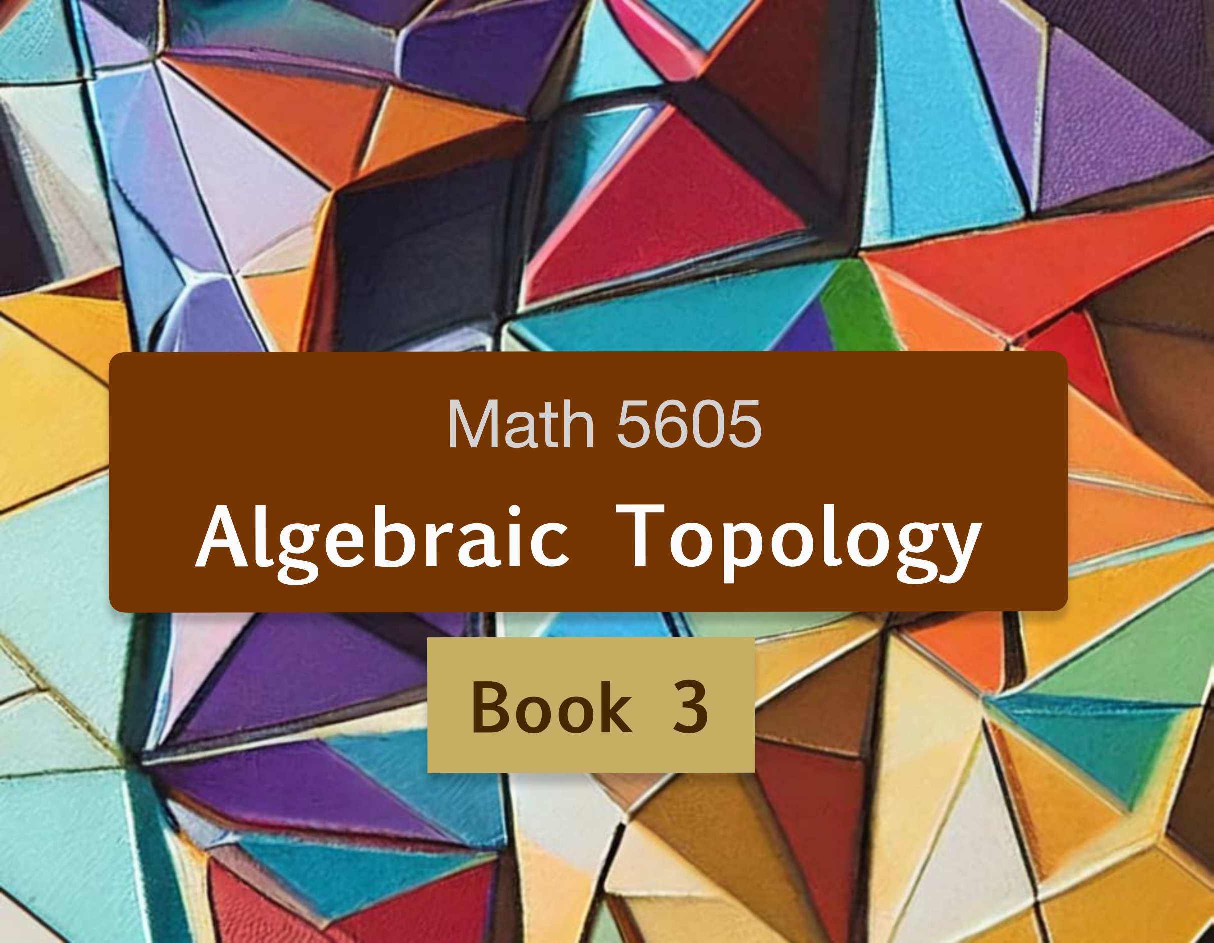


Math 5605

Algebraic Topology

Book 3

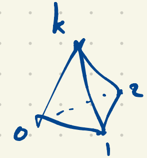
Cup product for simplicial cohomology $H^k \times H^l \xrightarrow{\cup} H^{k+l}$

makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains : $(k = 0, 1, 2, 3, \dots)$ ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex $\{$all subsets of $\{0,1,2,\dots,k\}\}$.

This has a geometric realization 

$\xrightarrow{\sigma} X$

$\Delta = \Delta^n = \{\underbrace{(v_0, v_1, \dots, v_n)}_{\substack{\text{barycentric}\\ \text{coordinates}}} : v_i \geq 0,\ \sum v_i = 1\} \subset \mathbb{R}^{n+1}$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \longrightarrow X$.

$C_n = \{n\text{-chains in } X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{n\text{-cochains in } X\} = \mathrm{Hom}(C_n, R) = \{R\text{-homomorphisms } C_n \longrightarrow R\}$

$\partial : C_n \longrightarrow C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma \,|\, [v_0, \dots, \hat{v}_i, \dots, v_n]$ $\qquad \partial^2 = 0$, $(\partial^*)^2 = 0$

$\partial^* : C^{n-1} \longrightarrow C^n$

If $\phi \in C^k$ k-cochain
$\psi \in C^l$ l-cochain

then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\,\psi(\sigma|[v_k, \dots, v_{k+l}])$$

$[v_0, \dots, v_{k+l}] \longmapsto \sigma(v_0, \dots, v_{k+l})$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)
making $H^*(X; R)$ into a graded ring
$$H^*(X;R) = \bigoplus_{i \geq 0} H^i(X; R).$$

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{$ 1-dim'l subspaces of $\mathbb{R}^{n+1} \} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$ 

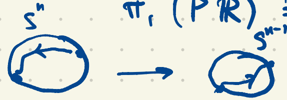
$H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{ a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2 \}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$

Proof is by contradiction

Suppose $f: S^n \to S^{n-1}$ is antipodal. $\quad (f(-x) = -f(x))$

Then f induces a well-defined map
$$\begin{array}{ccc} P^n\mathbb{R} & \xrightarrow{f} & P^{n-1}\mathbb{R} \\ \psi & & \psi \\ \pm x & & \pm f(x) \end{array}$$
$(x \in S^n)$



$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces
$$\begin{array}{ccc} f^*: H^*(P^{n-1}\mathbb{R};\mathbb{F}_2) & \longrightarrow & H^*(P^n\mathbb{R};\mathbb{F}_2) \\ \wr\wr & & \wr\wr \\ \mathbb{F}_2[x]/(x^n) & & \mathbb{F}_2[x]/(x^{n+1}) \end{array}$$
mapping $x \longmapsto x$

$$x^n \longmapsto x^{n+1} \quad ; \quad \text{contradiction.}$$

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A)} \oplus T(A)$ canonically. where $T(A) =$ torsion subgp of $A = \{$elements of A of finite order$\}$

$k = \operatorname{rank} A = \dim A$.

For any chain complex $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$ (over $\mathbb{Q}$ or $\mathbb{R}$)

we have homology groups $H_n = \ker \partial_{n-1} / \operatorname{im} \partial_n$ with well-defined rank $H_n(X;\mathbb{Z}) = \operatorname{rank} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{R})$

and Euler characteristic
$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i .$$

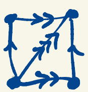
$C_n \xrightarrow{\partial_n} C_{n-1} \qquad \dim C_n = \dim \ker \partial_n + \dim \operatorname{im} \partial_n$

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1} \qquad \dim H_n = \dim \ker \partial_n - \dim \operatorname{im} \partial_{n+1}$

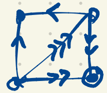
eg. $\chi(S^2) = 4 - 6 + 4 = 2$

Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

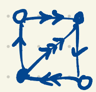
	S^2	T^2	$P^2\mathbb{R}$	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0

T^2 

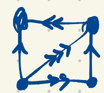
$1-3+2=0$

S^2 

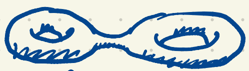
$3-3+2=2$

$P^2\mathbb{R}$ 

$2-3+2=1$

K^2 Klein bottle 

$1-3+2=0$

 #  = 

T^2 T^2 T^2 T^2

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ for any two closed surfaces S_1, S_2

$$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0+0-2 = -2$$

$\underbrace{T^2 \# \cdots \# T^2}_{g}$ = 

$\chi(T^2 \# \cdots \# T^2) = 2-2g$ g = genus of orientable surface

$$\chi(\underbrace{P^2\mathbb{R} \# P^2\mathbb{R}}_{K^2}) = 1+1-2 = 0$$

Exact sequences $\quad \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \qquad \ker \partial_n = \operatorname{im} \partial_{n+1}$

$0 \to C \to 0$ is exact iff $C = 0$

$0 \to A \to B \to 0$ is exact iff $A \cong B$

$0 \to A \to B \to C \to 0$ is exact (short exact) iff $C \cong B/A$

If $f : X \to X$ is an endomorphism of an abel. gp. X (or vector space) (at least in an abelian category) Some important short exact sequences are

$$0 \to \ker f \to X \to f(X) \to 0$$

$$0 \leftarrow \operatorname{coker} f \leftarrow X \leftarrow f(X) \leftarrow 0$$

If $f : X \to Y$ then $\operatorname{coker} f = Y / \underbrace{f(X)}_{\operatorname{im} f}$

If $\cdots \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to \cdots$ are exact then we get an exact seq.

$$\to A \to B \to D \to E \to \cdots$$

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & A \arrow[dr] & & & & E \arrow[r, dashed] & \cdots \\
& & B \arrow[rr, dashed] \arrow[dr] & & D \arrow[ur] & & \\
& & & C \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$0 \to \ker f \to X \xrightarrow{f} X \to \operatorname{coker} f \to 0 \quad \text{is exact.}$$

If X is a fin. dim'l vector space over F then the Euler char. of this sequence is

$$\dim \operatorname{coker} f - \dim X + \dim X - \dim \ker f = 0$$

If $T : X \to X$ is an operator (endomorphism) (don't worry about boundedness) the index of T is $\operatorname{ind} T = \dim \operatorname{coker} T - \dim \ker T$ when both of these terms are finite (i.e. T is Fredholm).

Theorem: Let $S, T : X \to X$ be operators (lin. transf). Of the three operators S, T, ST, then whenever two are Fredholm then so is the third and in this case $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

(or abel. gps)

In general (i.e. for any lin. transf. $S, T : X \to X$ we have an exact sequence

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0$$

So its Euler characteristic is zero. ie. $\operatorname{ind} S + \operatorname{ind} T - \operatorname{ind} ST = 0$.

<u>Snake Lemma</u> In an abel. category we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

then we have a six-term exact seq.

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[dddll, bend left] & \\
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} a \arrow[r] \arrow[d] & \operatorname{coker} b \arrow[r] \arrow[d] & \operatorname{coker} c \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Group Cohomology : used in the study of group extensions

If G and H are groups then an extension of H by G is a group X giving an exact sequence $1 \to H \to X \to G \to 1$

Note: Groups are not necessarily abelian. We are asking for a new group X having a normal subgp $\cong H$ s.t. $X/H \cong G$. G on top, H on the bottom.

Trivial: $X = G \times H$. (split extension)

G is now an arbitrary group and A is an abelian group (G multiplicative; A additive notation) on which G acts (each $g \in G$ gives $g \in GL(A)$ (automorphisms of A as an abel. gp or $\mathbb{Z}$-module).

$(g_1 g_2)(a) = g_1(g_2(a))$; $g(a+b) = ga + gb$; $1_{G}\, a = a$. $\quad G \xrightarrow[\text{(fixed)}]{\text{homo.}} \operatorname{Aut} A = GL(A)$

We construct an extension of A by G i.e. an exact sequence of gps

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \hat{G} \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \arrow[d] \\
1 \arrow[r] & A \arrow[r] & \hat{\hat{G}} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

i.e. $\hat{G}$ is a gp with normal subgp. iso. to A with $\hat{G}/A \cong G$

Two extensions $\hat{G}$ $\hat{\hat{G}}$ are equivalent if we have a commutative diagram as shown with α, β, γ isomorphisms of groups (with exact rows). Note that the action of G on A is fixed throughout.

Cohomology of groups is the tool for this.

$$\cdots \xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} 0$$ is an exact sequence of additive abel. gps where $C^k = C^k(G;A)$ is the set of all maps $\underbrace{G^k}_{G\times G\times\cdots\times G} \to A$ as an additive abel. gp. i.e. $\mathbb{Z}$-module

i.e. k-tuples of G

$C^0 = A$ (maps $\{1\} \to A$)

$C^1 = A^G$ = maps $G \to A$ i.e. $f: G \to A$

$C^2 = A^{G\times G}$ = maps $G\times G \to A$ etc.

Given $a \in C^0$ i.e. $a \in A$, $\delta a \in C^1$ is $\delta a : G \to A$

$g \mapsto ga - a$

Given $f \in C^1$ i.e. $f: G \to A$

construct $\delta f \in C^2$ i.e. $(\delta f): G\times G \to A$

$$(\delta f)(g,h) = g\underbrace{f(h)}_{\in A} - \underbrace{f(gh)}_{\in A} + \underbrace{f(g)}_{\in A} \in A.$$

Given $f \in C^2$ i.e. $f: G\times G \to A$

construct $(\delta f): G\times G\times G \to A$

$$(\delta f)(g,h,k) = g f(h,k) - f(gh,k) + f(g,hk) - f(g,h)$$

See p. 2 bottom of handout for $\delta: C^k \to C^{k+1}$ in general

Check: $C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0$ $\delta^2 = 0$?

Take $a \in C^0 = A$.

$(\delta a): G \to A$

$(\delta a)(g) = ga - a$.

$\delta^2 a : G\times G \to A$

$$(\delta^2 a)(f,g) = f(\delta a)(g) - (\delta a)(fg) + \delta a(f)$$

$$= f(ga-a) - ((fg)(a) - a) + (fa - a)$$

$$= \cancel{fga} - \cancel{fa} - \cancel{fga} + \cancel{a} + \cancel{fa} - \cancel{a}$$

$$= 0$$