Math 5605

Algebraic Topology

Book 1

If X, Y are top. spaces, $f: X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a homeomorphism if f is bijective and f, f^{-1} are continuous.

$X \cong Y$ are homeomorphic if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

$\mathbb{R}^2 \not\cong S^2$ Since S^2 is compact; $\mathbb{R}^2$ is not.
↳ 2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} = $

$S^2 \not\cong T^2 = S^1 \times S^1 = $ $ = $ S^2, T^2 are compact surfaces. They are locally homeomorphic
↳ S^1 but not globally homeomorphic.

$T^1 = S^1 = $ $ = $ circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not.
In S^2, every closed path can be "continuously shrunk" to a point
(homotopic to a point, i.e. null-homotopic)

X $T = T^2$
 $\not\cong$ although both surfaces are compact, connected, not simply connected

These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of homotopy equivalence.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$.

Given points $a, b \in X$ (a topological space), a ~~path~~ path from a to b is a function (continuous) $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.
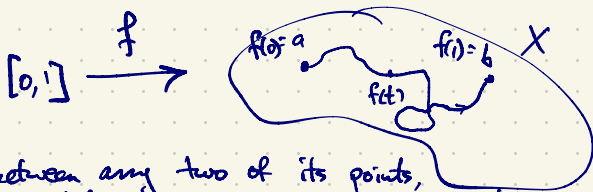
All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:

$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$
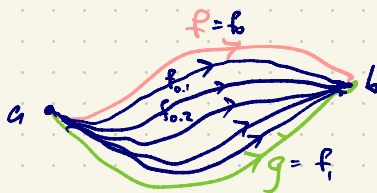
$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$
$(s,t) \mapsto f(s,t) = f_s(t)$
such that $f_0 = f$ ie. $f_0(t) = f(t)$
$f_1 = g$ ie. $f_1(t) = g(t)$
$f_s(0) = a$ for all
$f_s(1) = b$ $s \in [0,1]$

This is a homotopy from f to g.

f, g are homotopic but h is not homotopic to f, g

We say f, g are homotopic if there is a continuous family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi: [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f \circ \varphi: [0,1] \to X$ is homotopic to f. A homotopy from f to $f \circ \varphi$ is

$$[0,1]^2 \to X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{[0,1]}) = f_s(t)$$

$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
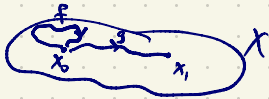$$f_s(0) = f((1-s)\cdot 0 + s\cdot\varphi(0)) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot\varphi(1)) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)

Fix g path in X from x_0 to x_1

An isomorphism $\phi: \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$
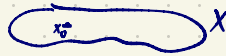
$$f \xrightarrow{\phi} \bar{g} f g$$
$$g f \bar{g} \xleftarrow{\phi^{-1}} h$$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f \gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is
$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$
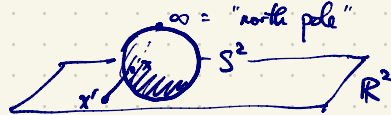(same path in the reverse direction)

$$f\bar{f} = \bar{f}f = \gamma = \text{null path}$$

$$\phi(f_1 f_2) = \bar{g}\, f_1 f_2\, g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$
$$f_1, f_2 \in \pi_1(X, x_0)$$

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

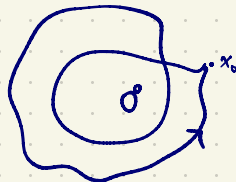$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)

∞ = "north pole"

$x \longmapsto x'$ is stereographic projection from the north pole ∞

See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 \smallsetminus \{0\}) \cong \mathbb{Z}$
   punctured plane

follows from the fact that

$\mathbb{R}^2 \smallsetminus \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 \smallsetminus \{0\} \simeq S^1$

$\mathbb{R}^3 \smallsetminus \{0\} \simeq S^2$

- retraction
- deformation retraction
- strong deformation retraction
- homotopy
- relative homotopy
- homotopy equivalence

"def. retraction in the weak sense"

"def. retraction"

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).

Hatcher writes $X \approx Y$ for homeomorphic.

Let $A \subseteq X$ (subspace of a top. space).

A <u>retraction</u> $f: X \longrightarrow A$ is a map (continuous) such that $f|_A = id_A = \mathbb{1}_A$ ie. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a <u>retract</u> of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \longrightarrow \{a\}$, $x \longmapsto a$ is a retraction.

$\mathbb{R}^2 \longrightarrow x\text{-axis}$ $(x, y) \longmapsto (x, 0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \longrightarrow S^1$.
(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to $\underset{X \to A}{\underset{\text{retraction}}{\hat{a}}}$), $A \subseteq X$.

ie. $f: [0,1] \times X \longrightarrow X$

$$f(t,x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f: [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$
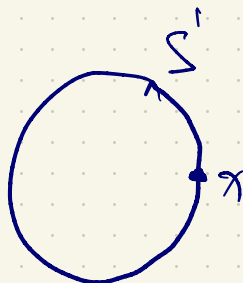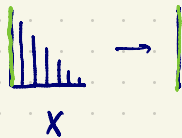
$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.

Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f: [0,1] \times X \to X$:

$f_0(x) = x$ i.e. $f_0 = id_X$

f_1 is a retraction $X \to A$

$f_t|_A = id_A$ for all $t \in [0,1]$

S^1

x

X

def. retract
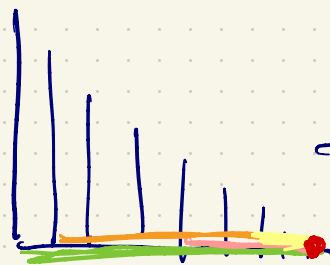but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $X = \left([0,1] \times \{0\}\right) \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} \left(\{r\} \times [0, 1-r]\right)$ X has a def. ret. to $A := \{0\} \times [0,1]$.

This is a def. retract but not a strong def. retract.

There is no strong def. retract $X \to A$.

$0 \leq t \leq \frac{1}{2}$ $\frac{1}{2} \leq t \leq 1$

Let $f_0, f_1 : X \to Y$ be maps. A __homotopy__ from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0,x) = f_0(x)$
$f(1,x) = f_1(x)$
$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 $(f_0, f_1 : X \to Y)$ is a homotopy $f : [0,1] \times X \to Y$ such that $f_t(a)$ is independent of $t \in [0,1]$ for all $a \in A$.
(constant)

A path from a to b in X is a homotopy from a to b.
Given two paths f_0, f_1 in X from a to b $(a, b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{}{\overset{f}{\rightleftarrows}} Y$
such that $f \circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$

$f(x) = 0$ for all $x \in \mathbb{R}^n$
$g(0) = 0 \in \mathbb{R}^n$

$g \circ f : \mathbb{R}^n \to \mathbb{R}^n$
$\quad\quad x \longmapsto 0$
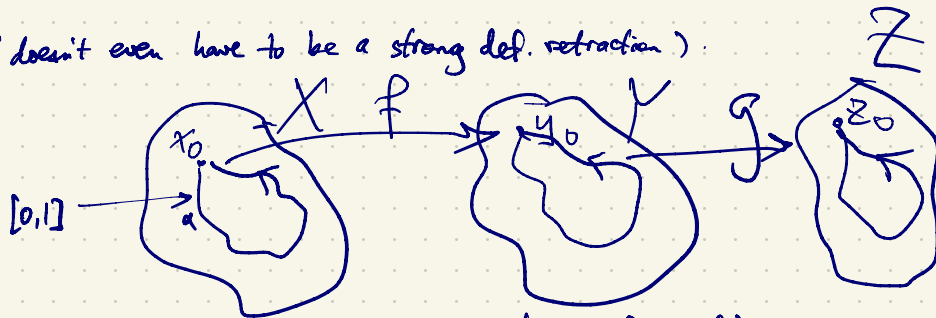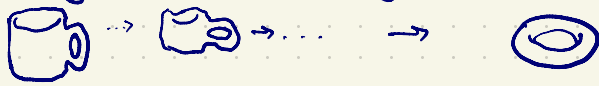
$f \circ g : \{0\} \to \{0\}$
$\quad\quad 0 \longmapsto 0$

A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ to $id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$
$\quad\quad\quad\quad\quad\quad\quad\quad x \longmapsto 0 \quad\quad\quad x \longmapsto x$

↳ Not relative
to any subspace
necessarily.

$h_0 = g \circ f$
$h_1 = id_{\mathbb{R}^n}$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).

S^1 is not homotopic to a point.
(S^1 is not null homotopic; not contractible)

If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

$\quad\quad \alpha : [0,1] \to X$ gives $f_* \alpha = f \circ \alpha : [0,1] \to Y$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (g \circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

D^n = closed ball in $\mathbb{R}^n$ $\dot{D}^0$ $\overline{D'}$

        "disk" D^2

$S^n = \partial D^{n+1}$ = n-sphere

A CW complex is formed $\dot{S}^0$

from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$ $= D^0 \sqcup D^0$ S'

   X^n is a union of copies of D^n with the boundaries of D^n_α attached to X^{n-1} via attaching maps.

        D^n_α

Eg. Torus $T^2 = S' \times S' = $ $=$ $=$

$X^0 = \; \cdot \; = D^0$

$X' = $ $\; \cong S' \vee S'$
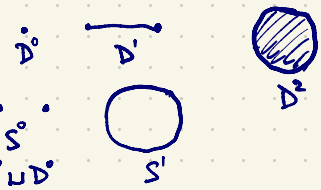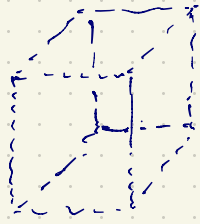
          D'_0 ω D'_1

X^2
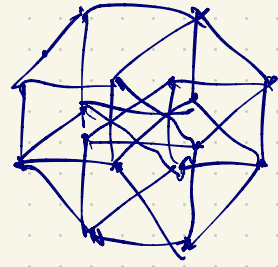
$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$
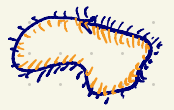
Möbius strip not homeomorphic to cylinder $S^1 \times [0,1] = $ $=$

$X^0 =$ $\cdot$ $\cdot$ Cylinder: $X^0 =$ $\cdot$ $\cdot$ $D^1 \sqcup D^0$

$X^1 = $ A   B   B   A $X^1 = $ orientable

$X^2 = $ A   B $X^2 = $ (def. retract to S^1)
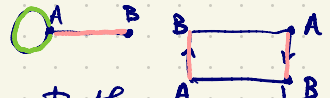
not orientable   Both are homotopy equivalent to S^1

Both have $\mathbb{Z}$ as fund. gp.
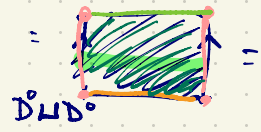
$P^2\mathbb{R}$ (or $\mathbb{R}P^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified

(non-orientable surface)

$P^2\mathbb{R} = D^2$ glued to a Möbius strip

D^2
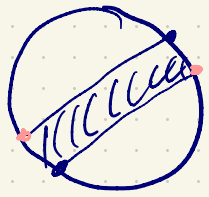
$\pi_1(\mathbb{P}\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ a is not homotopic to the null path γ. a^2 is homotopic to γ
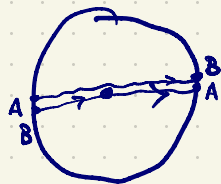
a^2
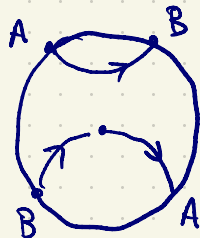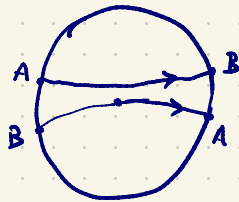
$\mathbb{R}^2 - \{0\} \simeq S^1$
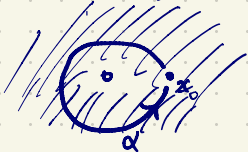
A homotopy equivalence

$\mathbb{R}^2 - \{0\} \xrightarrow{f_1} S^1$

$f_1((x,y)) = \left(\dfrac{x}{\sqrt{x^2+y^2}} , \dfrac{y}{\sqrt{x^2+y^2}} \right)$

$f_t(v) = (1-t)v + t\dfrac{v}{|v|}$ $f_1(v) = \dfrac{v}{|v|}$
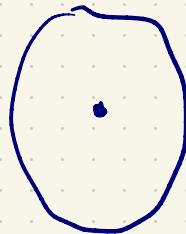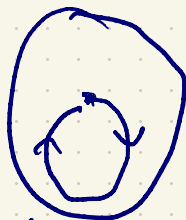
$f_t : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$

$f_0 = id_{\mathbb{R}^2 - \{0\}}$

f_1 is a retraction to S^1

strong def. retraction since

$f_t |_{S^1} = id_{S^1}$ for all $t \in [0,1]$.

$\{\alpha^n : n \in \mathbb{Z}\}$
$\parallel$
$\langle \alpha \rangle = \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$ free group on one generator

$$\pi_1(S^1) \cong \mathbb{Z}$$

Given a closed path $\overset{\alpha}{\beta}$ in S^1 with base point $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$

define $w(\beta) = \dfrac{1}{2\pi i} \displaystyle\int_\beta \dfrac{dz}{z} = \dfrac{1}{2\pi i} \displaystyle\int_{\beta(0)}^{\beta(1)} \dfrac{dz}{z}$

$\beta : [0,1] \longrightarrow S^1$

$\beta(0) = 1$

$\beta(e^{2\pi i \theta}) = \theta + n$

$w(\alpha) = 1$

$w(\alpha^n) = n$

w is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$.

$$\int \frac{dz}{z} = \ln|z| + 2\pi i \, \text{Arg } z$$

In $\mathbb{C} - \{0\}$ the same argument works

$\mathbb{R}^2 - \{0\}$ $O = (0,0)$

$\pi_1(\mathbb{R}^2 - \{0\}) = \langle \alpha \rangle \cong \mathbb{Z}$

k-punctured plane $X = \mathbb{R}^2 - \{A_1, \ldots, A_k\}$

distinct

$\pi_1(X) = F_k = \text{Free}(\{\alpha_1, \ldots, \alpha_k\})$

$X \simeq S^1 \vee S^1 \vee \cdots \vee S^1$ (k)

$X = \mathbb{R}^2 - \{A, B\}$

$\pi_1(X) = \{\alpha^{i_1} \beta^{j_0} \alpha^{i_1} \beta^{j_1} \cdots \alpha^{i_k} \beta^{j_k},$

$\alpha^{i_0} \beta^{j_0} \cdots \alpha^{i_{k-1}} \beta^{j_{k-1}} \alpha^{i_k},$

$\beta^{j_0} \alpha^{i_1} \beta^{j_1} \cdots \alpha^{i_k} \beta^{j_k},$

$\beta^{j_0} \alpha^{i_1} \beta^{j_1} \cdots \alpha^{i_k} \beta^{j_k} \alpha^{i_k} \}$

$: i_i, j_i \in \mathbb{Z} - \{0\}$

$k \geqslant 0 \}$

$\pi_1(X)$ is the free group on two generators i.e.

$F_2 = \text{Free}(\{\alpha, \beta\}) = \langle \alpha \rangle * \langle \beta \rangle = \mathbb{Z} * \mathbb{Z}$

α A_0 B β x_0

A_0 B $\alpha\beta$ x_0

A_0 B $\beta\alpha$ x_0

The Van Kampen Theorem gives a presentation for $\pi_1(X)$ when X is suitably described in terms of smaller pieces.

A presentation for a group G expresses G as a homomorphic image of a free group F i.e. $G \cong F/N$, $N \triangleleft F$.

Let X be a set of generators of G ($X \subseteq G$, $\langle X \rangle = G$).

$\text{free}(X) \longrightarrow G$ is a surjective homomorphism; N is its kernel.

$G = \langle x_1, \dots, x_k : \overbrace{r_1, \dots, r_m}^{\in F} \rangle$ is a presentation for G if $X = \{x_1, \dots, x_k\}$ is a set of k symbols, $F = \text{free}(X)$ (the free group on $x_1, \dots, x_k$).

Let N be the smallest normal subgp of F containing $r_1, \dots r_m$ i.e. the normal closure of $\langle r_1, \dots, r_m \rangle \leq F$

ie. the subgp. of F generated by $r_1, \dots, r_m$ and their conjugates in F

$N = \langle h r_i h^{-1} : i = 1, \dots, m; h \in F \rangle$. (When there are k generators and m relators, we say G is finitely presented.)

eg. the dihedral group of order 10 is $D_{10} \cong \langle a, b : a^2, b^2, (ab)^5 \rangle \cong \langle x, y : x^2, y^5, xyxy \rangle$

$xyxy = 1$
$\Leftrightarrow xy = y^{-1}x$
$\Leftrightarrow xyx^{-1} = y^{-1}$
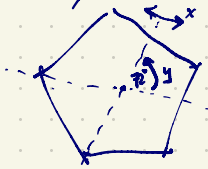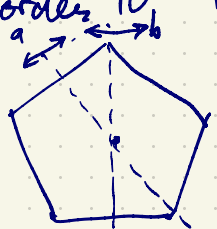
$D_{10} \stackrel{\sim}{=} \langle a,b : a^2, b^2, (ab)^5 \rangle \stackrel{\sim}{=} \langle x,y : x^2, y^5, xyx^{-1}y \rangle$

$x^2 = a^2 = 1$

$y^5 = (ab)^5 = 1$

$$\phi$$

$\longrightarrow$

$x = a$

$y = ab$

$xyx^{-1} = a \cdot ab \cdot a^{-1} = ba$

whereas $y^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba$

$\longleftarrow$

check.

A free product $G * H$ is the set of all words from elements in G and H having no relations between elements in G and elements in H (except if one of these elements is the identity).
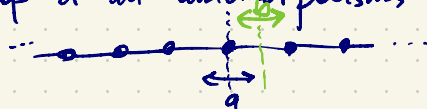
$\mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle = \text{free} \{a,b\} = \langle a,b \rangle = \{ a^{i_1} b^{j_1} a^{i_2} b^{j_2} \dots a^{i_k} b^{j_k}, \dots \}$

infinite cyclic   infinite cyclic

$= \langle a,b : a^2, b^2 \rangle$

$\langle a : a^2 \rangle * \langle b : b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \dots \} = D_\infty$

$\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle x,y : x^2, xyx^{-1} = y^{-1} \rangle = $ group of all automorphisms of

$xyx^{-1}y = 1$

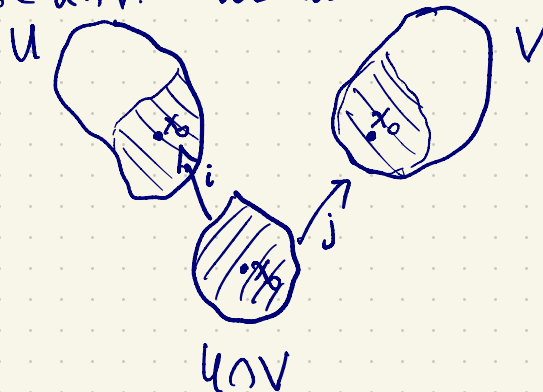ie. $\langle X : R \rangle = \langle x_1, \dots, x_m : r_1, \dots, r_k \rangle$

$\langle Y : S \rangle = \langle y_1, \dots, y_n : s_1, \dots, s_\ell \rangle$

$\langle X : R \rangle * \langle Y : S \rangle = \langle X \cup Y : R \cup S \rangle = \langle x_1, \dots, x_m, y_1, \dots, y_n : r_1, \dots, r_k, s_1, \dots, s_\ell \rangle$

Free products with amalgamation : add more relations involving x_i's and y_j's e.g.

$$D_{10} = \langle a, b : a^2, b^2, (ab)^5 \rangle = \underbrace{\langle a : a^2 \rangle}_{\text{cyclic of order 2}} \underset{(ab)^5}{*} \underbrace{\langle b : b^2 \rangle}_{\text{cyclic of order 2}} = D_\infty / \text{Normal closure of } \langle (ab)^5 \rangle$$

Let X be a path-connected top. space covered by two open sets U, V. Since X is connected, $U \cap V \neq \emptyset$. Pick $x_0 \in U \cap V$. We also assume $U \cap V$ is path-connected.

X:

i, j inclusion maps as shown (injective, continuous)

Theorem (Van Kampen, special case)

$$\pi_1(X) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$$

where the amalgamation over $\pi_1(U \cap V)$ is given by:
for all $\alpha \in \pi_1(U \cap V)$, identify $i_*(\alpha)$ with $j_*(\alpha)$ ie
$i_*(\alpha)^{-1} j_*(\alpha)$ is a new relator.
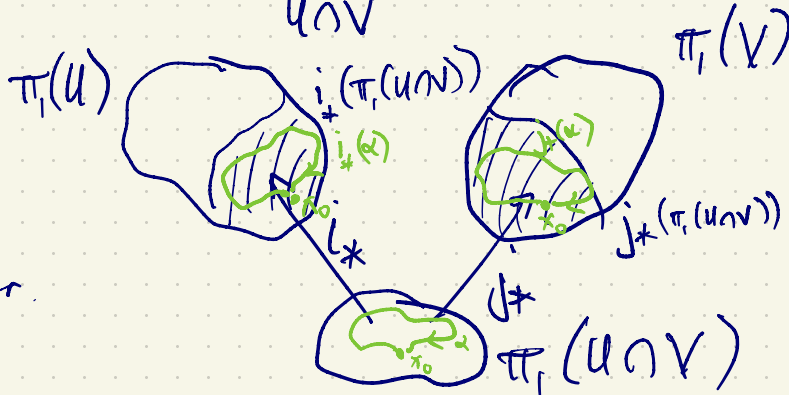
This induces group homomorphisms i_*, j_* as shown.

Eg. $X = S^2 = D \cup D'$

X

$\pi_1(S^2) = \pi_1(D) \underset{\square}{*} \pi_1(D') = 1$

$= \boxed{\pi_1(D) * \pi_1(D')} \cdot = 1.$ ← trivial

amalgamation

$D \cap D' \simeq S'$

$\pi_1(D) \cong \pi_1(D') = 1$ trivial

$\pi_1(D \cap D') = \langle \alpha \rangle$

$\pi_1(D) \quad \pi_1(D')$

i_* j_*
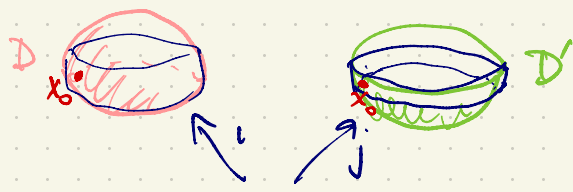
$\pi_1(D \cap D')$

i_* , j_*

not one-to-one

$i_*(\alpha) = 1$

$j_*(\alpha) = 1.$

$T^2 = S^1 \times S^1 =$ $=$

$T^2 =$

U

$\pi_1(U) = 1$

$V \simeq S^1 \vee S^1$

$\pi_1(V) = \langle a, b \rangle$
free group on two generators

$\pi_1(U) * \pi_1(V)$

$= 1 * \langle a, b \rangle$

$= \langle a, b \rangle$

$U \cap V \cong S^1 \times (-\varepsilon, \varepsilon)$

$\pi_1(U \cap V) = \langle \alpha \rangle$

Then identify α with 1
due to the inclusion $U \cap V \subset U$. In V, α is homotopic to $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$

Then identify $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$ with α.

So we identify $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$ with 1.

So identify $\alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0$ with 1.

identify $\alpha_1 \alpha_0^{-1}$ with $\alpha_0^{-1} \alpha_1$; $\alpha_0 \alpha_1$ with $\alpha_1 \alpha_0$.

So identify ab with ba.

$\pi_1(T^2) = \langle a, b \rangle / {}^{\text{"}}ab = ba{}^{\text{"}} \cong \mathbb{Z} \times \mathbb{Z}$

$= \langle a, b \rangle / \text{Normal closure of } aba^{-1}b^{-1}$

$=$ the abelianization of $\langle a, b \rangle$
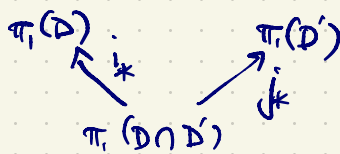
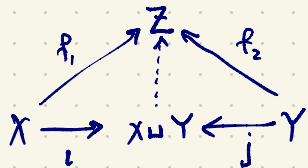$\pi_1(S' \vee S') = \pi_1(S') * \pi_1(S')$ also follows from Van Kampen's Theorem.

If X and Y are path-connected top. spaces then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.
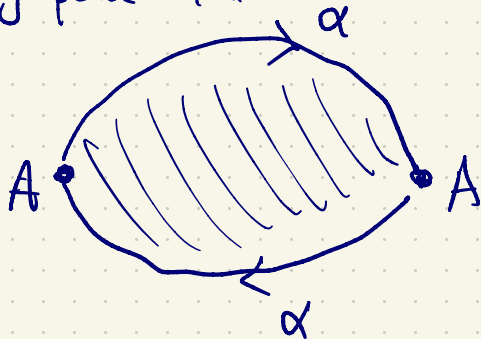

$U \cong V \cong S'$

$\pi_1(U \cap V) = 1$
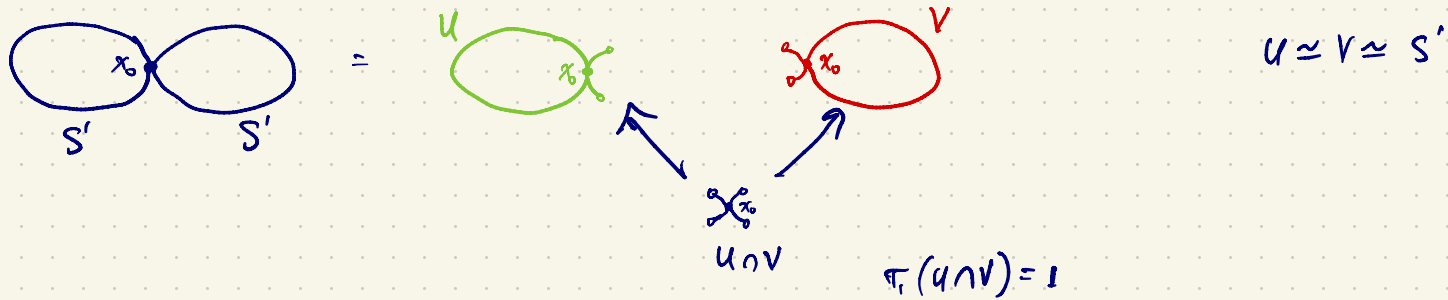
$\pi_1(S_1 \vee S_1) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z} = $ free group on two generators.

In T_p, the category of top. spaces, the coproduct of X and Y is $X \sqcup Y = $ disjoint union; product is $X \times Y$.

The category of pointed top. spaces i.e. pairs (X, x_0), $x_0 \in X$, X nonempty top. space
The coproduct $(X, x_0) \sqcup (Y, y_0) = $ disjoint union $X \sqcup Y$ with x_0, y_0 identified
    denoted as $(X, x_0) \vee (Y, y_0) = (X \sqcup Y, x_0 = y_0)$
A morphism $(X, x_0) \longrightarrow (Y, y_0)$ is a (continuous) map $X \to Y$ satisfying $x_0 \longmapsto y_0$.
We have a functor $\{$ pointed top. spaces $\} \longrightarrow Grp = \{$ groups $\}$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$$

$g \circ f$ (over the diagram)

$F \downarrow \qquad \downarrow F \qquad \downarrow$

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(Z, z_0)$$

$$(g \circ f)_* = g_* \circ f_*$$

$$(X, x_0) \vee (Y, y_0)$$

$$\downarrow F$$

$$F((X, x_0) \vee (Y, y_0)) = F((X, x_0)) * F((Y, y_0))$$

coproduct in pointed top. spaces is wedge sum.
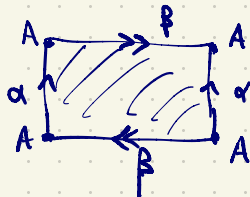
coproduct in Grp is just free product

Proj. plane $P^2\mathbb{R}$

$$\pi_1(P^2\mathbb{R}) = \langle \alpha : \alpha^2 \rangle = \langle\alpha\rangle/\langle\alpha^2\rangle = \text{group of order 2}$$
$$= \mathbb{Z}/2\mathbb{Z}$$

Klein Bottle K

$$\pi_1(K) = \langle \alpha, \beta : \alpha\beta\alpha^{-1}\beta \rangle = \langle\alpha,\beta\rangle/N$$

where N is the subgp. of $\langle\alpha,\beta\rangle$ generated by $\alpha\beta\alpha^{-1}\beta$ and its conjugates

Force $\alpha\beta\alpha^{-1}\beta = 1$ in the quotient gp.

ie. $\alpha\beta\alpha^{-1} = \beta^{-1}$

So $\pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

$\langle\beta\rangle \qquad \langle\alpha\rangle$ where α inverts β.

$(k,l)(r,s) = \begin{cases} (k+r, l+s) & \text{if } r \text{ is even} \\ (k+r, -l+s) & \text{if } r \text{ is odd} \end{cases}$

Every element can be uniquely expressed as $\alpha^k\beta^l$, $\quad k, l \in \mathbb{Z}$

$$(\alpha^k\beta^l)(\alpha^r\beta^s) = \begin{cases} \alpha^{r+k}\beta^{l+s} & \text{if } r \text{ is even;} \\ \alpha^{r+k}\beta^{-l+s} & \text{if } r \text{ is odd.} \end{cases}$$

$\alpha\beta^3\alpha^{-1} = \alpha\beta\beta\beta\alpha^{-1} = (\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1}) = \beta^{-1}\beta^{-1}\beta^{-1} = \beta^{-3}$ so $\alpha\beta^3 = \beta^{-3}\alpha$
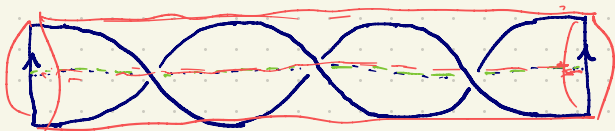
$\alpha\beta^k\alpha^{-1} = \beta^{-k}$
$\beta^k = \alpha^{-1}\beta^{-k}\alpha$

$\{ \alpha^k \beta^l : k, l \in \mathbb{Z} \} = \pi_1(K) = \mathbb{Z} \times \mathbb{Z}$

The elements with k even form an abelian subgp $\langle \alpha^2, \beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$ (free abelian of rank 2).

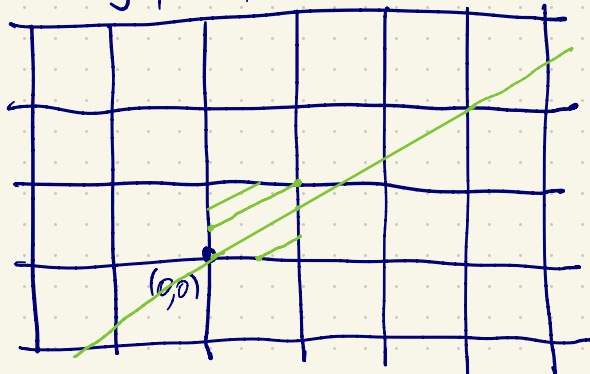and the quotient $\pi_1(K) / \langle \alpha^2, \beta \rangle$ is order 2.

Leading up to trefoil:

$T^2 \hookrightarrow \mathbb{R}^3 \, S^3$

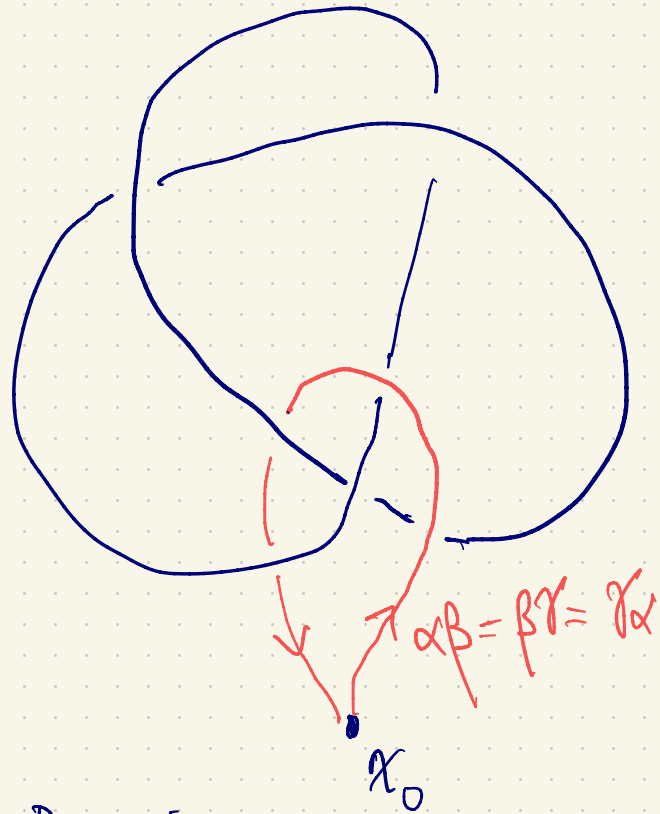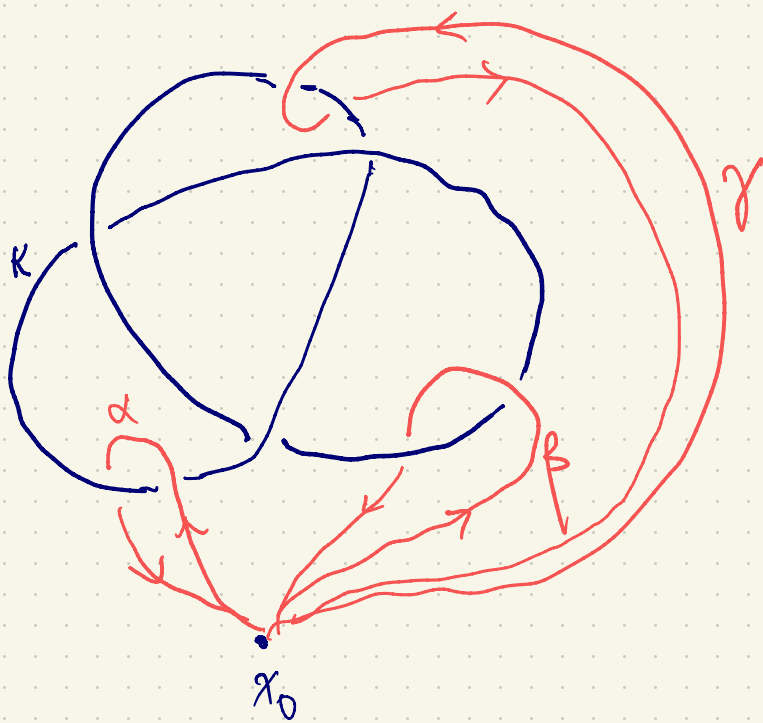This surface partitions $\mathbb{R}^3 \, S^3$ into two solid tori

for any $m, n > 1$ relatively prime, we have an (m,n) torus knot embedded in the torus

$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

Take the line $y = \frac{n}{m} x$

The trefoil is the $(3,2)$ torus knot.
or $(2,3)$

$(0,0)$

$\pi_1(\mathbb{R}^3 \smallsetminus K) = \langle \alpha, \beta, \gamma : \alpha\beta = \beta\gamma = \gamma\alpha \rangle = \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle$? Prove using Van Kampen's Theorem
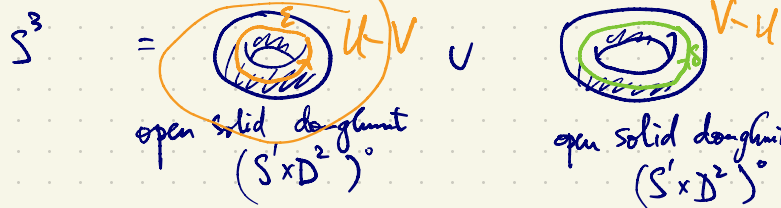
or S^3

$\gamma = \beta^{-1}\alpha\beta = \alpha\beta\alpha^{-1}$

$$S^3 \smallsetminus K = U \cup V$$

↑ trefoil knot

U, V open path-connected

$U \cap V$ .. .. ..

$$S^3 = $$ $U \cap V$ ∪ $V \smallsetminus U$

open solid doughnut $(S^1 \times D^2)^\circ$ open solid doughnut $(S^1 \times D^2)^\circ$

U open nbhd of torus

$$\pi_1(U) = \langle \varepsilon \rangle \cong \mathbb{Z}$$

$$\pi_1(V) = \langle \delta \rangle \cong \mathbb{Z}$$

$$\varepsilon^2 = \gamma = \delta^3$$

(exterior)   (interior)

$$S^3 \smallsetminus K = \begin{array}{c}\text{open solid}\\\text{doughnut}\end{array} \cup \begin{array}{c}\text{open solid}\\\text{doughnut}\end{array}$$

V open torus $-K$

 $=$ γ

$= U \cap V$

$$\pi_1(S^3 \smallsetminus K) = \underbrace{\pi_1(U) \ast \pi(V)}_{U \cup V} = \langle \varepsilon \rangle \ast \langle \delta \rangle \Big/ \begin{array}{c}\text{Normal closure}\\\text{of } \langle \varepsilon^2 \delta^3 \rangle\end{array}$$

$$= \langle \varepsilon, \delta : \varepsilon^2 = \delta^3 \rangle$$

$$\cong \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle \quad \leftarrow \text{No torsion}$$

$$\pi_1(U \cap V) = \langle \gamma \rangle \cong \mathbb{Z}$$

$$\varepsilon = \alpha\beta\alpha, \quad \delta = \alpha\beta \quad \Rightarrow \quad \delta^3 = \underbrace{\alpha\beta}_{\varepsilon}\alpha\underbrace{\beta\alpha\beta}_{\varepsilon} = \varepsilon^2$$

Where do torus knots arise "in nature"?

The (m,n)-torus knot for $m,n \geq 2$ rel. prime.

A knot is an embedding of S^1 in S^3 (or in $\mathbb{R}^3$).

The unit sphere in $\mathbb{C}^2$ is $\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \cong S^3$

Consider the algebraic variety $\{(z,w) \in \mathbb{C}^2 : z^m = w^n\}$ intersected with S^3 is the (m,n)-torus knot.
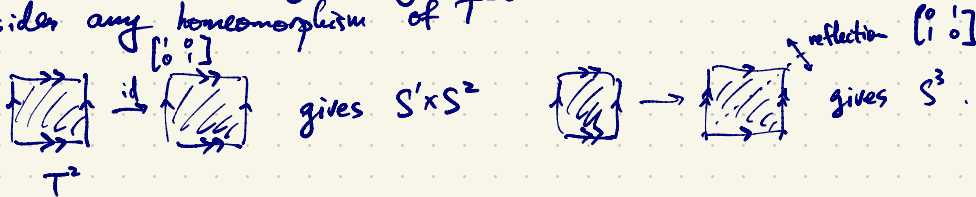
Recall: S^3 can be constructed by pasting together two solid tori $S^1 \times D^2$
(second torus having its meridian and longitude "reversed" as happens when turning it "inside out")

What if we instead glue together without this reversal? (ie. boundary points on first torus are pasted to the identical point on the second torus)? $S^1 \times S^2$. $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$

In fact there are many ways to glue together two solid tori along their boundaries!

Consider any homeomorphism of T^2.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

gives $S^1 \times S^2$ →   gives S^3.

T^2

Apply any $A \in SL_2(\mathbb{Z}) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}, \ ad - bc = \pm 1 \}$. to $\mathbb{R}^2$

A preserves $\mathbb{Z}^2 \subset \mathbb{R}^2$ so A acts on $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \Rightarrow A^{-1} = \pm \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det A = \pm 1$

A maps the "unit" paths in x, y directions to certain torus knots. But the real thing is what happens to the construction above (gluing together two solid tori). This gives a lens space. Even this construction generalizes quite far.

This leads to Dehn surgery: Start with a knot K in S^3

ε-nbhd of K has boundary $R \cong T^2$ embedded strangely in S^3.

$(S^3 \smallsetminus R)$ glued to $(S^3 \smallsetminus R')$ along their boundaries $R \cong R' \cong T^2$.