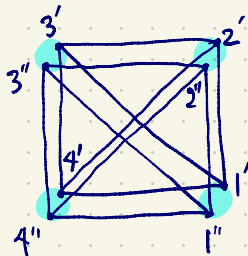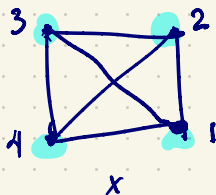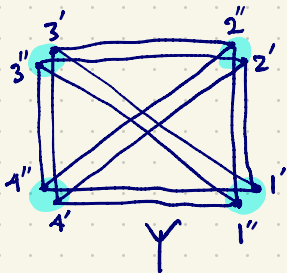
W

To construct an r-fold cover of X, created one copy of $[r] = \{1, 2, \ldots, r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

A triple cover $Y \to X$ is constructed as

X

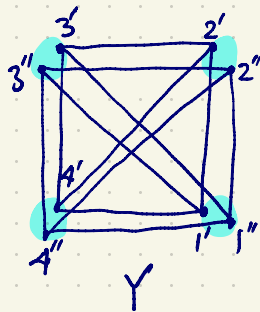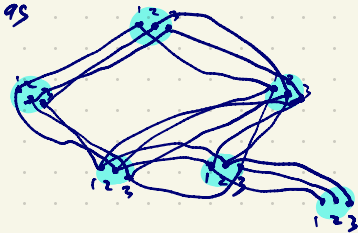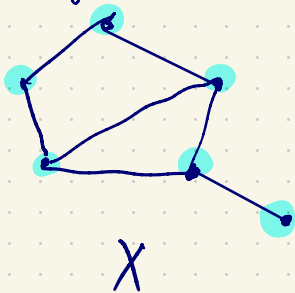Why is 2 more special than other positive integers (the oddest prime of all)?

Consider $X =$ has many triple covers including $Y_1 =$

The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

$Y_2 =$
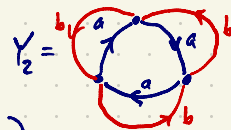
An equivalence between $Y \to X$ and itself (automorphism of the cover) is a _deck transformation_. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations) But $Y_1 \to X$ has only one (trivial) deck transformation.
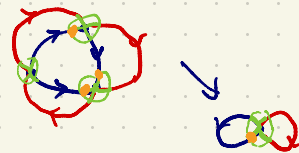
In a connected r-fold cover, there are at most r deck transformations. If equality holds, the covering space is _normal_ or _Galois_. (not the same as normal space in point set topology).
    Double covers are always normal.

$Y_3 =$

In group theory, subgroups of index 2 are normal.

In the case of (separable) extensions of fields, the extension is normal.

For a field extension $E \supseteq F$, the degree of the extension is $[E:F] = $ dimension of E as a vector space over F. The number of F-automorphisms of E (ie. $\sigma: E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers : examples
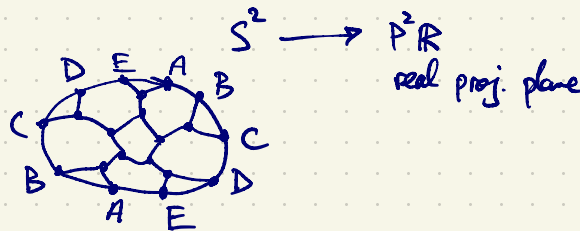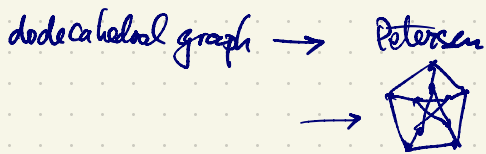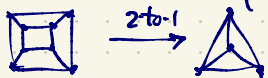
S^n is not a top. group unless $n \in \{1, 3\}$.

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$S^3 = \{z \in \mathbb{H} : |z| = 1\}$ $\mathbb{H} = \{a + bi + cj + dk : a,b,c,d \in \mathbb{R}\}$ $i^2 = j^2 = k^2 = ijk = -1$

$\cong SU_2(\mathbb{C}) = \{A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C}, \; AA^* = A^*A = I, \; \det A = 1\}$

$SO_3(\mathbb{R}) = \{A \in \mathbb{R}^{3 \times 3} : AA^T = A^TA = I, \; \det A = 1\}$

$O_3(\mathbb{R}) = \{A \in \mathbb{R}^{3 \times 3} : AA^T = A^TA = I\}$ has two connected components

Fact : $S^3 \cong SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R})$ is a double cover. $Z(S^3) = \{\pm 1\}$

$PSU_2(\mathbb{C}) = S^3 / Z(S^3) \overset{\sim}{=} SO_3(\mathbb{R}) \cong \mathbb{P}^3\mathbb{R}$ homeomorphism

$$S^3 \longrightarrow SO_3(\mathbb{R})$$
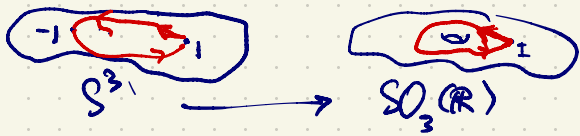
In general for $n \geq 3$, $\pi_1\left(\underline{SO_n(\mathbb{R})}\right) \cong \mathbb{Z}/2\mathbb{Z}$. simply connected

$$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R}) \text{ is its universal cover; a double cover}$$

constructed from ~~Clifford~~ Clifford Algebras (generalizing $\mathbb{H}$).

$$Y = T^2 \qquad \xrightarrow{6\text{-to-}1} \qquad X = T^2$$
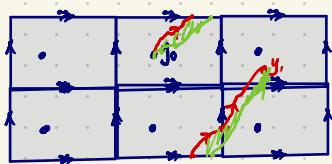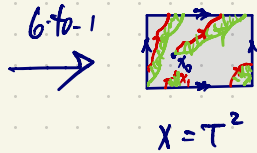
$f: [0,1] \to X$
$f_t: [0,1] \to X$
$f: [0,1]^2 \to X$

$f: [0,1] \to X$
path in X
from $f(0) = x_0$
to $f(1) = x_1$

$f_1: [0,1] \to X$
is another path in X
from x_0 to x_1
homotopic to f_0

Assuming X is path-connected and
$p: Y \to X$ is a path-connected covering space,
$X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff
$p(y_0) = p(y_1)$.

In any covering space $p: Y \to X$ and given
any path $f: [0,1] \to X$ starting at $f(0) = x_0$,
the path f can be lifted to Y
ie. there is a path $g: [0,1] \to Y$ such
that $f = p \circ g$ ie.

$$\begin{array}{ccc} & & Y \\ & \nearrow{\scriptstyle g} & \downarrow{\scriptstyle p} \\ [0,1] & \xrightarrow{f} & X \end{array}$$
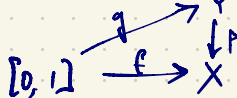
and this lift is
unique if we say which
of the points in $f^{-1}(x_0)$
to take as the starting
point for g.
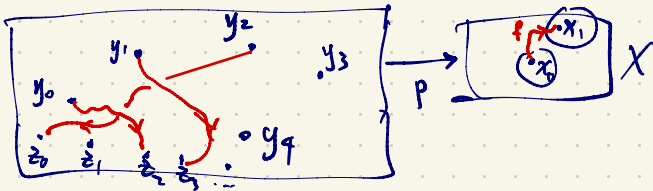
Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$.

In particular if p is k-to-1 at x_0 i.e. $|p^{-1}(x_0)| = k$ then it is k-to-1 everywhere ie. $|p^{-1}(x)| = k$ for all $x \in X$.

$$p^{-1}(x_0) = \{ y_0, y_1, y_2, \cdots \}$$

$$p^{-1}(x_1) = \{ z_0, z_1, z_2, \cdots \}$$

More generally, if f_t is a homotopy in X and we are given f_0, then every lifting of f_0 to Y extends to a lifting of f_t to Y.
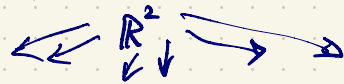
$\mathbb{R}^2$ is the universal cover of T^2

$$\mathbb{R}^2 \xrightarrow{\quad} T^2 = \mathbb{R}^2/\mathbb{Z}^2$$

$S^1 \times \mathbb{R}$

$T^2 = S^1 \times S^1$
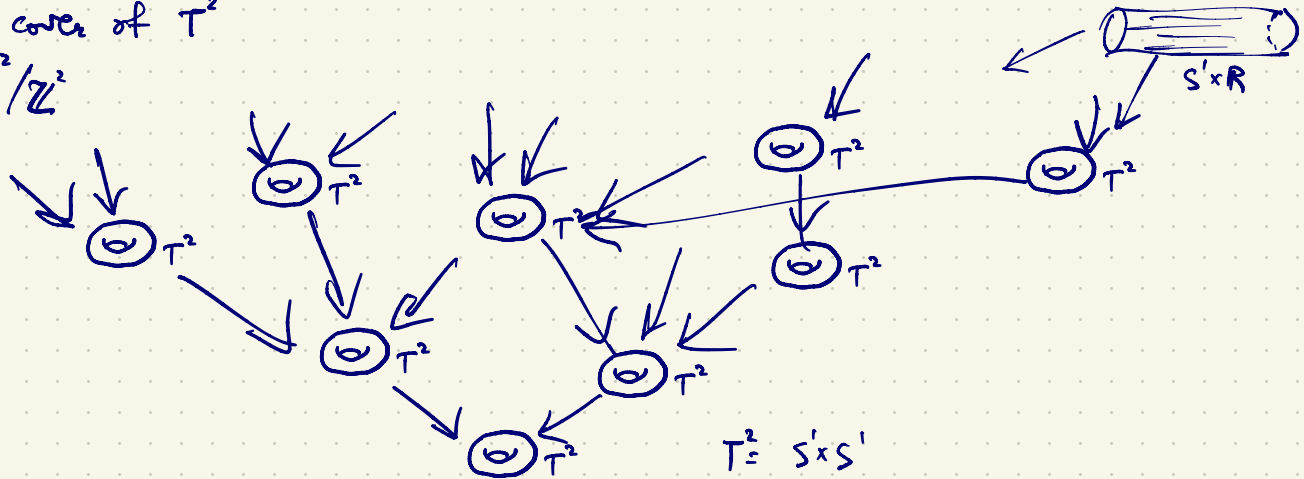
Which top. spaces have a universal cover?   (equivalently a simply connected cover)

Let X be a path-connected space. Then X has a ~~path-connected and~~ universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected

Example of a top. space without a universal cover: Hawaiian earring $\subset \mathbb{R}^2$
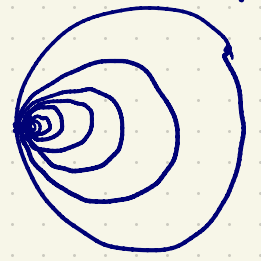
$$\neq S^1 \vee S^1 \vee S^1 \vee \ldots$$

countable
wedge sum
(CW complex)

(not a CW complex)

Universal cover of K_4 : trivalent tree
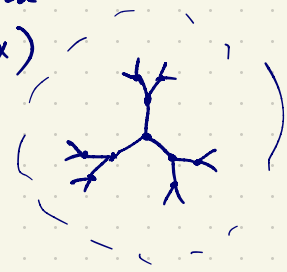
(also the universal cover
of any trivalent connected graph)
i.e. regular of degree 3 connected

Universal cover of any connected regular graph of degree 4 is

 $= \tilde{X}$ $\xrightarrow{P}$ X ∞
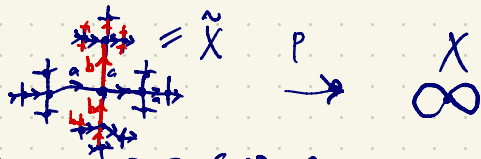
Cayley graph of Free $\{a,b\} = G$
Vertices correspond to elements of G.

Every vertex $w \in G$ has
edges to wa, wa^{-1}, wb, wb^{-1}.

Universal cover of $K_{3,4}$
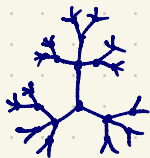
$\tilde{X} = X/G$
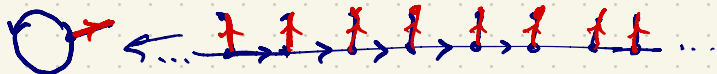
$P^2\mathbb{R}$ has S^2 as its universal cover.

$S^2 \longrightarrow P^2\mathbb{R}$
$\qquad = S^2/G$
quotient of S^2
by the antipodal
relation.

$G = \{1, -1\}$ acts on S^2

$1x = x$
$(-1)x = -x$ (antipode of x)

$X/_\sim$ = partition of X into equivalence classes of the equiv. relation "$\sim$"

X/G = partition of X into the orbits of G

$$(x \sim xg \text{ or } g(x))$$
$$\text{for all } g \in G.$$

$$X \longrightarrow X/_\sim$$

$$\mathbb{R}/\mathbb{Z} \cong S^1$$
$$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 = S^1 \times S^1$$

A non-discrete action of $\mathbb{Z}$ on $\mathbb{R}$ eg. $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\} < \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$
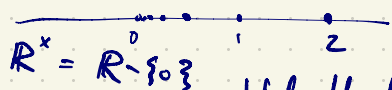
G acts discretely on X if for every $x \in X$ there is an open nbhd U of x such that the only $g \in G$ mapping $x \longmapsto x^g \in U$ is $g = 1$.

$G = \{x \longmapsto 2^k x + \ell : k, \ell \in \mathbb{Z}\}$ is non-discrete

If X is "nice" (path-connected, locally path-connected, SLSC) then X has a simply connected (and path-connected) cover which is a universal cover. It is unique up to isomorphism of covering spaces.

$\tilde{X}$ universal cover. Fix $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$.

$$G \cong \pi_1(X, x_0).$$

$$\begin{array}{ccc} \tilde{X} & \rightarrow & Y \\ p \downarrow & \swarrow & \\ X & = & \tilde{X}/G \end{array}$$

eg. $\mathbb{R} = \tilde{S}^1 \quad t \mapsto e^{2\pi i t/k}$

$$\begin{array}{ccc} & & \mathbb{R}/H = S^1 \\ p \downarrow & \swarrow & \\ S^1 & & P_k \end{array}$$
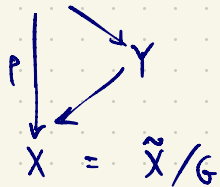
$p : t \mapsto e^{2\pi i t}$

$P_k : z \mapsto z^k$

Every other covering space $Y \rightarrow X$ (path-connected)

has the form $Y = \tilde{X}/H$, $\quad H \leq G$.
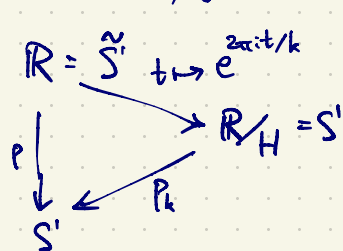
$G = \mathbb{Z}$

$H \leq G$ has the form $H = k\mathbb{Z}$, $k \in \mathbb{Z}$.

Construction of the universal cover $\tilde{X} \rightarrow X$?
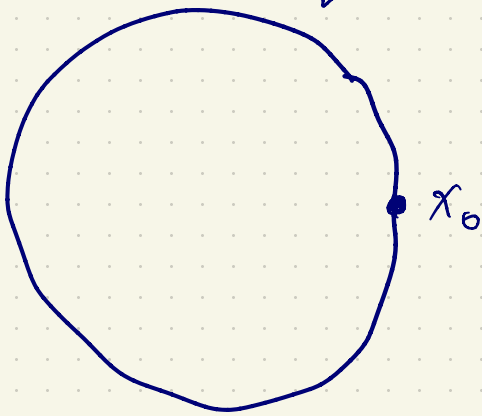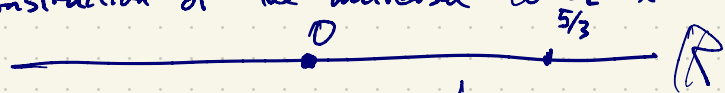
$\tilde{X} = \{ \text{paths in } X \text{ starting at the chosen base point } x_0 \}/\sim$

ie. paths up to homotopy with fixed starting and ending point

If $f \in \tilde{X}$ then

$$P(f) = f(1) \in X \text{ (the endpoint of } f)$$

Cohomology

Consider a sequence of vector spaces over F given by

$$\cdots \xrightarrow{\partial_4} V_3 \xrightarrow{\partial_3} V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} 0$$

or

$$0 \xrightarrow{0} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} \cdots$$

V^i or V_i has i just an index for purposes of reference.

If $\partial_i \circ \partial_{i+1} = 0$ then ∂ is a boundary map and the sequence of V_i's is a **complex**.
(similarly if $d^{i+1} \circ d^i = 0$, d is a **coboundary** map.)

∂_i, d^i linear transformations
(more generally V_i, V^i are modules over a ring R and ∂_i, d^i are R-homomorphisms
i.e. $d(av + bw) = a\,dv + b\,dw$
$a, b \in F$; $v, w \in V$

Notable example : differential forms

Let X be a real (smooth) n-manifold. In a nbhd of each point $x \in X$, $x \in U \subseteq X$, we have local coordinates $(x_1, \ldots, x_n) = x$.
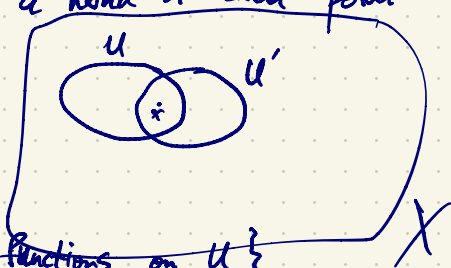
$R = C^\infty(U) = \{ \text{smooth real-valued functions on } U \}$.
$V^0 = R$. $d : V^0 \to V^1 = \{ \text{differential 1-forms on } U \} = \{ f_0\, dx_1 + f_2\, dx_2 + f_3\, dx_3 + \cdots + f_n\, dx_n \; : \; f_i \in R \}$
V^1 is a vector space over $\mathbb{R}$ (∞-dimensional)
but n-dimensional as module over R

Eg. $X = \mathbb{R}^2 - \{(0,0)\}$.

$$0 \xrightarrow{0} V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} 0$$

$V^0 = \{$ smooth functions $X \to \mathbb{R}\} = R = $ " 0-forms "

$V^1 = \{$ 1-forms on $X\}$ i.e. smooth differential 1-forms

$V^2 = \{$ 2-forms on $X\}$

$V^1 = \{f\,dx + g\,dy : f, g \in R\}$

ω is closed but not exact

Eg. $\omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ $w = \omega$

Integrate ω over the path $\gamma(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

$$\int_\gamma \omega = \int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = \int_0^{2\pi} \frac{\overbrace{\cos^2 t\,dt + \sin^2 t\,dt}^{1}}{} = \int_0^{2\pi} dt = 2\pi$$

$x = \cos t$
$dx = -\sin t\,dt$

$y = \sin t$
$dy = \cos t\,dt$

x, y : global coordinates in X

r, θ : local coordinates (not global)

$$\int_\gamma \omega = \int \cdots = 2\pi$$

$x, y, r, \theta \in R = V^0$

coordinate functions $X \to \mathbb{R}$

or on $U \subset X$

$x = r\cos\theta$
$y = r\sin\theta$

$\omega = \cos\theta$

$dx = \cos\theta\,dr - r\sin\theta\,d\theta$
$dy = \sin\theta\,dr + r\cos\theta\,d\theta$

$\gamma(t): \; r(t) = 1$
$\theta(t) = t$

$$V^0 \xrightarrow{\ d\ } V^1 \xrightarrow{\ d\ } V^2$$

$$f \longmapsto df$$
$$x \longmapsto dx$$
$$y \longmapsto dy$$
$$r \longmapsto dr$$
etc.

d is R-linear but not R-linear

$$d(fg) \neq f \, dg$$

$$\underline{V^2 = \{ f \, dx \wedge dy : f \in R \}}$$

If X has local coordinates $x_1, \cdots, x_n$ then $V^1 = \{ f_1 dx_1 + \cdots + f_n dx_n : f_i \in R \}$

$$V^2 = \{ f_{12} \, dx_1 \wedge dx_2 + f_{13} \, dx_1 \wedge dx_3 + \cdots : f_{ij} \in R \}$$

$$dx_i \wedge dx_i = 0$$
$$dx_i \wedge dx_j = - dx_j \wedge dx_i$$

Wedge products are R-multilinear e.g. $(fw + g\omega') \wedge \eta = f \, w \wedge \eta + g \, \omega' \wedge \eta$ $f, g \in R$

ω, ω', η
diff. forms

$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$$

$$= (- dy \wedge dx) \wedge dz) = -dy \wedge (dx \wedge dz) = - dy \wedge (- dz \wedge dx) = dy \wedge dz \wedge dx$$

$$dx \wedge (dy \wedge dz) = (dy \wedge dz) \wedge dx$$

If w is an i-form and η is a j-form then
$$w \wedge \eta = (-1)^{ij} \eta \wedge w \quad \text{is an } i+j\text{-form.}$$

If X is an n-manifold then $V^k = \{ k\text{-forms on } X \}$ is an R-module of dimension $\binom{n}{k}$.

We need X to be orientable

V^k is spanned by terms like $f \, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} =: \omega \in V^k$ WLOG $\leq i_1 < i_2 < \cdots < i_k \leq n$

$$V^k \xrightarrow{\ d\ } V^{k+1}$$

$$d\omega = d(f \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \underbrace{df} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in V^{k+1}$$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

In $\mathbb{R}^3$ with (global) coordinates x, y, z

$R = V^0 = \{$ smooth functions $\mathbb{R}^3 \to \mathbb{R} \}$

$d: V^k \to V^{k+1}$ is $\mathbb{R}$-linear but not R-linear

Pick $f \in V^0$ i.e. $f: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in V^1$ This is a very special 1-form because it is __exact__. ($\in dV^0$)

$$d(df) = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz$$

$$= \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial x} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial x} dz \right) \wedge dx + \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial y} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial y} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial y} dz \right) \wedge dy$$
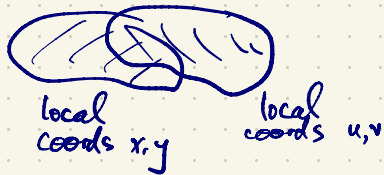
$$+ \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial z} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial z} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial z} dz \right) \wedge dz = 0$$

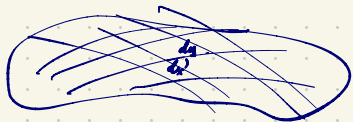$$\boxed{d^2 = 0}$$ i.e. $d^2\omega = d(d\omega) = 0$ for all ω.

Imagine a surface $S \subset \mathbb{R}^3$. We integrate an arbitrary 2-form $\omega \in V^2$ over S.

If $\omega = f(x,y,z) \, dx \wedge dy + g(x,y,z) \, dx \wedge dz + h(x,y,z) \, dy \wedge dz \in V^2$ then $\displaystyle\int_S \omega$

$$= \int_S f(x,y,z) \, dx \wedge dy + \cdots$$

If $\quad x = x(u,v)$
$\quad\quad y = y(u,v)$

then $\quad f(x,y) \, dx \wedge dy$

$$= f(x(u,v), y(u,v)) \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv$$

local coords x,y local coords u,v

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge$$

$$\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right)$$

$$= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv$$

For a region $X \subset \mathbb{R}^2$, γ path in X from P to Q, $\omega \in V^1$, we define the path integral $\displaystyle\int_\gamma \omega$.

If $\omega = df$ (an __exact__ 1-form) then $\displaystyle\int_\gamma \omega = \int_\gamma df = f(Q) - f(P)$ by the ~~Fundamental~~ Fundamental Theorem of calculus

~~But for~~

But if $\gamma' \sim \gamma$ in X then $\displaystyle\int_{\gamma'} \omega = \int_\gamma \omega = f(Q) - f(P)$ whenever $\omega = df$.

Stokes' Theorem (general Fundamental Theorem of Calculus)

Let X be an orientable n-manifold with boundary ∂X which is also orientable $(n-1)$-manifold. Let $\omega \in \Lambda^{n-1}$, so that $d\omega \in \Lambda^n$. Then

$$\int_{\partial X} \omega = \int_X d\omega$$

Special case: $X = [a,b] \subset \mathbb{R}$, $\partial X = \{a, b\}$, $\omega = f \in \mathbb{R}$ (smooth function $X \longrightarrow \mathbb{R}$)

$$\int_a^b f' = \int_a^b f'(t)\, dt = f(b) - f(a) \qquad d\omega = f'(x)\, dx$$

$\gamma \sim \gamma'$ in X $\qquad \partial A = \gamma - \gamma'$

If $\omega \in \Lambda^1(X)$ then $d\omega \in \Lambda^2(X)$

$$\int_\gamma \omega - \int_{\gamma'} \omega = \int_{\partial A} \omega = \int_A d\omega$$

If in particular $d\omega = 0$ (ω a <u>closed</u> t-form) then RHS $= 0$ giving $\int_\gamma \omega = \int_{\gamma'} \omega$.

Exact forms are automatically closed (if $\omega = df$ then $d\omega = d^2 f = 0$).

Not conversely! unless X is simply connected.

The "gap" between {closed forms} and {exact forms} is measured by cohomology.

$$V^{n-1} \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} V^{n+1}$$

(n-1)·forms   n·forms   (n+1)·forms

image of $d^{n-1}: V^{n-1} \longrightarrow V^n$ is $B^n = \{$ exact n-forms $\}$

kernel of $d^n: V^n \longrightarrow V^{n+1}$ is $Z^n = \{$ closed n-forms $\}$

$H^n = Z^n / B^n = n^{th}$ cohomology group (or vector space over $\mathbb{R}$)

dim H^n is the number of "n-dim'l holes" in X.

$X = \mathbb{R}^2 - \{0\}$ punctured plane

$0 \xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} 0$

$\omega = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ is closed (i.e. $d\omega = 0$)

but ω is not exact i.e. $\omega \neq df$
for any $f \in C^0$.

$H^1 = \{$ closed 1-forms $\} / \{$ exact 1-forms $\} = H^1(X; \mathbb{R})$

dim $H^1 = 1$.   Proof of this follows.

C stands for cochains; d is the coboundary map

$C^k = \{$ diff. k-forms on X $\} = C^k(X; \mathbb{R}) = C^k(X)$

$C^0 = \{$ smooth functions $X \to \mathbb{R} \}$

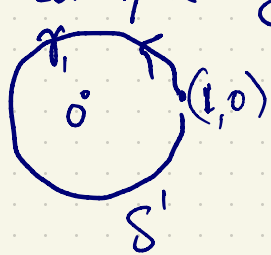$C^1 = \{ f(x,y)\,dx + g(x,y)\,dy : f, g \in C^0 \}$

$C^2 = \{ h(x,y)\,dx \wedge dy : h \in C^0 \}$

$df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$

$d(f\,dx) = df \wedge dx$, $d(g\,dy) = dg \wedge dy$.

Proof that dim $H' = 1$.

Let η be any closed 1-form on X ie. $\eta \in C'$, $d\eta = 0$.

Let $c = \int_{S'} \eta = \int_0^{2\pi} (\quad) d\theta$ and set $\tilde{\eta} = \eta - \frac{c}{2\pi}\omega$

$d\tilde{\eta} = 0 - 0 = 0$ so $\tilde{\eta}$ is closed

and we claim $\tilde{\eta}$ is exact

ie. $\eta = \underbrace{\tilde{\eta}}_{exact} + \frac{c}{2\pi}\omega$

To show $\tilde{\eta}$ is exact use:

For any two paths γ, γ' in X which are homotopic in X, (with common endpoints),

$$\int_\gamma \eta = \int_{\gamma'} \eta$$

$$0 = \int_A d\eta = \int_{\partial A} \eta = \int_\gamma \eta - \int_{\gamma'} \eta$$

Stokes' Theorem

Here is our candidate $f \in C^0$ for which $df = \tilde{\eta}$.

For each $Q \in X$, define $f(Q) = \int_\gamma \tilde{\eta} = \int_{(1,0)}^x \tilde{\eta}$ where γ is any path in X from $(1,0)$ to Q.

To see that this f is well-defined, first fix one path γ_0 from $(1,0)$ to Q. Then any path γ in X from $(1,0)$ to Q is homotopic to γ_0 concatenated with γ_1^k so $\int_\gamma \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} + k\int_{\gamma_1} \tilde{\eta} = \int_{\gamma_0} \tilde{\eta} \pm 0$
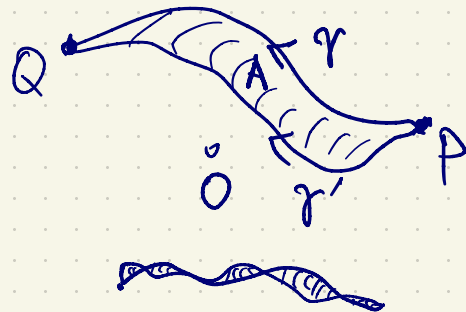
Finally,

$df = \tilde{\eta}$.

For $X = \mathbb{R}^2 - \{0\}$, (punctured plane), $\pi_1(X) \cong \mathbb{Z}$ since X is is contractible to S^1 and $\dim H^1(X; \mathbb{R}) = 1$. These two facts are related by the theorem of Hurewicz. $\mathbb{R}$ If we define our cohomology groups in a more universal way then $H^1(X) = H^1(X; \mathbb{Z})$ is an additive abelian group ie. $\mathbb{Z}$-module. $(\cong \mathbb{Z}$ in the case of $X = \mathbb{R}^2 - \{0\}$).

Hurewicz gave a homomorphism $\pi_1(X) \longrightarrow H_1(X) = H_1(X; \mathbb{Z})$ which is surjective; its kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ so $H_1(X)$ is the abelianization of $\pi_1(X)$.

0-Simplex   1-simplex   2-simplex   3-simplex   Simplicial complex
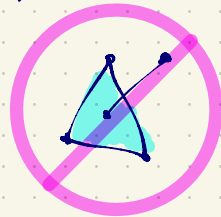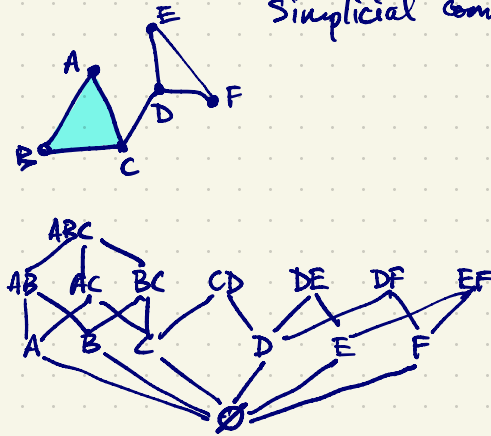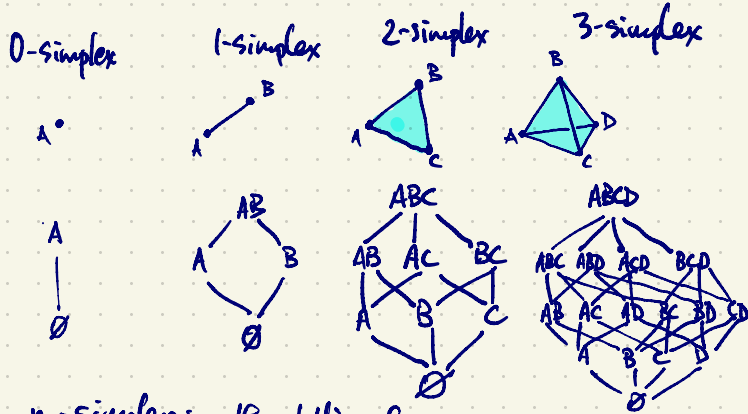
n-simplex: the lattice of subsets of an (n+1)-set.

eg. $X = $ 2-skeleton of a 3-simplex
(i.e. surface of a solid tetrahedron)
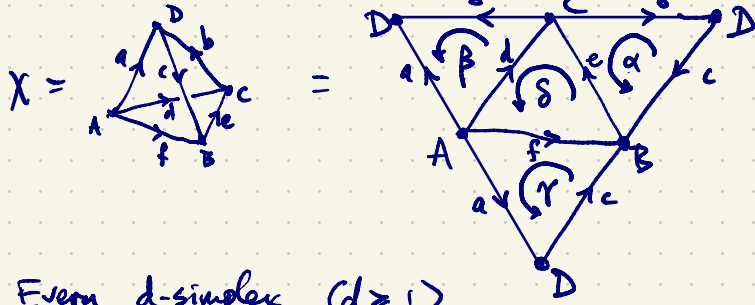
Every d-simplex $(d \geq 1)$ is orientable. Orient the faces arbitrarily

$X \cong S^2$ (homeomorphic top. spaces)

$\sim \mathbb{R}^3 - \{0\}$ (homotopic equivalent)

so we get the same algebraic invariants including homology groups) as we'll point out later

$\partial a = D - A$ $\partial \alpha = -b - c - e$
$\partial b = D - C$ $\partial \beta = -a + b + d$
$\partial c = B - D$ $\partial \gamma = a + c - f$
$\partial e = C - B$ $\partial \delta = -d + e + f$
$\partial f = B - A$
$\partial d = C - A$

$\partial^2 \alpha = \partial(\partial \alpha) = \partial(-b - c - e)$
$= -(D-C) - (B-D) - (C-B) = 0$

$$\partial_2 = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \\ \left[\begin{array}{cccc} 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \end{array}$$

$$\partial_1 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{array}{cccccc} a & b & c & d & e & f \\ \left[\begin{array}{cccccc} -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

We have a chain complex

$$0 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

where $C_k = \{k\text{-chains in } X\}$ $\partial = $ boundary map

A k-chain is a $\mathbb{Z}$-linear combination of k-faces of X (k-simplices)

$C_2 = \{x_1 \alpha + x_2 \beta + x_3 \gamma + x_4 \delta \ : \ x_1, x_2, x_3, x_4 \in \mathbb{Z}\}$

$C_1 = \{x_1 a + x_2 b + x_3 c + \cdots + x_6 f \ : \ x_1, \cdots, x_6 \in \mathbb{Z}\}$

$C_0 = \{x_1 A + x_2 B + x_3 C + x_4 D \ : \ x_1, \cdots, x_4 \in \mathbb{Z}\}$

C_k is an additive abelian group   ie. $\mathbb{Z}$-module.

$\partial_1 \circ \partial_2 = 0$

$\partial_n : C_k \longrightarrow C_{k-1}$ is additive ie.

$\partial(u \pm v) = \partial u \pm \partial v$

$\Rightarrow \partial(ru + sv) = r\partial u + s\partial v$

for all $r, s \in \mathbb{Z}$

i.e. ∂ is a $\mathbb{Z}$-module homomorphism

C_k is a free $\mathbb{Z}$-module

A number line marked: $-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

$\mathbb{R}/\sim \qquad x \sim y \iff \left(x = y \quad \text{or} \quad x, y \in \mathbb{Z} \right)$

$\cong \overset{\infty}{\underset{n=-\infty}{\bigvee}} S^1$

homeomorphic

$\mathbb{R}/\mathbb{Z} \cong S^1 \qquad$ quotient of top. groups.

not the same
(one is a group,
the other not)

For any chain complex $\quad \cdots \xrightarrow{\partial_{k+2}} C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \qquad$ i.e. $\partial_{k-1} \circ \partial_k = 0$

$\partial = (\partial_k)_k$ homomorphisms of additive abelian groups

$Z_k, B_k \leq C_k \qquad$ subgroups of $C_k = \{ k\text{-chains} \}$

$B_k = \partial_{k+1} C_{k+1} = $ image of $\partial_{k+1} : C_{k+1} \longrightarrow C_k = \{ k\text{-boundaries} \}$

$B_k \leq Z_k \leq C_k$

$Z_k = \ker \left(\partial_k : C_k \longrightarrow C_{k-1} \right) = \{ k\text{-cycles} \}$

The kth homology group $H_k = Z_k / B_k$

Two elements $z, z' \in Z_k$ are homologous if
$z + B_k = z' + B_k \iff z - z' \in B_k$.

eg. $X =$ 2-skeleton of a 3-simplex
(i.e. surface of a solid tetrahedron)

$$0 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

$$H_1 = Z_1 / B_1 =$$

$$Z_1 = \ker \partial_1 : C_1 \longrightarrow C_0$$

$$z = z_1 a + z_2 b + z_3 c + z_4 d + z_5 e + z_6 f \in Z^1 \Longleftrightarrow$$

$$0 = \partial z = z_1(D-A) + z_2(D-C) + z_3(B-D) + z_4(C-A) + z_5(C-B) + z_6(B-A)$$

$$= (-z_1 - z_4 - z_6) A + (z_3 - z_5 + z_6) B + (z_2 + z_4 + z_5) C + (z_1 + z_2 - z_3) D$$
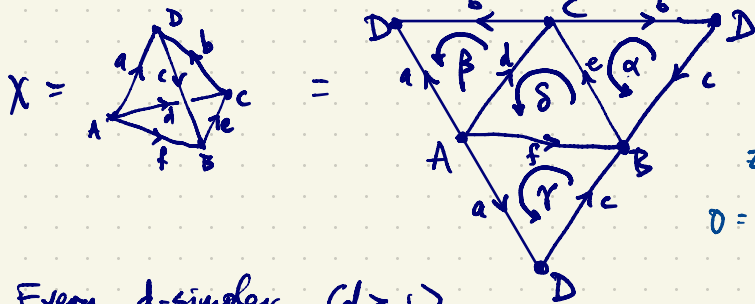
Every d-simplex $(d \geq 1)$ is orientable. Orient the faces arbitrarily

"null space" of
$$\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$
over $\mathbb{Z}$

$$\partial_1 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} a & b & c & d & e & f \\ -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

$X \cong S^2$ (homeomorphic top. spaces)

$\sim \mathbb{R}^3 - \{0\}$ (homotopy equivalent)

so we get the same algebraic invariants including homology groups)
as we'll point out later

$$\partial a = D - A \qquad \partial \alpha = -b - c - e$$
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$$\partial e = C - B \qquad \partial \delta = -d + e + f$$
$$\partial f = B - A$$
$$\partial d = C - A$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$a = -d - f$
$b = d + e$
$c = e - f$

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -d-f \\ d+e \\ e-f \\ d \\ e \\ f \end{bmatrix} = d \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\partial_2 = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$B_1 = \langle$ cols of $\partial_2 \rangle$
$= Z_1$

$$Z_1 = \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle_{\mathbb{Z}} \cong \mathbb{Z}^3$$

$H_1 = Z_1 / B_1 = 0$

$$H_0 = Z_0/B_0 \cong \mathbb{Z} \qquad H_0 = Z_0/B_0 = \{ kA + B_0 : k \in \mathbb{Z} \}.$$

$$B_0 = \langle A-B, A-C, A-D, B-C, B-D, C-D \rangle = \{ x_1 A + x_2 B + x_3 C + x_4 D : x_1, \cdots, x_4 \in \mathbb{Z}, \; x_1 + x_2 + x_3 + x_4 = 0 \}$$

$$Z_0 = \langle A, B, C, D \rangle$$

$$x_1 A + x_2 B + x_3 C + x_4 D = \underbrace{(x_1 + x_2 + x_3 + x_4)}_{k} A + \underbrace{(-x_2 - x_3 - x_4)A + x_2 B + x_3 C + x_4 D}_{B_0}$$