Math 5605

Algebraic Topology

Book 3

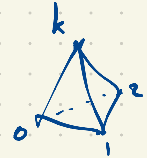
Cup product for simplicial cohomology $\quad H^k \times H^l \xrightarrow{\cup} H^{k+l}$
makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains : $(k = 0, 1, 2, 3, \dots)$ ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex $\{$all subsets of $\{0,1,2,\dots,k\}\}$.

This has a geometric realization



$\xrightarrow{\sigma} X$

$\Delta = \Delta^n = \{\underbrace{(v_0, v_1, \dots, v_n)}_{\text{barycentric coordinates}} : v_i \geq 0,\ \Sigma v_i = 1\} \subset R^{n+1}$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \longrightarrow X$.

$C_n = \{$n-chains in $X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{$n-cochains in $X\} = \mathrm{Hom}(C_n, R) = \{R$-homomorphisms $C_n \longrightarrow R\}$

$\partial : C_n \longrightarrow C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]$ $\qquad \partial^2 = 0$, $(\partial^*)^2 = 0$

$\partial^* : C^{n-1} \longrightarrow C^n$

If $\phi \in C^k$ k-cochain then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$
$\psi \in C^l$ l-cochain

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\, \psi(\sigma|[v_k, \dots, v_{k+l}])$$

$[v_0, \dots, v_{k+l}] \longmapsto \sigma(v_0, \dots, v_{k+l})$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)
making $H^*(X; R)$ into a graded ring
$$H^*(X;R) = \bigoplus_{i \geq 0} H^i(X; R).$$

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{ \text{1-dim'l subspaces of } \mathbb{R}^{n+1} \} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$ 

$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{ a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2 \}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$

Proof is by contradiction

Suppose $f: S^n \to S^{n-1}$ is antipodal. $\quad (f(-x) = -f(x))$

Then f induces a well-defined map $P^n\mathbb{R} \xrightarrow{f} P^{n-1}\mathbb{R}$
$$\pm x \mapsto \pm f(x) \qquad (x \in S^n)$$

$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

$S^n \longrightarrow S^{n-1}$

f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces $f^*: H^*(P^{n-1}\mathbb{R}; \mathbb{F}_2) \longrightarrow H^*(P^n\mathbb{R}; \mathbb{F}_2)$ mapping $x \longmapsto x$

$$H^*(P^{n-1}\mathbb{R}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n), \qquad H^*(P^n\mathbb{R}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$$

$$x^n \longmapsto x^{n+1} \quad ; \quad \text{contradiction.}$$

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A)} \oplus T(A)$ canonically, where $T(A)$ = torsion subgp of A = {elements of A of finite order}

$k = \operatorname{rank} A = \dim A$.

For any chain complex $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$ (over $\mathbb{Q}$ or $\mathbb{R}$)

we have homology groups $H_n = \ker \partial_{n-1} / \operatorname{im} \partial_n$ with well-defined rank $H_n(X;\mathbb{Z}) = \operatorname{rank} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{R})$

and Euler characteristic
$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i.$$

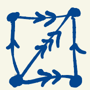
$C_n \xrightarrow{\partial_n} C_{n-1} \qquad \dim C_n = \dim \ker \partial_n + \dim \operatorname{im} \partial_n$

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1} \qquad \dim H_n = \dim \ker \partial_n - \dim \operatorname{im} \partial_{n+1}$

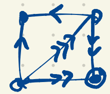
eg. $\chi(S^2) = 4 - 6 + 4 = 2$

Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

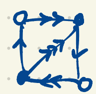
	S^2	T^2	$P^2\mathbb{R}$	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0

T^2 

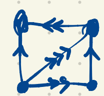
$1-3+2=0$

S^2 

$3-3+2=2$

$P^2\mathbb{R}$ 

$2-3+2=1$

K^2 Klein bottle 

$1-3+2=0$

T^2 $\quad$ T^2 $\quad$ T^2 $\quad$ T^2

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ for any two closed surfaces S_1, S_2

$$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0+0-2 = -2$$

$\underbrace{T^2 \# \cdots \# T^2}_{g} =$ 

$\chi(T^2 \# \cdots \# T^2) = 2-2g$ $\qquad$ g = genus of orientable surface

$$\chi(\underbrace{P^2\mathbb{R} \# P^2\mathbb{R}}_{K^2}) = 1+1-2 = 0$$

Exact sequences

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \qquad \ker \partial_n = \operatorname{im} \partial_{n+1}$$

$0 \to C \to 0$ is exact iff $C = 0$

$0 \to A \to B \to 0$ is exact iff $A \cong B$

$0 \to A \to B \to C \to 0$ is exact (short exact) iff $C \cong B/A$

If $f : X \to X$ is an endomorphism of an abel. gp. X (or vector space) (at least in an abelian category) Some important short exact sequences are

$$0 \to \ker f \to X \to f(X) \to 0$$

$$0 \leftarrow \operatorname{coker} f \leftarrow X \leftarrow f(X) \leftarrow 0$$

If $f : X \to Y$ then $\operatorname{coker} f = Y / \underbrace{f(X)}_{\operatorname{im} f}$

If $\cdots \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to \cdots$ are exact then we get an exact seq.

$$\to A \to B \to D \to E \to \cdots$$

$$\begin{tikzcd}
\cdots \arrow[dr, dashed] & & & & & \cdots \\
& A \arrow[dr] & & & E \arrow[ur] & \\
& & B \arrow[dr] \arrow[rr, dashed] & & D \arrow[u] & \\
& & & C \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

$$0 \to \ker f \to X \xrightarrow{f} X \to \operatorname{coker} f \to 0 \quad \text{is exact.}$$

If X is a fin. dim'l vector space over F then the Euler char. of this sequence is

$$\dim \operatorname{coker} f - \dim X + \dim X - \dim \ker f = 0$$

If $T : X \to X$ is an operator (endomorphism) (don't worry about boundedness) the index of T is $\operatorname{ind} T = \dim \operatorname{coker} T - \dim \ker T$ when both of these terms are finite (i.e. T is Fredholm).

Theorem: Let $S, T: X \to X$ be operators (lin. transf). Of the three operators S, T, ST, then whenever two are Fredholm then so is the third and in this case $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

(or abel. gps)

In general (i.e. for any lin. transf. $S, T: X \to X$ we have an exact sequence

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0$$

So its Euler characteristic is zero. i.e. $\operatorname{ind} S + \operatorname{ind} T - \operatorname{ind} ST = 0$.

__Snake Lemma__ In an abel. category we have a commutative diagram with exact rows

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

then we have a six-term exact seq.

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[dddll, bend left] & \\
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
& \operatorname{coker} a \arrow[r] \arrow[d] & \operatorname{coker} b \arrow[r] \arrow[d] & \operatorname{coker} c \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Group Cohomology : used in the study of group extensions

If G and H are groups then an extension of H by G is a group X giving an exact sequence $1 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 1$

Note: Groups are not necessarily abelian. We are asking for a new group X having a normal subgp $\cong H$ s.t. $X/H \cong G$. G on top, H on the bottom.

Trivial: $X = G \times H$. (split extension)

G is now an arbitrary group and A is an abelian group (G multiplicative ; A additive notation) on which G acts (each $g \in G$ gives $g \in GL(A)$ (automorphisms of A as an abel. gp or $\mathbb{Z}$-module).

$(g_1 g_2)(a) = g_1(g_2(a))$; $g(a+b) = ga + gb$; $\underset{\hat{G}}{1} a = a$. $G \xrightarrow[\text{(fixed)}]{\text{homo.}} \mathrm{Aut}\, A = GL(A)$

We construct an extension of A by G i.e. an exact sequence of gps i.e. $\hat{G}$ is a gp with normal subgp. iso. to A with $\hat{G}/A \cong G$

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \hat{G} \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \arrow[d] \\
1 \arrow[r] & A \arrow[r] & \hat{\hat{G}} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Two extensions $\hat{G}$ $\hat{\hat{G}}$ are equivalent if we have a commutative diagram as shown with α, β, γ isomorphisms of groups (with exact rows). Note that the action of G on A is fixed throughout.

Cohomology of groups is the tool for this.

$\cdots \xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} 0$ is an exact sequence of additive abel. gps where $C^k = C^k(G;A)$ is the set of all maps $\underbrace{G^k}_{G\times G\times\cdots\times G} \to A$ as an additive abel. gp. i.e. $\mathbb{Z}$-module

$G\times G\times\cdots\times G$ i.e. k-tuples of G

$C^0 = A$ (maps $\{1\} \to A$)
$C^1 = A^G$ = maps $G \to A$ i.e. $f: G \to A$
$C^2 = A^{G\times G}$ = maps $G\times G \to A$ etc.

Given $a \in C^0$ i.e. $a \in A$, $\delta a \in C^1$ is $\delta a: G \to A$
$g \mapsto ga - a$

Given $f \in C^1$ i.e. $f: G \to A$
Construct $\delta f \in C^2$ i.e. $(\delta f): G\times G \to A$

$$(\delta f)(g,h) = g\underbrace{f(h)}_{\in A} - \underbrace{f(gh)}_{\in A} + \underbrace{f(g)}_{\in A} \in A.$$

Given $f \in C^2$ i.e. $f: G\times G \to A$
Construct $(\delta f): G\times G\times G \to A$

$$(\delta f)(g,h,k) = gf(h,k) - f(gh,k) + f(g,hk) - f(g,h)$$

See p. 2 bottom of handout for $\delta: C^k \to C^{k+1}$ in general

Check: $C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0$ $\delta^2 = 0$?

Take $a \in C^0 = A$.

$(\delta a): G \to A$
$(\delta a)(g) = ga - a$.

$\delta^2 a: G\times G \to A$

$$(\delta^2 a)(f,g) = f(\delta a)(g) - (\delta a)(fg) + \delta a(f)$$
$$= f(ga - a) - ((fg)(a) - a) + (fa - a)$$
$$= \cancel{fga} - \cancel{fa} - \cancel{fga} + \cancel{a} + \cancel{fa} - \cancel{a}$$
$$= 0$$

Classify extensions $1 \to A \to \hat{G} \to G \to 1$ where G is a group acting on an abelian gp A
i.e. $\hat{G}$ is a group with normal subgp A with $\hat{G}/A \cong G$, using cohomology. Start with a split extension
i.e. A has a complementary subgp in $\hat{G}$. So $\hat{G}$ acts on the subgps complementary to A by conjugation.
$H^1(G;A)$ classifies the complementary subgps up to conjugacy.

Fix an action of G on A.
$\underset{\text{mult.}}{G}$ $\underset{\text{additively}}{A}$

$a(g_1 g_2) = (ag_1)g_2$

$a1 = a$ (1 = id. of G)

$(a+a')g = ag + a'g$ for $a, a' \in A$; $1, g, g_1, g_2 \in G$.

here A is a right G-module.

$\hat{G}$ is isomorphic to the semidirect product $A \rtimes G = \{(a,g) : a \in A, g \in A\}$

$G \ltimes A$ for left action

$(a_1, g_1)(a_2, g_2) = (a_1 g_2 + a_2, g_1 g_2)$ identity $(\underset{\text{in } A}{0}, \underset{\text{in } G}{1})$

Alternative notation: $A \rtimes G = \left\{ \begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} : a \in A, g \in G \right\}$

$$\begin{bmatrix} g_1 & 0 \\ a_1 & 1 \end{bmatrix}\begin{bmatrix} g_2 & 0 \\ a_2 & 1 \end{bmatrix} = \begin{bmatrix} g_1 g_2 & 0 \\ a_1 g_2 + a_2 & 1 \end{bmatrix}$$

Complements of A in $\hat{G}$ are given by 1-cocycles.

$$C^2 \xleftarrow{\delta^1} C^1 \xleftarrow{\delta^0} \underset{A}{\underset{\shortparallel}{C^0}} \longleftarrow 0$$

For $a \in \overset{C^0}{\overset{\shortparallel}{A}}$, $(\delta a)(g) = ag - a$

How do we construct a subgp of $A \rtimes G$ complementary to A?
Any such subgp $H \leq A \rtimes G$ has the form $\{(t_g, g) : g \in G\}$. $= \left\{ \begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix} : g \in G \right\}$
Here $g \overset{f}{\longmapsto} t_g$, $G \longrightarrow A$. This will automatically be a complement to A as long as it is a subgp. eg. $t_1 = 0$ but most importantly, closure.

$$\begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix}\begin{bmatrix} g' & 0 \\ t_{g'} & 1 \end{bmatrix} = \begin{bmatrix} gg' & 0 \\ t_{gg'} & 1 \end{bmatrix}$$

i.e. $\underset{A}{\underbrace{t_{gg'}}} = \underset{A}{\underbrace{t_g}}\,\underset{G}{\underbrace{g'}} + \underset{A}{\underbrace{t_{g'}}}$ so $(\delta f)(g, g') = -f(gg') + f(g)g' + f(g') = 0$.

When are two complements of A conjugate in $\hat{G}$?

If A has complementary subgps $H_1, H_2 \leq \hat{G}$ given by $H_i = \{(f_i(g), g) : g \in G\}$, $f_i \in Z^1(G; A)$

$(\delta f_i)(g, g') = f_i(g') - f_i(gg') + f_i(g)g' = 0$

when are $H_1 H_2$ conjugate in $\hat{G}$?

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}^{-1} = \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix}$$

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use $\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \in \hat{G}$ (a, g fixed) to conjugate H_1:

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ f_1(x) & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gx & 0 \\ ax + f_1(x) & 1 \end{bmatrix} \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ axg^{-1} + f_1(x)g^{-1} - ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ f_2(gxg^{-1}) & 1 \end{bmatrix}$$

The 1-cycle defining this conjugate subgp would have to be f_2: so

$$f_2(gxg^{-1}) = axg^{-1} + f_1(x)g^{-1} - ag^{-1}.$$

$f_2(gxg^{-1}) = f_2(g(xg^{-1}))$

$= f_2(g)xg^{-1} + f_2(xg^{-1})$

$= f_2(g)xg^{-1} + f_2(x)g^{-1} + f_2(g^{-1})$

$= f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1}$

$f \in C^1$ ie. $f : G \to A$

$f(xy) = f(x)y + f(y)$

$f(1) = f(1 \cdot 1) = f(1)1 + f(1)$
$\Rightarrow f(1) = 0$

$0 = f(1) = f(gg^{-1}) = f(g)g^{-1} + f(g^{-1})$
$\Rightarrow f(g^{-1}) = -f(g)g^{-1}$