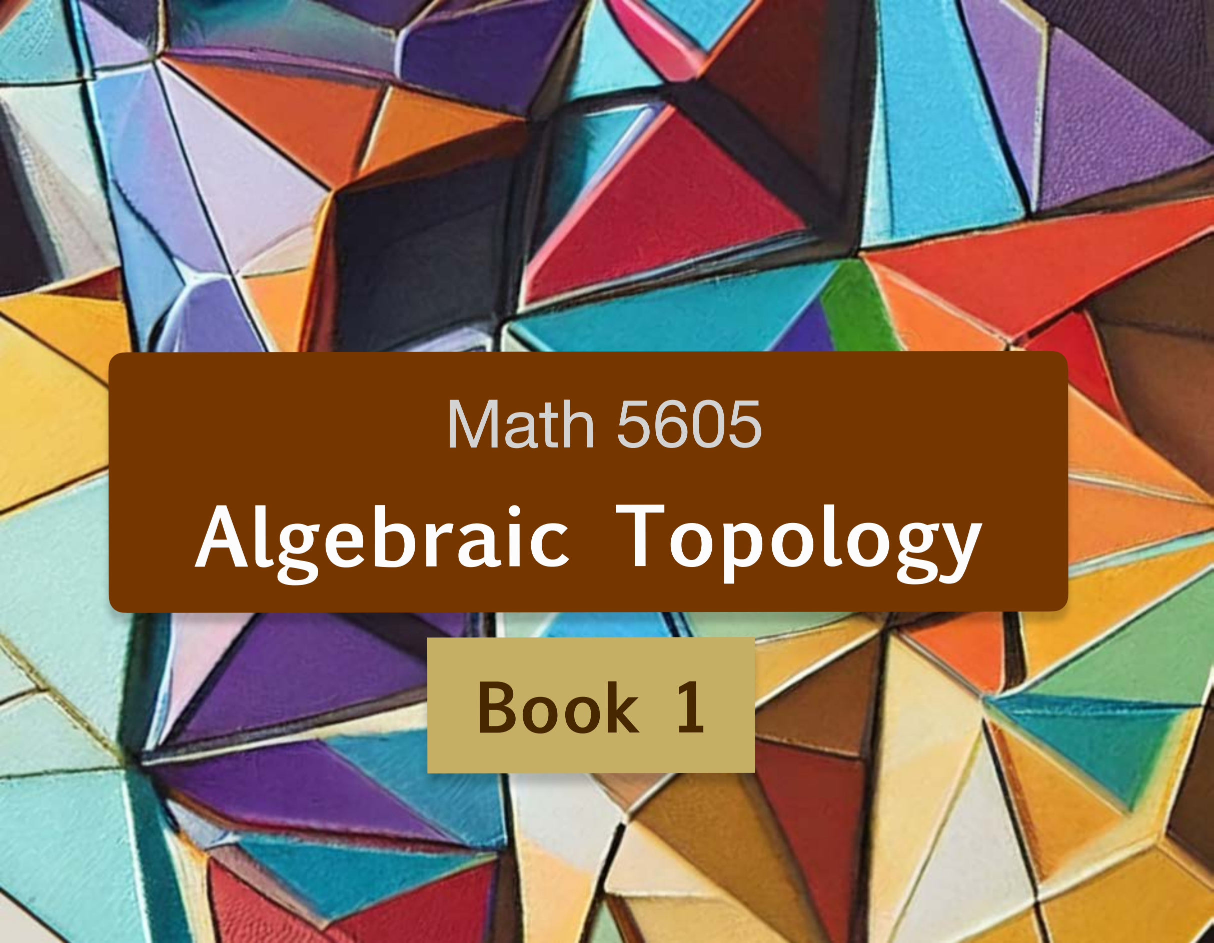


Math 5605

Algebraic Topology

Book 1

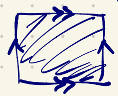
If X, Y are top. spaces, $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a <u>homeomorphism</u> if f is bijective and f, f^{-1} are continuous.

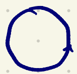
$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

$\mathbb{R}^2 \not\cong S^2$ since S^2 is compact; $\mathbb{R}^2$ is not.

2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ = 

$S^2 \not\cong T^2 = \underset{S}{\underset{\shortparallel}{S^1}} \times S^1 =$  = 

S^2, T^2 are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$T^1 = S^1 =$  = circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not.

In S^2, every closed path can be "continuously shrunk" to a point (homotopic to a point, i.e. null-homotopic)

X

 $\not\cong$

$T = T^2$

although both surfaces are compact, connected, not simply connected

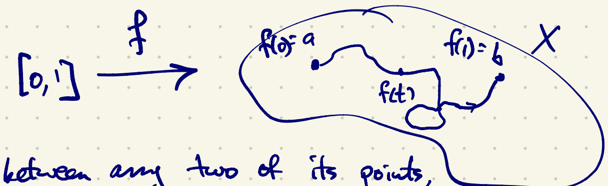
These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of <u>homotopy equivalence</u>.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\bullet\}$.

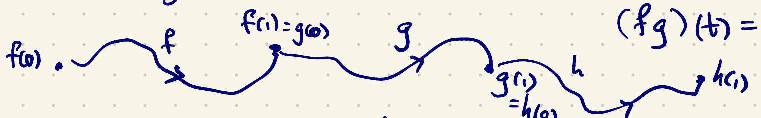
Given points $a, b \in X$ (a topological space), a path from a to b is a (continuous) function $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.



All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:



$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$

$(s,t) \mapsto f(s,t) = f_s(t)$

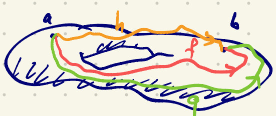
such that $f_0 = f$ i.e. $f_0(t) = f(t)$

$f_1 = g$ i.e. $f_1(t) = g(t)$

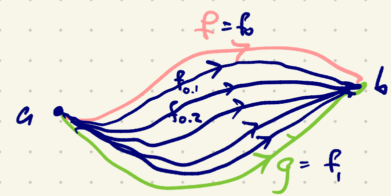
$f_s(0) = a$

$f_s(1) = b$ for all $s \in [0,1]$

This is a homotopy from f to g.



f, g are homotopic but h is not homotopic to f, g



We say f, g are homotopic if there is a 'continuous' family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi: [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f\circ\varphi : [0,1] \to X$ is homotopic to f. A homotopy from f to $f\circ\varphi$ is

$$[0,1]^2 \longrightarrow X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{\in [0,1]}) = f_s(t)$$

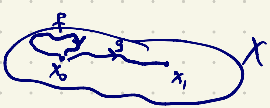
$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
$$f_s(0) = f(\cancel{(1-s)\cdot 0} + \cancel{s\cdot\varphi(0)}) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot \underline{\varphi(1)}) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X,x_0) \cong \pi_1(X,x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)



Fix g path in X from x_0 to x_1

An isomorphism $\phi: \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$f \overset{\phi}{\longmapsto} \bar{g} f g$$
$$g f \bar{g} \overset{\phi^{-1}}{\longleftarrow} h$$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f\gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X,x_0)$ is

$$\bar{f}(t) = f(1-t), \quad t\in[0,1]$$

(same path in the reverse direction)

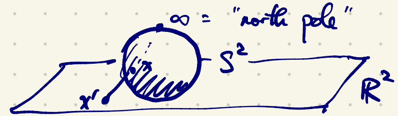
$f\bar{f} = \bar{f}f = \gamma$ = null path

$$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$

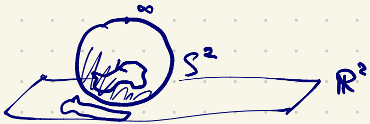
$f_1, f_2 \in \pi_1(X, x_0)$

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)



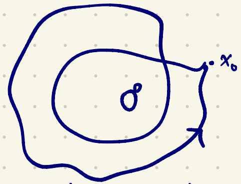
$x \longmapsto x'$ is stereographic projection from the north pole ∞



See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
punctured plane



follows from the fact that

$\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S^1$

$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).

Hatcher writes $X \approx Y$ for homeomorphic.

- retraction
- deformation retraction — "def. retraction in the weak sense"
- strong deformation retraction — "def. retraction".
- homotopy
- relative homotopy
- homotopy equivalence

Let $A \subseteq X$ (subspace of a top. space).

A retraction $f : X \to A$ is a (continuous) map such that $f|_A = \mathrm{id}_A = \mathbb{1}_A$ i.e. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a retract of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \to \{a\}$, $x \mapsto a$ is a retraction.

$\mathbb{R}^2 \to x\text{-axis}$, $(x,y) \mapsto (x,0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \to S^1$.

(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to a retraction $X \to A$), $A \subseteq X$.

i.e. $f : [0,1] \times X \longrightarrow X$

$$f(t,x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

$f_1(a) = a$ for all $a \in A$.

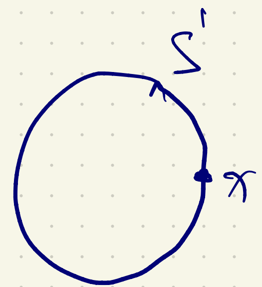
If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$

$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.

Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

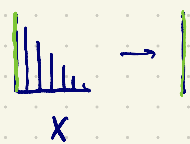


A strong def. retract $f : [0,1] \times X \to X$:

$f_0(x) = x$ i.e. $f_0 = id_X$

f_1 is a retraction $X \to A$

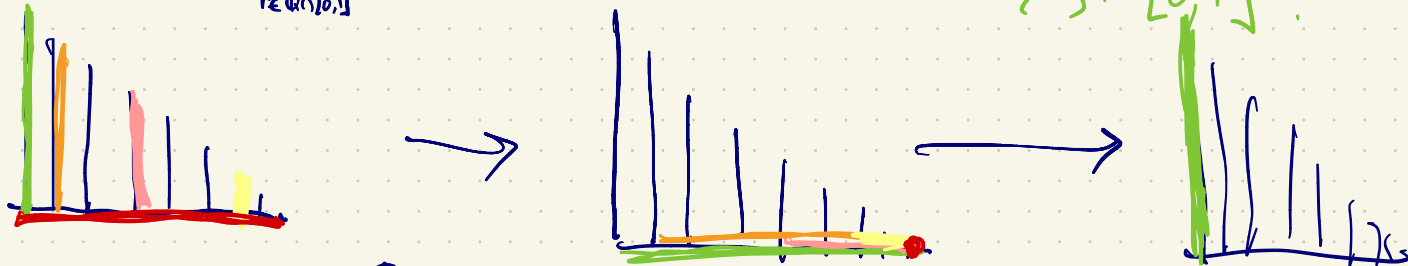
$f_t|_A = id_A$ for all $t \in [0,1]$



def. retract but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $\quad X = ([0,1]\times\{0\}) \cup \bigcup_{r\in\mathbb{Q}\cap[0,1]} (\{r\}\times[0,1-r])$

X has a def. ret. to $A := \{0\}\times[0,1]$.



$0 \le t \le \frac{1}{2}$ $\qquad \frac{1}{2} \le t \le 1$

This is a def. retract but not a strong def. retract.

There is no strong def. retract $X \to A$.

("continuous deformation")

Let $f_0, f_1 : X \to Y$ be maps. A homotopy from f_0 to f_1 is a map $f : [0,1]\times X \to Y$ such that $f(0,x) = f_0(x)$, $f(1,x) = f_1(x)$.

$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 ($f_0, f_1 : X \to Y$) is a homotopy $f : [0,1]\times X \to Y$ such that $f_t(a)$ is independent (constant) of $t \in [0,1]$ for all $a \in A$.

A path from a to b in X is a homotopy from a to b.

Given two paths f_0, f_1 in X from a to b $(a,b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$
such that $f \circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "f"] & \{0\} \arrow[l, bend left, "g"] \end{tikzcd}$$

$f(x) = 0$ for all $x \in \mathbb{R}^n$
$g(0) = 0 \in \mathbb{R}^n$

$$g \circ f : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto 0$$

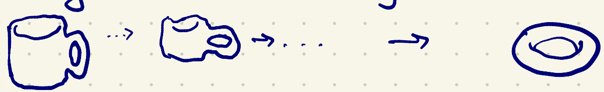
$$f \circ g : \{0\} \to \{0\}$$
$$0 \mapsto 0$$

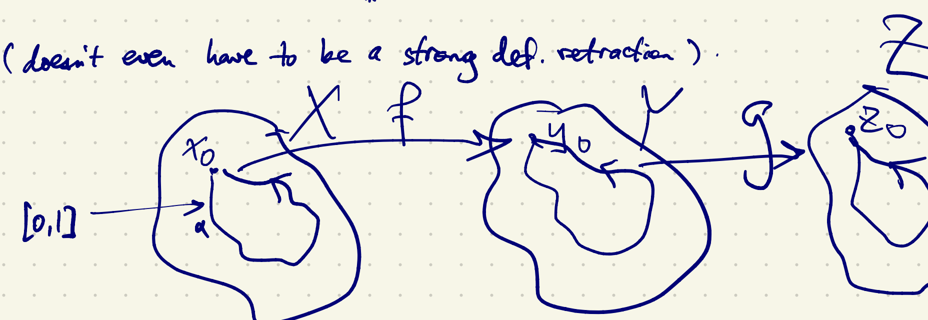
A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ to $id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$

Not relative to any subspace necessarily.

$h_0 = g \circ f$
$h_1 = id_{\mathbb{R}^n}$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).





S^1 is not homotopic to a point.
(S^1 is not null homotopic; not contractible)

If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

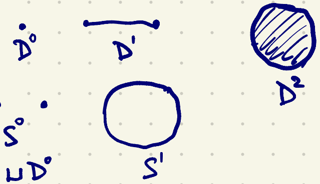
$\alpha : [0,1] \to X$ gives $f_*\alpha = f \circ \alpha : [0,1] \to Y$

$(g \circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

D^n = closed ball in $\mathbb{R}^n$
"disk"

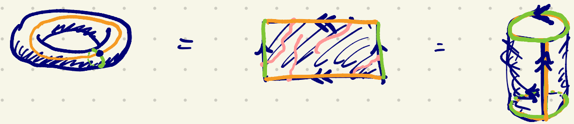
$S^n = \partial D^{n+1}$ = n-sphere

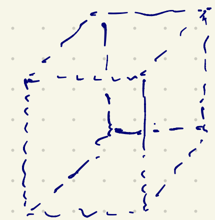


A CW complex is formed from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$

X^n is a union of $\underbrace{\text{copies of } D^n}_{D^n_\alpha}$ with the boundaries of D^n_α attached to X^{n-1} via attaching maps.

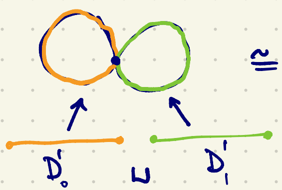
Eg. Torus $T^2 = S^1 \times S^1 =$



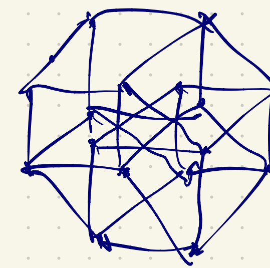


$X^0 = \bullet = D^0$

$X^1 =$

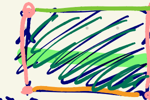


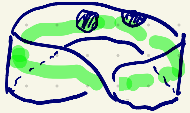
$\cong S^1 \vee S^1$



X^2

$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

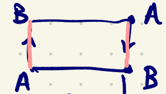
Möbius strip  not homeomorphic to cylinder $S^1 \times [0,1]$ =  = 



not orientable

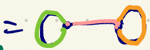
$X^0 =$ $\cdot \quad \cdot$

$X^1 =$ 



$X^2 =$

Cylinder: $X^0 =$ $\cdot \quad \cdot$ $D^0 \sqcup D^0$

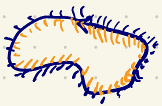
$X^1 =$ 

$X^2 =$ 

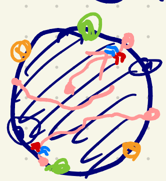
orientable

Both are homotopy equivalent to S^1 (def. retract to S^1)

Both have $\mathbb{Z}$ as fund. gp.



$P^2\mathbb{R}$ (or $\mathbb{R}P^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified



D^2

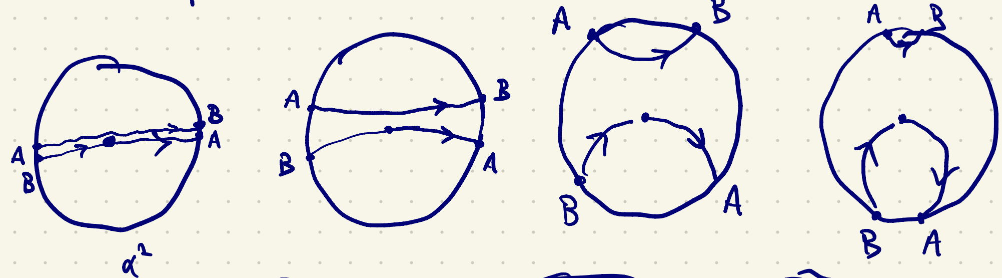
(non-orientable surface)

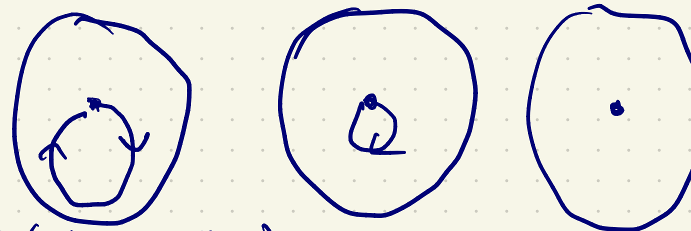
$P^2\mathbb{R} = D^2$ glued to a Möbius strip

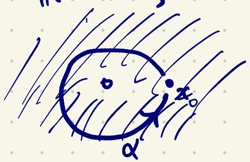
$\pi_1(P^2\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

α is not homotopic to the null path γ. α^2 is homotopic to γ





$\mathbb{R}^2 - \{0\} \simeq S^1$



A homotopy equivalence

$\mathbb{R}^2 - \{0\} \xrightarrow{f_1} S^1$

$$f_1((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$$

$$f_t(v) = (1-t)v + t\frac{v}{|v|} \qquad f_1(v) = \frac{v}{|v|}$$

$f_t : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$ strong def. retraction since $f_t|_{S^1} = id_{S^1}$ for all $t \in [0,1]$.

$f_0 = id_{\mathbb{R}^2 - \{0\}}$

f_1 is a retraction to S^1

$\{\alpha^n : n \in \mathbb{Z}\}$

$\|$

$\langle \alpha \rangle = \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$ free group on one generator

$\pi_1(S^1) \cong \mathbb{Z}$

Given a closed path α in S^1 with base point $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$

define $w(\beta) = \underrightarrow{\frac{1}{2\pi i}\int_\beta \frac{dz}{z}} = \frac{1}{2\pi i}\int_{\beta(0)}^{\beta(1)} \frac{dz}{z}$

$\beta : [0,1] \to S^1$

$\beta(0) = 1$

$\beta(e^{2\pi i\theta}) = \theta + n$

$w(\alpha) = 1$

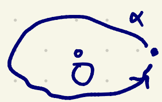
$w(\alpha^n) = n$

w is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$.

$\int \frac{dz}{z} = \ln|z| + 2\pi i \arg z$

In $\mathbb{C} - \{0\}$ the same argument works

$\mathbb{R}^2 - \{O\}$ $\quad O = (0,0)$



$\pi_1(\mathbb{R}^2 - \{O\}) = \langle\alpha\rangle \cong \mathbb{Z}$

k-punctured plane $X = \mathbb{R}^2 - \{\overbrace{A_1, \cdots, A_k}^{\text{distinct}}\}$

$\pi_1(X) = F_k = \mathrm{Free}(\{\alpha_1, \cdots, \alpha_k\})$

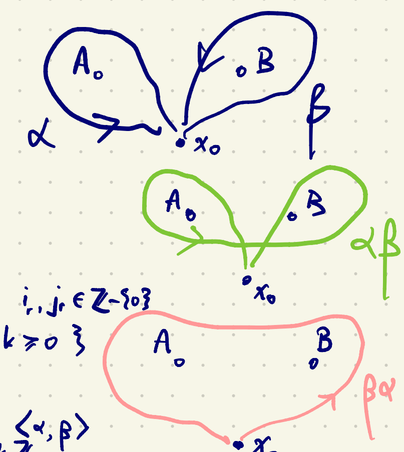
$X \simeq \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{k}$

$X = \mathbb{R}^2 - \{A, B\}$

$\pi_1(X) = \{\alpha^{i_0}\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{j_k},$
$\alpha^{i_0}\beta^{j_0}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{j_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k} \ : \ i_r, j_r \in \mathbb{Z} - \{0\}$
$k \geq 0\}$

$\pi_1(X)$ is the free group on two generators i.e.

$F_2 = \mathrm{Free}(\{\alpha, \beta\}) = \langle\alpha, \beta\rangle$
$= \langle\alpha\rangle * \langle\beta\rangle = \mathbb{Z} * \mathbb{Z}$



<u>The Van Kampen Theorem</u> gives a presentation for $\pi_1(X)$ when X is suitably described in terms of smaller pieces.

A <u>presentation</u> for a group G expresses G as a homomorphic image of a free group F i.e. $G \cong F/N$, $N \trianglelefteq F$.

Let X be a set of generators of G ($X \subseteq G$, $\langle X \rangle = G$).

$\mathrm{Free}(X) \longrightarrow G$ is a surjective homomorphism; N is its kernel.

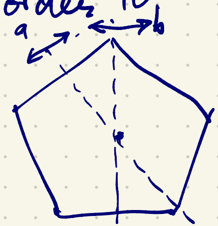
$G = \langle x_1, \dots, x_k : \overbrace{r_1, \dots, r_m}^{\in F} \rangle$ is a <u>presentation</u> for G if $X = \{x_1, \dots, x_k\}$ is a set of k symbols, $F = \mathrm{Free}(X)$ (the free group on $x_1, \dots, x_k$).

Let N be the smallest normal subgp of F containing $r_1, \dots r_m$ i.e. the normal closure of $\langle r_1, \dots, r_m \rangle \leq F$

i.e. the subgp. of F generated by $r_1, \dots, r_m$ <u>and</u> their conjugates in F

$N = \langle h r_i h^{-1} : i = 1, \dots, m ;\ h \in F \rangle$. (When there are k generators and m relators, we say G is <u>finitely presented</u>.)

eg. the dihedral group of order 10 is $D_{10} \cong \langle a, b : a^2, b^2, (ab)^5 \rangle \cong \langle x, y : x^2, y^5, xyx^{-1}y \rangle$



$xyx^{-1}y = 1$
$\Longleftrightarrow xy = y^{-1}x$
$\Longleftrightarrow xyx^{-1} = y^{-1}$

$$D_{10} \cong \langle a,b : a^2, b^2, (ab)^5 \rangle \cong \langle x,y : x^2, y^5, xyx^{-1}y \rangle$$

$\xrightarrow{\phi}$

$x = a$

$y = ab$

$x^2 = a^2 = 1$

$y^5 = (ab)^5 = 1$

$xyx^{-1} = a \cdot ab \cdot a^{-1} = ba$

whereas $y^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba$

$\longleftarrow$

check.

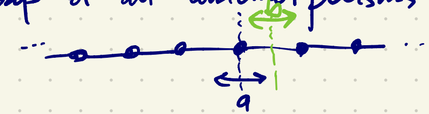
A free product $G*H$ is the set of all words from elements in G and H having no relations between elements in G and elements in H (except if one of these elements is the identity).

$$\underset{\text{infinite cyclic}}{\mathbb{Z}} * \underset{\text{infinite cyclic}}{\mathbb{Z}} = \langle a \rangle * \langle b \rangle = \mathrm{Free}\{a,b\} = \langle a,b \rangle = \{a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_k}b^{j_k}, \dots \}$$

$$= \langle a,b : a^2, b^2 \rangle$$

$$\langle a : a^2 \rangle * \langle b : b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \dots\} = D_\infty$$

$$\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle x,y : x^2, \underbrace{xyx^{-1} = y^{-1}}_{xyx^{-1}y = 1} \rangle = \text{group of all automorphisms of}$$



i.e. $\langle X : R \rangle = \langle x_1, \dots, x_m : r_1, \dots, r_k \rangle$

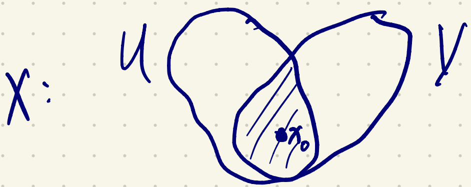
$\langle Y : S \rangle = \langle y_1, \dots, y_n : s_1, \dots, s_l \rangle$

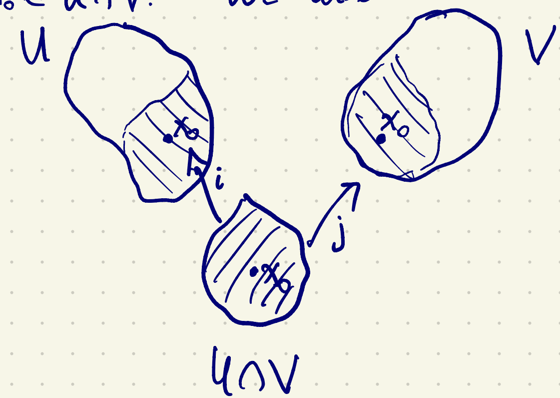
$$\langle X : R \rangle * \langle Y : S \rangle = \langle X \cup Y : R \cup S \rangle = \langle x_1, \dots, x_m, y_1, \dots, y_n : r_1, \dots, r_k, s_1, \dots, s_l \rangle$$

Free products with amalgamation: add more relations involving x_i's and y_j's e.g.

$$D_{10} = \langle a, b : a^2, b^2, (ab)^5 \rangle = \underbrace{\langle a : a^2 \rangle}_{\text{cyclic of order 2}} *_{(ab)^5} \underbrace{\langle b : b^2 \rangle}_{\text{cyclic of order 2}} = D_\infty / \text{Normal closure of } \langle (ab)^5 \rangle$$

Let X be a path-connected top. space covered by two open sets U, V. Since X is connected, $U \cap V \neq \emptyset$. Pick $x_0 \in U \cap V$. We also assume $U \cap V$ is path-connected.





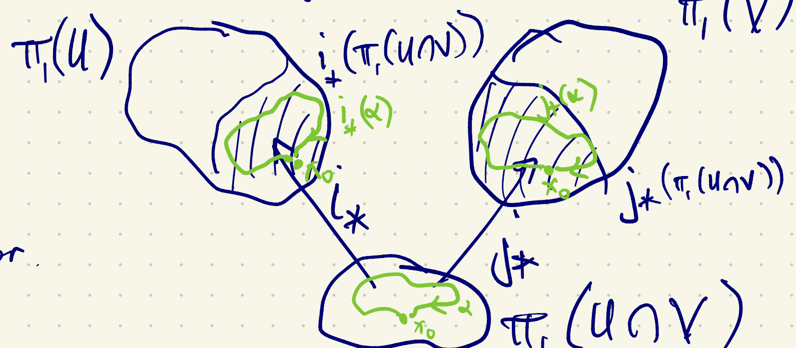
i, j inclusion maps as shown (injective, continuous)

Theorem (Van Kampen, special case)

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

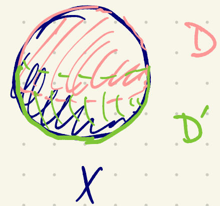
where the amalgamation over $\pi_1(U \cap V)$ is given by:

for all $\alpha \in \pi_1(U \cap V)$, identify $i_*(\alpha)$ with $j_*(\alpha)$ i.e. $i_*(\alpha)^{-1} j_*(\alpha)$ is a new relator.



This induces group homomorphisms i_*, j_* as shown.

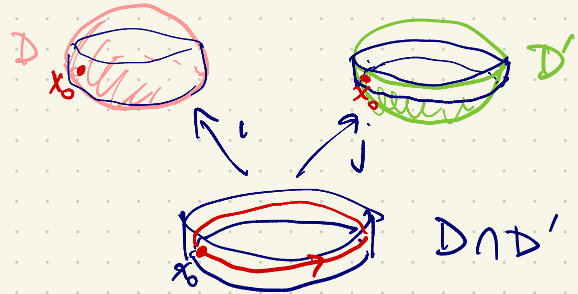
Eg. $X = S^2 = D \cup D'$



$\pi_1(S^2) = \pi_1(D) \underset{\square}{*} \pi_1(D') = 1$

$= \underbrace{\pi_1(D) * \pi_1(D')}_{\text{amalgamation}} = 1.$

trivial



$D \cap D' \simeq S^1$

$\pi_1(D) \cong \pi_1(D') = 1$ trivial

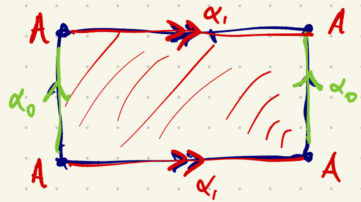
$\pi_1(D \cap D') = \langle \alpha \rangle$

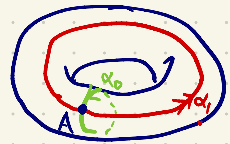
$$\begin{tikzcd} \pi_1(D) & & \pi_1(D') \\ & \pi_1(D \cap D') \arrow[ul, "i_*"] \arrow[ur, "j_*"'] & \end{tikzcd}$$

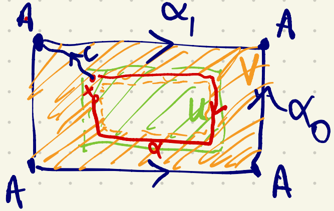
i_*, j_* not one-to-one

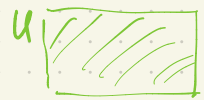
$i_*(\alpha) = 1$

$j_*(\alpha) = 1.$

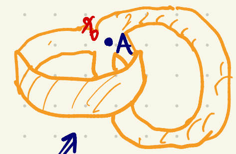
$T^2 = S^1 \times S^1 =$ 

$=$ 

$T^2 =$ 



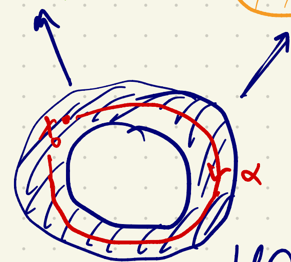
$\pi_1(U) = 1$



$V \simeq S^1 \vee S^1$

$\pi_1(V) = \langle a, b \rangle$ Free group on two generators

$$\pi_1(U) * \pi_1(V)$$
$$= 1 * \langle a, b \rangle$$
$$= \langle a, b \rangle$$



$U \cap V \cong S_1 \times (-\varepsilon, \varepsilon)$

$\pi_1(U \cap V) = \langle \alpha \rangle$

Then identify α with 1 due to the inclusion $U \cap V \subset U$. In V, α is homotopic to $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$

Then identify $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$ with α.

So we identify $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$ with 1.

So identify $\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0$ with 1.

identify $\alpha_1\alpha_0^{-1}$ with $\alpha_0^{-1}\alpha_1$; $\alpha_0\alpha_1$ with $\alpha_1\alpha_0$.

So identify ab with ba.

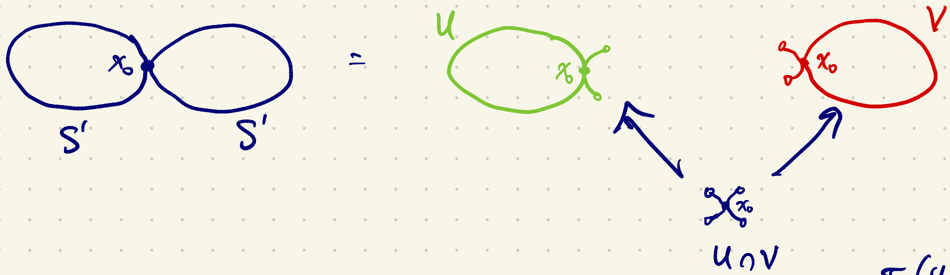
$\pi_1(T^2) = \langle a, b \rangle / \text{"}ab = ba\text{"} \cong \mathbb{Z} \times \mathbb{Z}$

$= \langle a, b \rangle / \text{Normal closure of } aba^{-1}b^{-1}$

$=$ the abelianization of $\langle a, b \rangle$

$\pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1)$ also follows from Van Kampen's Theorem.

If X and Y are path-connected top. spaces then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.



$U \simeq V \simeq S^1$

$\pi_1(U \cap V) = 1$

$$\pi_1(S_1 \vee S_1) = \pi_1(U) \underset{\boxed{\ }}{*} \pi_1(V) = \mathbb{Z} * \mathbb{Z} = \text{free group on two generators.}$$

In Tp, the category of top. spaces, the coproduct of X and Y is $X \sqcup Y$ = disjoint union; product is $X \times Y$.

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f_1"] \arrow[r, "i"'] & X \sqcup Y \arrow[u, dotted] & Y \arrow[ul, "f_2"'] \arrow[l, "j"]
\end{tikzcd}$$

The category of pointed top. spaces i.e. pairs (X, x_0), $x_0 \in X$, X nonempty top. space

The coproduct $(X, x_0) \sqcup (Y, y_0)$ = disjoint union $X \sqcup Y$ with x_0, y_0 identified

denoted as $(X, x_0) \vee (Y, y_0) = (X \sqcup Y, x_0 = y_0)$

A morphism $(X, x_0) \longrightarrow (Y, y_0)$ is a (continuous) map $X \to Y$ satisfying $x_0 \mapsto y_0$.

We have a functor $\{\text{pointed top. spaces}\} \longrightarrow Grp = \{\text{groups}\}$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

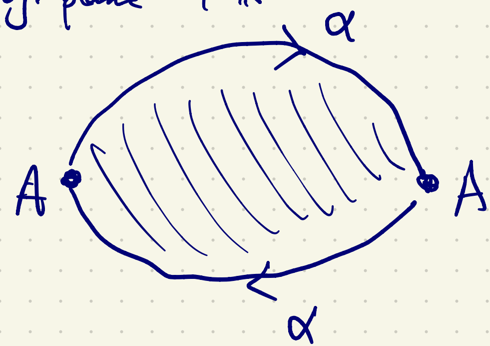
$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] \arrow[d, "F"'] \arrow[rr, bend left, "g \circ f"] & (Y, y_0) \arrow[r, "g"] \arrow[d, "F"] & (Z, z_0) \arrow[d] \\
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g \circ f)_* = g_* \circ f_*"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] & \pi_1(Z, z_0)
\end{tikzcd}$$

$$\begin{tikzcd}
(X, x_0) \vee (Y, y_0) \arrow[d, "F"] \\
F((X, x_0) \vee (Y, y_0)) = F((X, x_0)) * F((Y, y_0))
\end{tikzcd}$$

coproduct in pointed top. spaces is wedge sum.

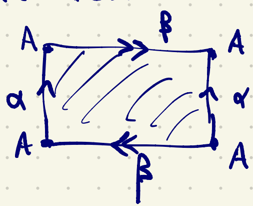
coproduct in Grp is just free product

Proj. plane $P^2\mathbb{R}$



$$\pi_1(P^2\mathbb{R}) = \langle \alpha : \alpha^2 \rangle = \langle\alpha\rangle / \langle\alpha^2\rangle = \text{group of order } 2$$
$$= \mathbb{Z}/2\mathbb{Z}$$

Klein Bottle K



$\pi_1(K) = \langle \alpha, \beta : \alpha\beta\alpha^{-1}\beta \rangle = \langle\alpha,\beta\rangle / N$ where N is the subgp of $\langle\alpha,\beta\rangle$ generated by $\alpha\beta\alpha^{-1}\beta$ and its conjugates

Force $\alpha\beta\alpha^{-1}\beta = 1$ in the quotient gp.

ie. $\alpha\beta\alpha^{-1} = \beta^{-1}$

so $\pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

$\langle\beta\rangle$ $\quad$ $\langle\alpha\rangle$ where α inverts β.

Every element can be uniquely expressed as $\alpha^k\beta^l$, $\quad k, l \in \mathbb{Z}$

$$(k,l)(r,s) = \begin{cases} (k+r, l+s) & \text{if } r \text{ is even} \\ (k+r, -l+s) & \text{if } r \text{ is odd} \end{cases}$$

$$(\alpha^k\beta^l)(\alpha^r\beta^s) = \begin{cases} \alpha^{r+k}\beta^{l+s} & \text{if } r \text{ is even;} \\ \alpha^{r+k}\beta^{-l+s} & \text{if } r \text{ is odd.} \end{cases}$$

$\alpha\beta^3\alpha^{-1} = \alpha\beta\beta\beta\alpha^{-1} = (\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1}) = \beta^{-1}\beta^{-1}\beta^{-1} = \beta^{-3}$ so $\alpha\beta^3 = \beta^{-3}\alpha$

$\alpha\beta^k\alpha^{-1} = \beta^{-k}$

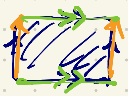
$\beta^k = \alpha^{-1}\beta^{-k}\alpha$

$\{\alpha^k \beta^l : k, l \in \mathbb{Z}\} = \pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

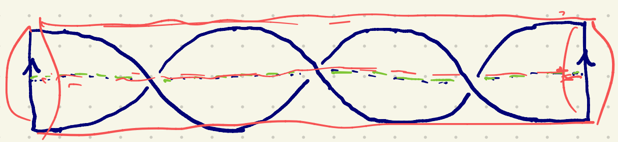
The elements with k even form an abelian subgp $\langle \alpha^2, \beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$ (free abelian of rank 2) and the quotient $\pi_1(K) / \langle \alpha^2, \beta \rangle$ is order 2.

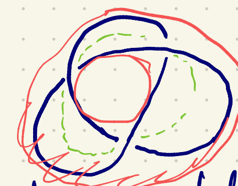
Leading up to trefoil:

$T^2 \hookrightarrow \cancel{\mathbb{R}^3}\ S^3$



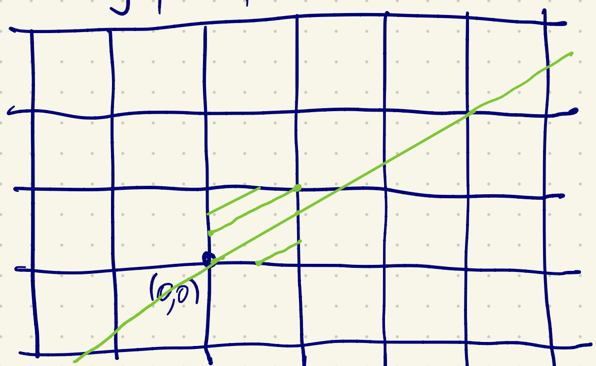
This surface partitions $\cancel{\mathbb{R}^3}\ S^3$ into two solid tori





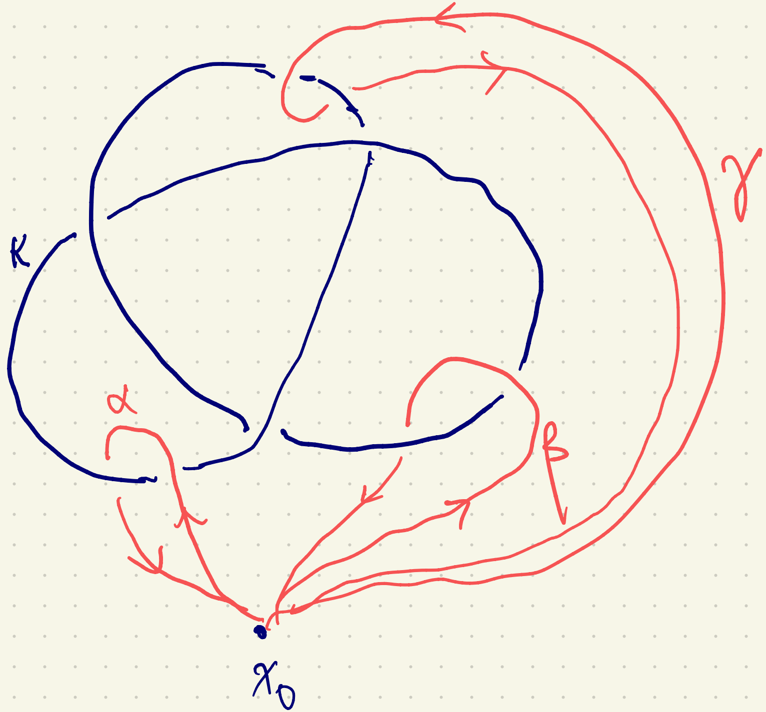
For any $m, n > 1$ relatively prime, we have an (m,n) torus knot embedded in the torus

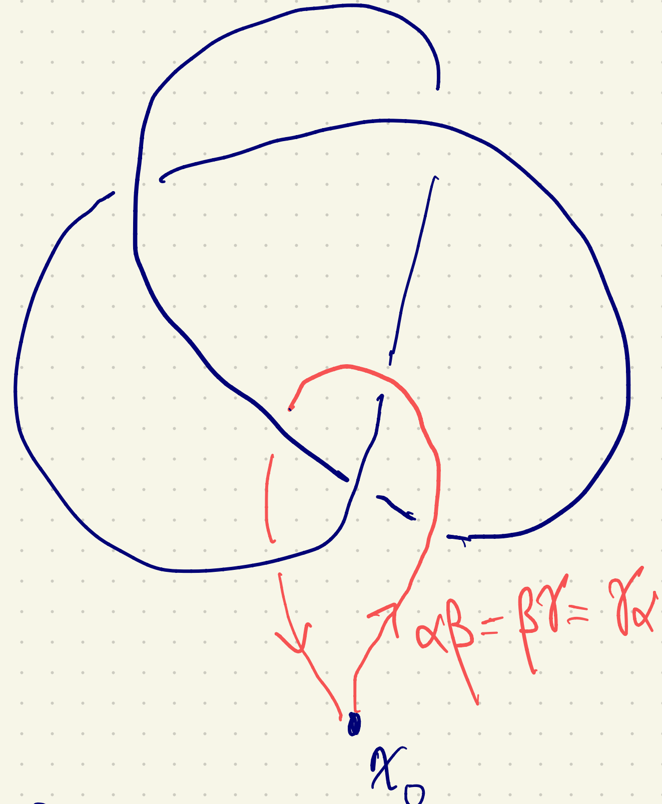
$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$



Take the line $y = \frac{n}{m} x$

The trefoil is the $(3,2)$ torus knot.
or $(2,3)$ " "





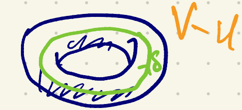
$\pi_1(\underset{\text{or } S^3}{\mathbb{R}^3} - K) = \langle \alpha, \beta, \gamma : \alpha\beta = \beta\gamma = \gamma\alpha \rangle = \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle$? Prove using Van Kampen's Theorem

$\gamma = \beta^{-1}\alpha\beta = \alpha\beta\alpha^{-1}$

$S^3 - K = U \cup V$

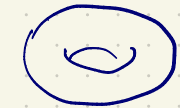
trefoil knot

U, V open path-connected
$U \cap V$ .. ...

$S^3 =$  $\cup$ 

open solid doughnut $(S^1 \times D^2)^\circ$ $\cup$ open solid doughnut $(S^1 \times D^2)^\circ$

$\cup$ open nbhd of torus



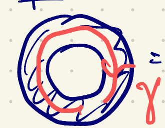
$\pi_1(U) = \langle \varepsilon \rangle \cong \mathbb{Z}$

$\pi_1(V) = \langle \delta \rangle \cong \mathbb{Z}$

$\varepsilon^2 = \gamma = \delta^3$

$S^3 - K =$ (exterior) open solid doughnut $\cup$ (interior) open solid doughnut

$\cup$ open torus $- K$

 $=$ 

$= U \cap V$

$\pi_1(U \cap V) = \langle \gamma \rangle \cong \mathbb{Z}$

$\pi_1(\underbrace{S^3 - K}_{U \cup V}) = \pi_1(U) \underset{\square}{*} \pi(V) = \langle \varepsilon \rangle * \langle \delta \rangle \big/$ Normal closure of $\langle \varepsilon^2 \delta^{-3} \rangle$

$= \langle \varepsilon, \delta : \varepsilon^2 = \delta^3 \rangle$

$\cong \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle$ ← No torsion

$\varepsilon = \alpha\beta\alpha, \quad \delta = \alpha\beta \quad \Rightarrow \quad \delta^3 = \underbrace{\alpha\beta\alpha}_{\varepsilon}\underbrace{\beta\alpha\beta}_{\varepsilon} = \varepsilon^2$

If $X \subseteq \mathbb{R}^n$, when can $\pi_1(X)$ have torsion (nontrivial elements of finite order)?

Eg. $P^2\mathbb{R} \hookrightarrow \mathbb{R}^4$, $\pi_1(P^2\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

For $n = 2$, no torsion in $\pi_1(X)$.

For $n = 3$, conjecturally $\pi_1(X)$ has no torsion.

Where do torus knots arise "in nature"?

The (m,n)-torus knot for $m,n \geq 2$ rel. prime.

A knot is an embedding of S^1 in S^3 (or in $\mathbb{R}^3$).

The unit sphere in $\mathbb{C}^2$ is $\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \cong S^3$

Consider the algebraic variety $\{(z,w) \in \mathbb{C}^2 : z^m = w^n\}$ intersected with S^3 is the (m,n)-torus knot.

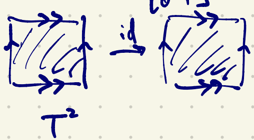
Recall: S^3 can be constructed by pasting together two solid tori $S^1 \times D^2$ (second torus having its meridian and longitude "reversed" as happens when turning it "inside out")

What if we instead glue together without this reversal? (i.e. boundary points on first torus are pasted to the identical point on the second torus)? $S^1 \times S^2$. $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$

In fact there are <u>many</u> ways to glue together two solid tori along their boundaries!

Consider any homeomorphism of T^2

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

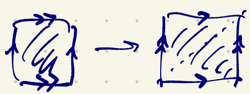


id

T^2

gives $S^1 \times S^2$

reflection $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



gives S^3.

Apply any $A \in SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc = \pm 1 \right\}$. to $\mathbb{R}^2$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \Rightarrow A^{-1} = \pm \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det A = \pm 1$

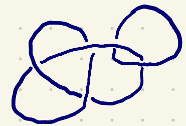
A preserves $\mathbb{Z}^2 \subset \mathbb{R}^2$ so A acts on $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$.

A maps the "unit" paths in x,y directions to certain torus knots. But the real thing is what happens to the construction above (gluing together two solid tori). This gives a lens space. Even this construction generalizes quite far.

This leads to Dehn surgery: Start with a knot K in S^3

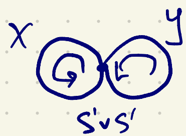
ε-nbhd of K has boundary $R \cong T^2$ embedded strangely in S^3.

$(S^3 - R)$ glued to $(S^3 - R')$ along their boundaries $R \cong R' \cong T^2$.



$G = \langle x, y : xy^2x^{-1}y,\ xy^4x \rangle$

Construct X having $\pi_1(X) \cong G$:



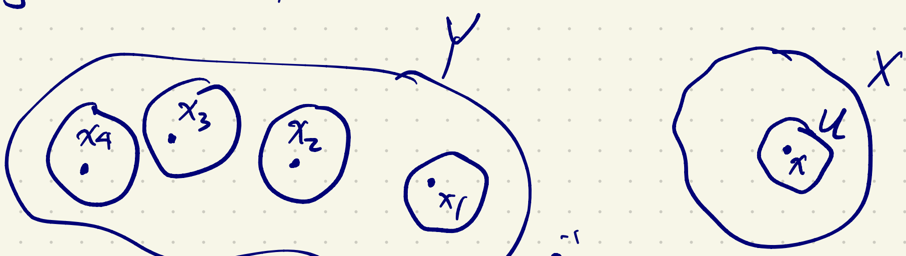
with two closed disks

glued on to the 1-skeleton $S^1 \vee S^1$

<u>Covering Spaces</u> Let $f: Y \to X$ be a map. The <u>fibre</u> over $x \in X$ is $f^{-1}(x) = \{y \in Y : f(y) = x\}$.

Given $A \subseteq X$, its preimage is $f^{-1}(A) = \{y \in Y : f(y) \in A\}$. (Hatcher instead writes $f^{-1}[A]$.)
I'm being more casual.

The map $f: Y \to X$ is a <u>covering map</u> if every point $x \in X$ has an open nbhd U such that $f^{-1}(U)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically to U by f.



$f^{-1}(x) = \{x_1, x_2, x_3, x_4\}$

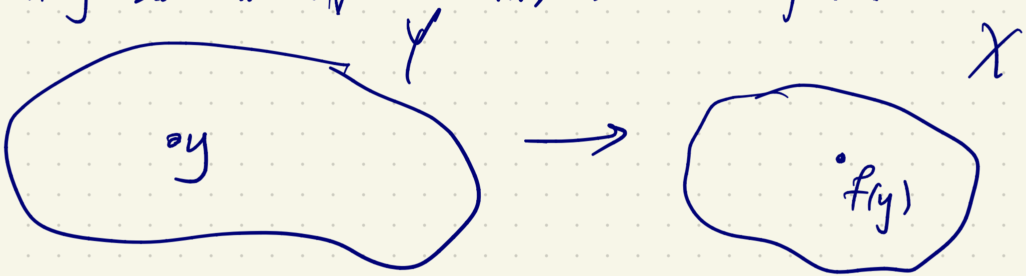
In particular, this requires $f^{-1}(x) \subset Y$ is a discrete subset.

Eg. $f : \mathbb{R} \to \mathbb{R}$
$$x \longmapsto \begin{cases} x \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

$f^{-1}(0) = \{0, \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \dots\}$

f is not a covering map

If $f : Y \to X$ is a covering map then f is locally a homeomorphism: for every $y \in Y$, there is an open nbhd V of y such that $f|_V : V \to f(V)$ is a homeomorphism.

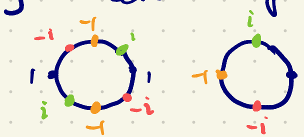


But the converse is not true: not every locally injective/homeomorphic map is a covering map.

Eg. Covers of S^1: There are many covering maps of S^1, some of which are connected; let's start with these.

Think $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. For every non-zero integer n, $\varphi_n(z) = z^n$, $\varphi_n : S^1 \to S^1$ is a covering map.

Eg. $\varphi_2 : S^1 \to S^1$, $z \longmapsto z^2$

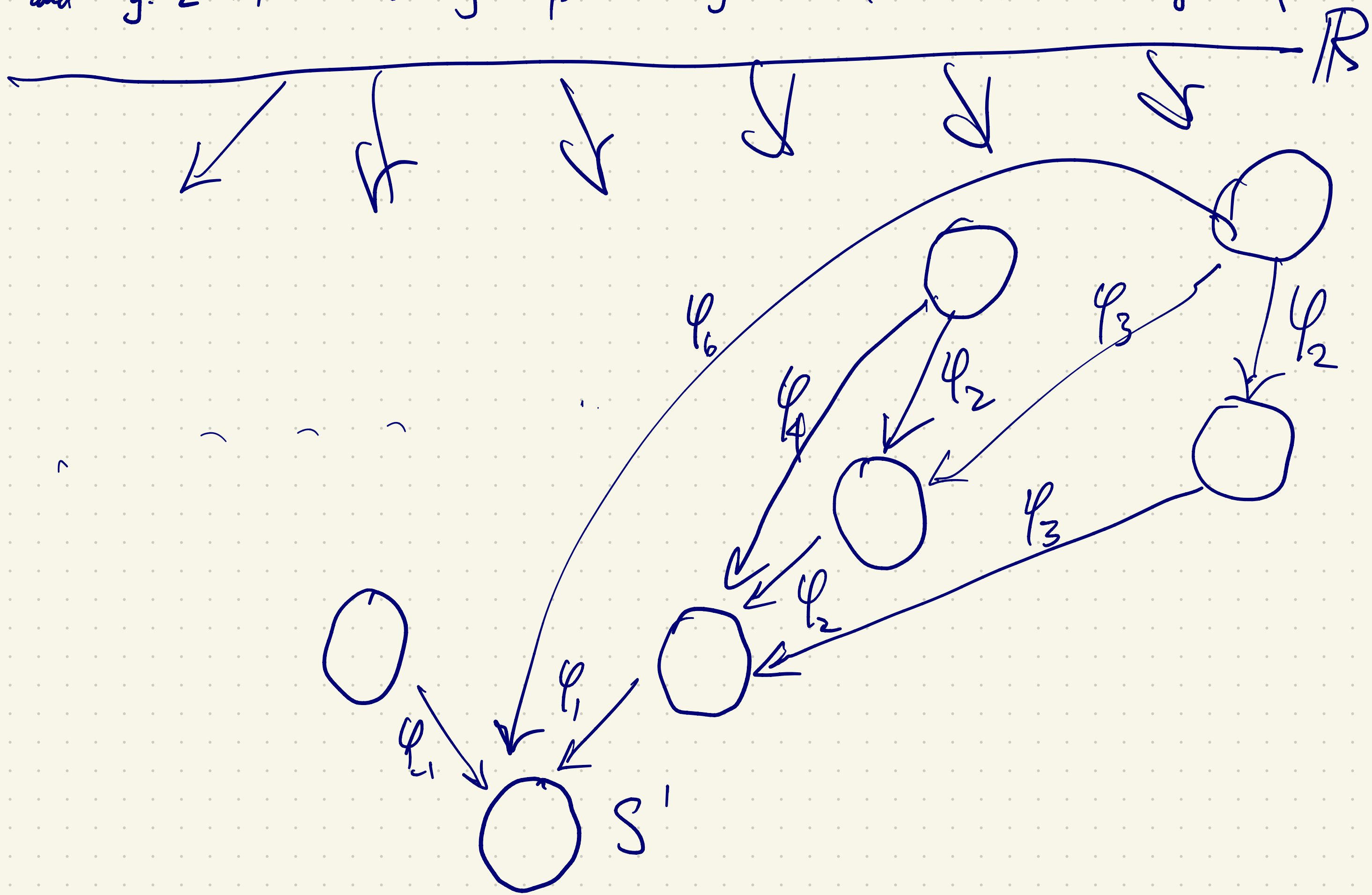


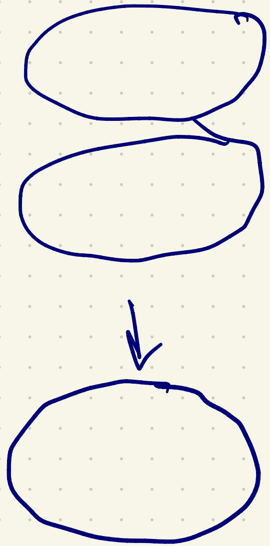
$f^{-1}(1) = \{1, -1\}$
$f^{-1}(-1) = \{i, -i\}$
$f^{-1}(i) = \{\frac{1+i}{\sqrt{2}}, -\frac{1+i}{\sqrt{2}}\}$
etc.

All fibres have size 2: this is a 2-to-1 covering map (double cover or map of degree 2 or a 2-sheeted cover)

$\varphi_\infty : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a cover of infinite degree (∞-to-1)

If $f: Y \to X$ and $g: Z \to Y$ are covering maps then $f \circ g : Z \to X$ is also a covering map.





$S^1 \sqcup S^1 = S^1 \times \{0,1\}$

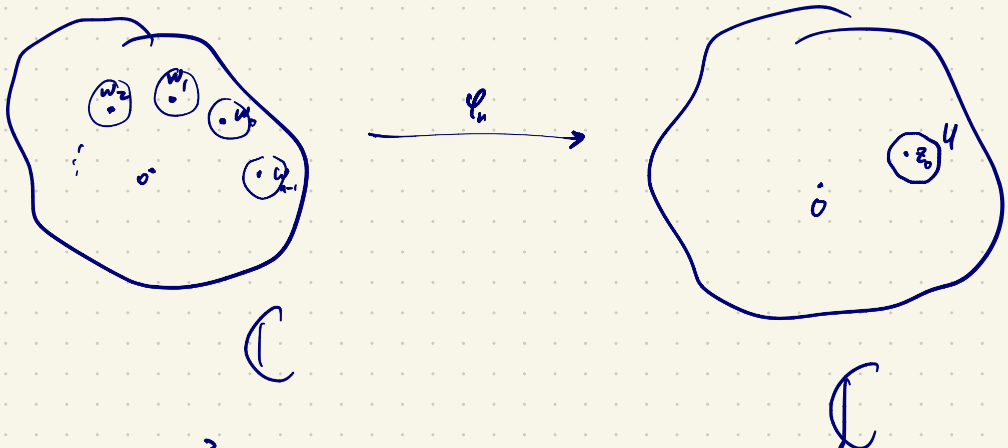
trivial double cover (disconnected)

S^1

A covering map $Y \xrightarrow{p} X$ is sometimes described as "Y is a cover of X". But really it's $Y \xrightarrow{p} X$ that we mean to be the cover.

The map $\varphi_n : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$ is not quite a covering map for $n > 1$; instead it is a branched cover.

For $z_0 \in \mathbb{C}$, $z_0 \neq 0$, φ_n covers nbhds of z_0 (n-fold cover).



$$\varphi_n^{-1}(z_0) = \{w_0, w_1, w_2, \ldots, w_{n-1}\}$$

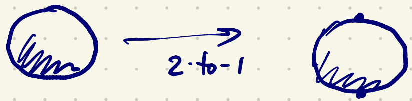
φ_n is a covering map $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$. ($n$-fold cover)

$$\cong S^1 \times \mathbb{R}$$
$$\cong S^1 \times (0,1)$$

More generally an algebraic curve of degree n (zero set of $f(x,y) \in \mathbb{C}[x,y]$ of degree n or projectively, zero set of a homogeneous poly. of degree n $f(x,y,z) \in \mathbb{C}[x,y,z]$) defines an n-fold cover of the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1\mathbb{C} \cong S^2$ (Riemann surface of f)

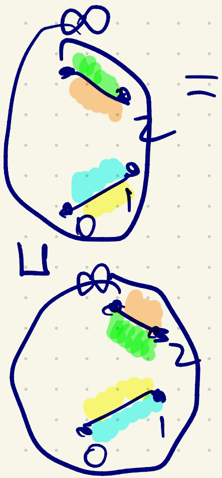
Eg. the curve $y - x^2 = 0$ has Riemann surface $\{(x,y) \in \mathbb{C}^2 : y = x^2\}$ is topologically $\cong S^2$ after compactification.

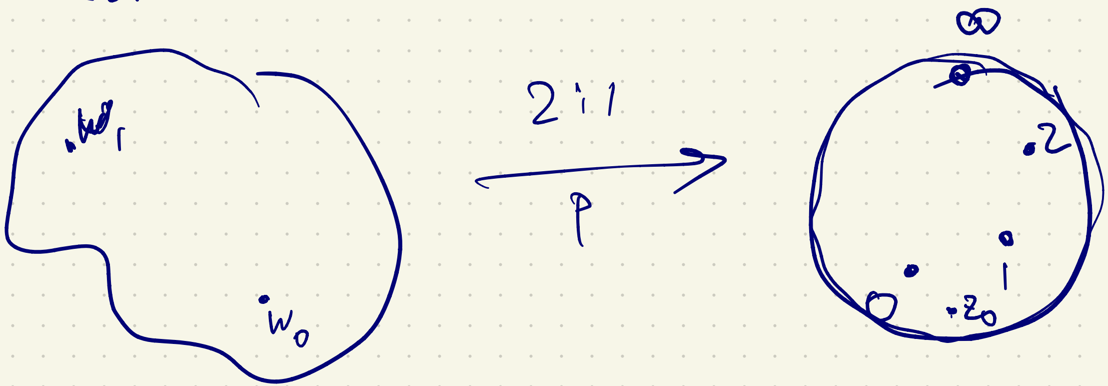
$(x, x^2) \to x^2$ is 2-to-1. Branched double cover with branch points $0, \infty \in \mathbb{C} \cup \{\infty\}$.



Eg. $y^2 = x(x-1)(x-2)$ is an elliptic curve. It defines a double cover $(x,y) \xrightarrow{p} x$.
This is branched at $0, 1, 2, \infty$.

The Riemann surface of this curve

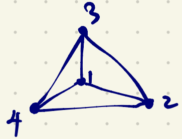




$p^{-1}(z_0) = \{w_0, w_1\}$

$\mathbb{C} \cup \{\infty\}$

When are two covering maps of X equivalent? Say $Y \xrightarrow{p} X$, $Y' \xrightarrow{p'} X$ are covering maps.

eg. $X =$ 

Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \dots, \{3,4\}\}$.

X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.