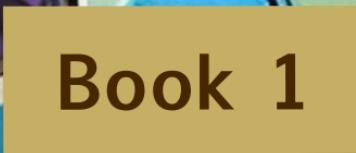
The background of the image is a vibrant, abstract geometric pattern composed of numerous triangles in various colors, including shades of blue, green, yellow, orange, red, purple, and pink. These triangles are arranged in a way that creates a sense of depth and perspective, resembling a stylized landscape or a complex crystal structure.

# Math 5605

# Algebraic Topology

A solid yellow rectangular box is positioned in the lower right quadrant of the central text area. It contains the word "Book" in a bold, black, sans-serif font, followed by the number "1" in a slightly smaller size.

Book 1

If  $X, Y$  are top. spaces,  $f: X \rightarrow Y$  is continuous if  $f^{-1}(U) \subseteq X$  is open whenever  $U \subseteq Y$  is open.  
 $f: X \rightarrow Y$  is a homeomorphism if  $f$  is bijective and  $f, f^{-1}$  are continuous.

$X \cong Y$  are homeomorphic if there exists a homeomorphism  $\tilde{f}: X \xrightarrow{\sim} Y$ .

$$\mathbb{R}^2 \neq S^2$$

2-sphere  $S^2 \cong \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$  — Since  $S^2$  is compact;  $\mathbb{R}^2$  is not.

$S^2 \neq T^2 = S' \times S' = \begin{array}{c} \text{---} \\ S \end{array} = \begin{array}{c} \text{---} \\ S' \times S' \end{array} = \begin{array}{c} \text{---} \\ S' \end{array}$   $S^2, T^2$  are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$$T = S = \text{circle} \cong \{z \in \mathbb{C} : |z| = 1\}$$

$S^2 \neq T^2$  because  $S^2$  is simply connected whereas  $T^2$  is not.

In  $S^1$ , every closed path can be "continuously shrunk" to a point  
 (homotopic to a point, i.e. null-homotopic)



$$\neq$$

although both surfaces are compact, connected, not simply connected

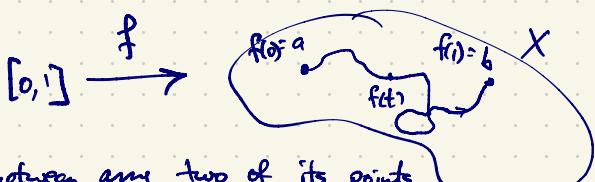
These two surfaces have different fundamental group:  $\pi_1(X)$  is nonabelian,  $\pi_1(T) \cong \mathbb{Z}^2$  is a (nontrivial) abelian group.

If  $X \cong Y$  (homeomorphic) then  $\pi_1(X) \cong \pi_1(Y)$ .

for much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of homotopy equivalence.

E.g. for every  $n > 0$ ,  $\mathbb{R}^n$  is homotopy equivalent to  $\mathbb{R}^0 = \{*\}$ .

Given points  $a, b \in X$  (a topological space), a path from  $a$  to  $b$  is a function  $f: [0, 1] \rightarrow X$  such that  $f(0) = a$ ,  $f(1) = b$ .



All maps (unless indicated otherwise) are assumed to be continuous.

If  $X$  has a path between any two of its points, then  $X$  is path-connected. For the time being, we'll assume  $X$  is path-connected. (In general, we instead define the fundamental groupoid of  $X$ .) If  $\varphi: [0, 1] \rightarrow [0, 1]$  (recall: continuous) such that  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ , then  $f \circ \varphi: [0, 1] \rightarrow X$  is just a reparameterization of the same path and we don't distinguish it from  $f$ .

If  $f, g: [0, 1] \rightarrow X$  are paths such that  $f(1) = g(0)$  then we can concatenate them to form a new path from  $f(0)$  to  $g(1)$ :

$$(fg)h = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

$(fg)h$  is the same path as  $f(gh)$  after reparameterization:

$$(fg)h(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(4t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$



$f, g$  are homotopic

but  $h$  is not homotopic to  $f, g$

$$(f(gh))t = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(2t-1), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$



More precisely, we require a map  $[0, 1]^2 \rightarrow X$

$$(s, t) \mapsto f(s, t) = f_s(t)$$

such that  $f_0 = f$  i.e.  $f_0(t) = f(t)$

$$f_1 = g \text{ i.e. } f_1(t) = g(t)$$

$$\begin{aligned} f_s(0) &= a \\ f_s(1) &= b \end{aligned} \text{ for all } s \in [0, 1]$$

This is a homotopy from  $f$  to  $g$ .

We say  $f, g$  are homotopic if there is a continuous family of paths from  $a$  to  $b$  in  $X$ ,  $f_s$  ( $s \in [0, 1]$ ) with  $f_0 = f$ ,  $f_1 = g$ .

If  $\varphi: [0, 1] \rightarrow [0, 1]$  is a map with  $\varphi(0) = 0$ ,  $\varphi(1) = 1$  then the reparameterized path  $f \circ \varphi: [0, 1] \rightarrow X$  is homotopic to  $f$ . A homotopy from  $f$  to  $f \circ \varphi$  is

$$[0, 1]^2 \rightarrow X$$

$$(s, t) \mapsto f(\underbrace{(1-s)t + s\varphi(t)}_{\in [0, 1]}) = f_s(t)$$

$$f_0(t) = f(t)$$

$$f_1(t) = f(\varphi(t))$$

$$f_s(0) = f((1-s)0 + s\varphi(0)) = f(0)$$

$$f_s(1) = f((1-s)1 + s\varphi(1)) = f(1)$$

Fix  $x_0 \in X$ . Assume  $X$  is path-connected.  $\pi_1(X, x_0)$  is the group of all homotopy classes of paths from  $x_0$  to  $x_0$  in  $X$  under concatenation. It turns out  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$  for all  $x_0, x_1 \in X$ .

This gives the fundamental group  $\pi_1(X)$ .

$$\pi_1(\mathbb{R}^n) = 1 \quad (\text{trivial group}).$$

$$\pi_1(S^1) \cong \mathbb{Z} \quad (\text{free group on one generator})$$