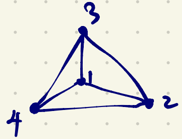
Math 5605

Algebraic Topology

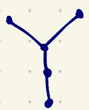
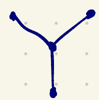
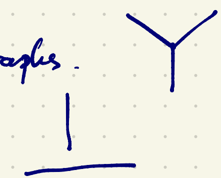
Book 2

When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

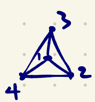
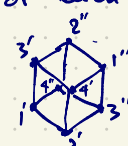
eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \dots, \{3,4\}\}$.

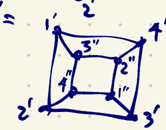
X is the <u>geometric</u> <u>realization</u> of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

 and  have the same geometric realization although they are different graphs. 

A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in Γ ($x, y \in V(\Gamma)$, $\{x,y\} \in E(\Gamma)$) $\Rightarrow$ $f(x) \sim f(y)$ in Γ'. A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

Eg. a double cover of $X =$  is the 1-skeleton of the cube $Y =$  $Y \longrightarrow X$

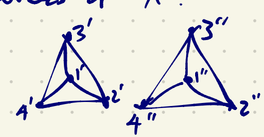
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

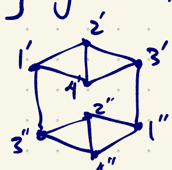
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



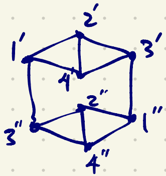
$Z \to X$

$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

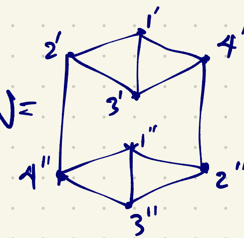
Let $p_1: X_1 \to X$, $p_2: X_2 \to X$ be covering spaces of X. We say $\theta: X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg. 

$Z \to X$
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

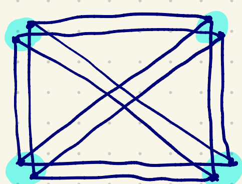
is not equivalent to $Y \to X$. But what about

$W =$ 

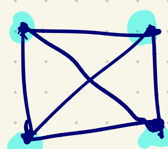
$W \to X$
$1', 1'' \mapsto 1$
$2', 2'' \mapsto 2$
$3', 3'' \mapsto 3$
$4', 4'' \mapsto 4$

Is this equivalent to $Z \to X$? No...

Another picture of these covers:



Y



X