Math 5605

Algebraic Topology

Book 1

If X, Y are top. spaces, $f: X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a <u>homeomorphism</u> if f is bijective and f, f^{-1} are continuous.

$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \overset{\cong}{\to} Y$.

$\mathbb{R}^2 \not\cong S^2$ Since S^2 is compact; $\mathbb{R}^2$ is not.

    2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} = $

$S^2 \not\cong T^2 = S^1 \times S^1 = $ $ = $ S^2, T^2 are compact surfaces. They are locally homeomorphic
      S^1 but not globally homeomorphic.

$T^1 = S^1 = $ $ = $ circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not.
In S^2, every closed path can be "continuously shrunk" to a point
                    (homotopic to a point, i.e. null-homotopic)

X $T = T^2$ although both surfaces are compact, connected, not simply connected
 $\not\cong$

These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of <u>homotopy equivalence</u>.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\bullet\}$.

Given points $a, b \in X$ (a topological space), a path from a to b is a function (continuous) $f: [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.
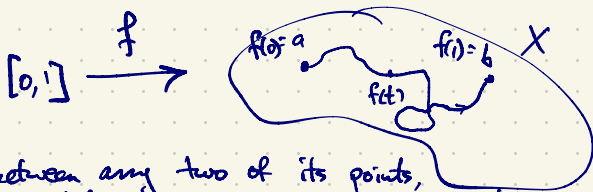
All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi: [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi: [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g: [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:

$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$
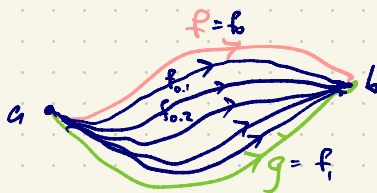
$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$
$(s,t) \mapsto f(s,t) = f_s(t)$
such that $f_0 = f$ ie. $f_0(t) = f(t)$
$f_1 = g$ ie. $f_1(t) = g(t)$
$f_s(0) = a$ for all
$f_s(1) = b$ $s \in [0,1]$

This is a homotopy from f to g.

f, g are homotopic but h is not homotopic to f, g

We say f, g are homotopic if there is a continuous family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi: [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f\circ\varphi: [0,1]\to X$ is homotopic to f. A homotopy from f to $f\circ\varphi$ is

$$[0,1]^2 \to X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{[0,1]}) = f_s(t)$$

$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
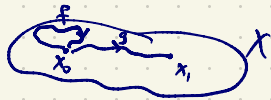$$f_s(0) = f((1-s)\cdot 0 + s\cdot\varphi(0)) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot\underbrace{\varphi(1)}{}) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$. This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)

Fix g path in X from x_0 to x_1
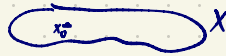
An isomorphism $\phi: \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$f \overset{\phi}{\longmapsto} \bar{g}fg$$
$$g f \bar{g} \overset{\phi^{-1}}{\longleftarrow} h$$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t\in[0,1]$

$\gamma f = f\gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is
$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$
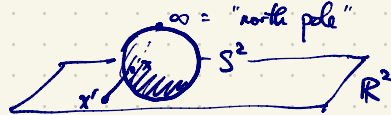(same path in the reverse direction)

$$f\bar{f} = \bar{f}f = \gamma = \text{null path}$$

$$\phi(f_1 f_2) = \bar{g}f_1 f_2 g = (\bar{g}f_1 g)(\bar{g}f_2 g)$$
$$f_1, f_2 \in \pi_1(X, x_0)$$

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

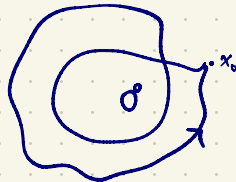$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)

∞ = "north pole"

S^2

$\mathbb{R}^2$

$x \longmapsto x'$ is stereographic projection from the north pole ∞

S^2

$\mathbb{R}^2$

See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 \smallsetminus \{0\}) \cong \mathbb{Z}$
  punctured plane

$\cdot x_0$

follows from the fact that

$\mathbb{R}^2 \smallsetminus \{0\}$ and S' have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 \smallsetminus \{0\} \simeq S'$

$\mathbb{R}^3 \smallsetminus \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).
    Hatcher writes $X \approx Y$ for homeomorphic.

- retraction
- deformation retraction
- strong deformation retraction
- homotopy
- relative homotopy
- homotopy equivalence

"def. retraction in the weak sense"

"def. retraction"

Let $A \subseteq X$ (subspace of a top. space).

A retraction $f: X \longrightarrow A$ is a map (continuous) such that $f|_A = \text{id}_A = 1_A$ ie. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a retract of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points.   If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \longrightarrow \{a\}$, $x \longmapsto a$ is a retraction.

$\mathbb{R}^2 \longrightarrow x\text{-axis}$ $(x,y) \longmapsto (x,0)$.

If $S' \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \longrightarrow S'$. (But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to $\underset{\underset{X \to A}{\text{refraction}}}{\hat{a}}$), $A \subseteq X$.

ie. $f : [0,1] \times X \longrightarrow X$

$$f(t,x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$

     $f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.

Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f : [0,1] \times X \to X$:

     $f_0(x) = x$ i.e. $f_0 = id_X$

     f_1 is a retraction $X \to A$

     $f_t|_A = id_A$ for all $t \in [0,1]$

$$S^1$$

x

X
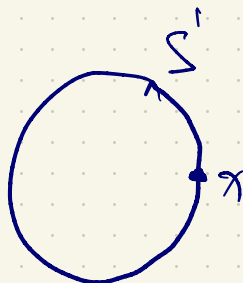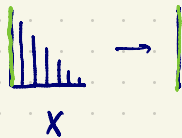
def. retract
but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $X = \left([0,1] \times \{0\} \right) \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} \left(\{r\} \times [0, 1-r] \right)$ X has a def. ret. to $A := \{0\} \times [0,1]$.

This is a def. retract, but not a strong def. retract.

There is no strong def. retract $X \to A$.
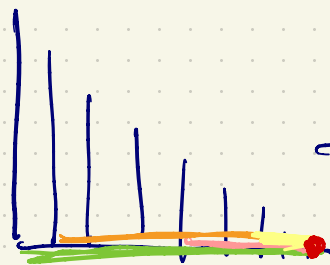
$0 \leq t \leq \frac{1}{2}$

$\frac{1}{2} \leq t \leq 1$

Let $f_0, f_1 : X \to Y$ be maps. A **homotopy** from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0,x) = f_0(x)$
$f(1,x) = f_1(x)$
$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy **relative to A** from f_0 to f_1 $\left(f_0, f_1 : X \to Y \right)$ is a homotopy $f : [0,1] \times X \to Y$ such that $f_t(a)$ is independent of $t \in [0,1]$ for all $a \in A$.
(constant)

A **path** from a to b in X is a homotopy from a to b.
Given two paths f_0, f_1 in X from a to b $(a, b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{}{\overset{f}{\rightleftarrows}} Y$
such that $f \circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)
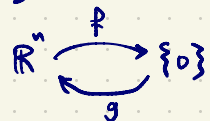
$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$ $f(x) = 0$ for all $x \in \mathbb{R}^n$
$\qquad\qquad\qquad g(0) = 0 \in \mathbb{R}^n$

$g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad\qquad\qquad f \circ g : \{0\} \to \{0\}$
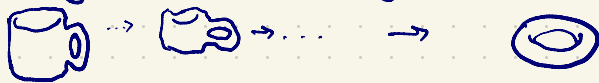$\qquad x \mapsto 0$ $\qquad\qquad\qquad\qquad\qquad 0 \mapsto 0$

A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad$ to $\quad id_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ is $h_t(x) = tx$, $\quad 0 \le t \le 1$, $\quad x \in \mathbb{R}^n$
$\qquad\qquad\qquad\qquad\qquad\; x \mapsto 0$ $\qquad\qquad\qquad\qquad x \mapsto x$
$\quad$↳ Not relative $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_0 = g \circ f$
$\qquad$ to any subspace $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad h_1 = id_{\mathbb{R}^n}$
$\qquad$ necessarily.

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).
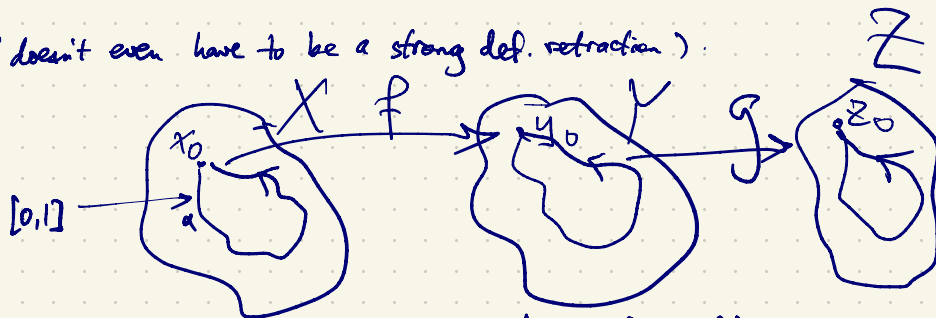
S^1 is not homotopic to a point.
(S^1$ is not null homotopic; not contractible)

If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

$\quad \alpha : [0,1] \to X$ gives $f_* \alpha = f \circ \alpha : [0,1] \to Y$ $\qquad\qquad\qquad\qquad\qquad\qquad (g \circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

D^n = closed ball in $\mathbb{R}^n$ $\overset{\bullet}{D^0}$ $\overline{D^1}$

    "disk" D^2

$S^n = \partial D^{n+1}$ = n-sphere

A CW complex is formed $\overset{\bullet}{S^0}\overset{\bullet}{}$

from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$ $= D^0 \sqcup D^0$ S^1

  X^n is a union of copies of D^n with the boundaries of D^n_α attached to X^{n-1} via attaching maps.
                              $\underline{}$
                      D^n_α
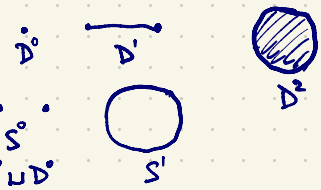
Eg. Torus $T^2 = S^1 \times S^1 = $ $= $ $=$
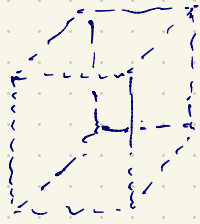
$X^0 = \quad \bullet \quad = D^0$

$X^1 = $ $\cong S^1 \vee S^1$

  $\underset{D^1_0}{\underline{}} \quad \omega \quad \underset{D^1_1}{\underline{}}$

X^2
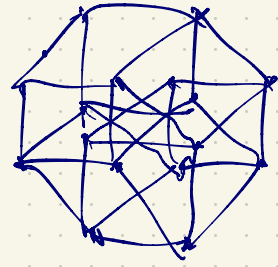
$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

Möbius strip not homeomorphic to cylinder $S^1 \times [0,1] =$ $=$

$X^0 = \cdot \quad \cdot$

$X^1 =$ A ——— B

Cylinder: $X^0 = \cdot \quad \cdot$

$X^1 =$

$D^1 \sqcup D^0$

$X^2 =$

$X^2 =$

orientable

not orientable

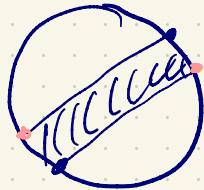Both are homotopy equivalent to S^1 (def. retract to S^1)

Both have $\mathbb{Z}$ as fund. gp.

$P^2 \mathbb{R}$ (or $\mathbb{R}P^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified
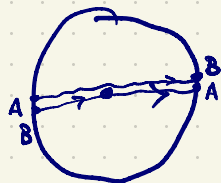

(non-orientable surface)

$P^2 \mathbb{R} = D^2$ glued to a Möbius strip

D^2

$\pi_1(\mathbb{PR}) \cong \mathbb{Z}/2\mathbb{Z}$ a is not homotopic to the null path γ. a^2 is homotopic to γ

a^2
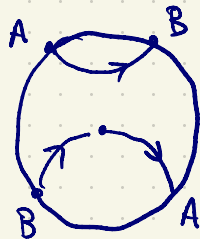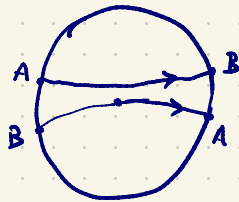
$\mathbb{R}^2 - \{0\} \simeq S^1$
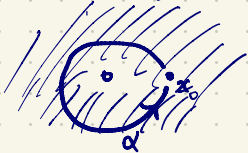
A homotopy equivalence

$\mathbb{R}^2 - \{0\} \xrightarrow{\; f_1 \;} S^1$

$f_1((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}} \, , \, \frac{y}{\sqrt{x^2+y^2}} \right)$

$f_t(v) = (1-t)v + t\frac{v}{|v|}$ $\qquad f_1(v) = \frac{v}{|v|}$

$f_t : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$ strong def. retraction since

$f_0 = id_{\mathbb{R}^2 - \{0\}}$

$\left. f_t \right|_{S^1} = id_{S^1}$ for all $t \in [0,1]$.

f_1 is a retraction to S^1

$\{ \alpha^n : n \in \mathbb{Z} \}$
$\|$
$\langle \alpha \rangle = \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$ free group on one generator

$$\pi_1(S^1) \cong \mathbb{Z}$$

Given a closed path $\overset{\alpha}{\text{in}}$ S^1 with base point $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$

define $w(\beta) = \dfrac{1}{2\pi i} \displaystyle\int_\beta \dfrac{dz}{z} = \dfrac{1}{2\pi i} \displaystyle\int_{\beta(0)}^{\beta(1)} \dfrac{dz}{z}$

$\beta : [0,1] \longrightarrow S^1$

$\beta(0) = 1$

$\beta(e^{2\pi i \theta}) = \theta + n$

$w(\alpha) = 1$

$w(\alpha^n) = n$

w is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$.

$$\int \frac{dz}{z} = \ln|z| + 2\pi i \, \text{Arg } z$$

In $\mathbb{C} - \{0\}$ the same argument works

$\mathbb{R}^2 - \{0\}$ $O = (0,0)$

$\pi_1(\mathbb{R}^2 - \{0\}) = \langle \alpha \rangle \cong \mathbb{Z}$

distinct

k-punctured plane $X = \mathbb{R}^2 - \{A_1, \dots, A_k\}$

$\pi_1(X) = F_k = \text{Free}(\{\alpha_1, \dots, \alpha_k\})$
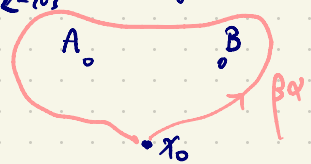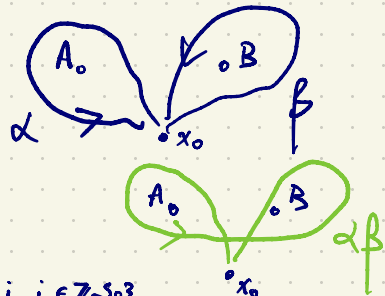
$X \simeq S^1 \vee S^1 \vee \dots \vee S^1$

(k)

$X = \mathbb{R}^2 - \{A, B\}$

$\pi_1(X) = \{ \alpha^{i_1} \beta^{j_1} \alpha^{i_2} \beta^{j_2} \dots \alpha^{i_k} \beta^{j_k},$
$\alpha^{i_1} \beta^{j_1} \dots \alpha^{i_k} \beta^{j_k} \alpha^{i_k},$
$\beta^{j_0} \alpha^{i_1} \beta^{j_1} \dots \alpha^{i_k} \beta^{j_k},$
$\beta^{j_0} \alpha^{i_1} \beta^{j_1} \dots \alpha^{i_k} \beta^{j_k} \alpha^{i_k} :$ $i_r, j_r \in \mathbb{Z} - \{0\}$
$k \geqslant 0\}$

$\pi_1(X)$ is the free group on two generators i.e.
$F_2 = \text{Free}(\{\alpha, \beta\}) = \langle \alpha \rangle * \langle \beta \rangle = \mathbb{Z} * \mathbb{Z} = \langle \alpha, \beta \rangle$

The Van Kampen Theorem gives a presentation for $\pi_1(X)$ when X is suitably described in terms of smaller pieces.

A presentation for a group G expresses G as a homomorphic image of a free group F i.e. $G \cong F/N$, $N \triangleleft F$.

Let X be a set of generators of G ($X \subseteq G$, $\langle X \rangle = G$).

free$(X) \longrightarrow G$ is a surjective homomorphism; N is its kernel.

$G = \langle x_1, \ldots, x_k : \underbrace{r_1, \ldots, r_m}_{\in F} \rangle$ is a presentation for G if $X = \{x_1, \ldots, x_k\}$ is a set of k symbols, $F = $ free(X) (the free group on $x_1, \ldots, x_k$).

Let N be the smallest normal subgp of F containing $r_1, \ldots r_m$ i.e. the normal closure of $\langle r_1, \ldots, r_m \rangle \leq F$
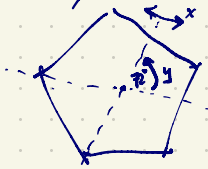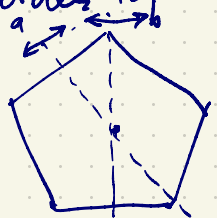
ie. the subgp. of F generated by $r_1, \ldots, r_m$ <u>and</u> their conjugates in F

$N = \langle h r_i h^{-1} : i = 1, \ldots, m; h \in F \rangle$. (When there are k generators and n relators, we say G is <u>finitely presented</u>.)

eg. the dihedral group of order 10 is $D_{10} \cong \langle a, b : a^2, b^2, (ab)^5 \rangle \cong \langle x, y : x^2, y^5, xyxy \rangle$

$xyxy = 1$
$\Leftrightarrow xy = y^{-1}x$
$\Leftrightarrow xyx^{-1} = y^{-1}$

$D_{10} \overset{\sim}{=} \langle a, b : a^2, b^2, (ab)^5 \rangle \overset{\sim}{=} \langle x, y : x^2, y^5, xyx^{-1}y \rangle$ $\quad x^2 = a^2 = 1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y^5 = (ab)^5 = 1$

$\xrightarrow{\phi}$

$\qquad\qquad\qquad\qquad\qquad x = a$ $\qquad\qquad xyx^{-1} = a \cdot ab \cdot a^{-1} = ba$

$\qquad\qquad\qquad\qquad\qquad y = ab$ $\qquad\quad$ whereas $\quad y^{-1} = (ab)^{-1} = b^{-1} a^{-1} = ba$

$\overset{\longleftarrow}{\text{check}}$

A free product $G * H$ is the set of all words from elements in G and H having no relations between elements in G and elements in H (except f one of these elements is the identity).
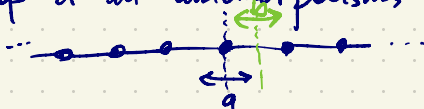
$\mathbb{Z} * \mathbb{Z} = \langle a \rangle * \langle b \rangle = \text{free} \{a, b\} = \langle a, b \rangle = \{ a^{i_1} b^{j_1} a^{i_2} b^{j_2} \dots a^{i_k} b^{j_k}, \dots \}$

$\qquad\underbrace{\quad}_{\substack{\text{infinite}\\\text{cyclic}}} \quad \underbrace{\quad}_{\substack{\text{infinite}\\\text{cyclic}}}$

$\qquad\qquad\qquad\qquad\qquad = \langle a, b : a^2, b^2 \rangle$

$\langle a : a^2 \rangle * \langle b : b^2 \rangle = \{ 1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \dots \} = D_\infty$

$\overset{\sim}{=} (\mathbb{Z}/_{2\mathbb{Z}}) * (\mathbb{Z}/_{2\mathbb{Z}}) = \langle x, y : x^2, xyx^{-1} = y^{-1} \rangle = \text{group of all automorphisms of}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad xyx^{-1}y = 1$

ie. $\langle X : R \rangle = \langle x_1, \dots, x_m : r_1, \dots, r_k \rangle$
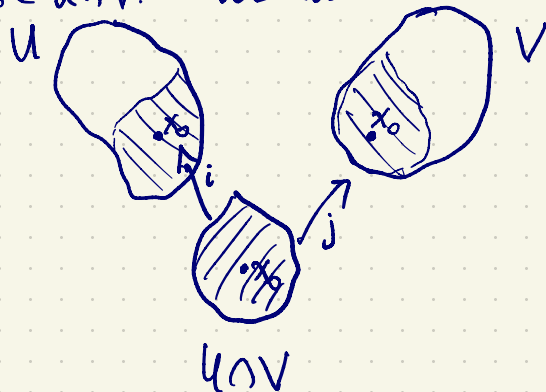
$\qquad \langle Y : S \rangle = \langle y_1, \dots, y_n : s_1, \dots, s_\ell \rangle$

$\qquad \langle X : R \rangle * \langle Y : S \rangle = \langle X \cup Y : R \cup S \rangle = \langle x_1, \dots, x_m, y_1, \dots, y_n : r_1, \dots, r_k, s_1, \dots, s_\ell \rangle$

Free products with amalgamation : add more relations involving x_i's and y_j's e.g.

$$D_{10} = \langle a, b : a^2, b^2, (ab)^5 \rangle = \langle a : a^2 \rangle \underset{(ab)^5}{*} \langle b : b^2 \rangle = D_\infty / \text{Normal closure of } \langle (ab)^5 \rangle$$

$\underbrace{\qquad\qquad}$ cyclic of order 2 $\underbrace{\qquad\qquad}$ cyclic of order 2

Let X be a path-connected top. space covered by two open sets U, V. Since X is connected, $U \cap V \neq \emptyset$. Pick $x_0 \in U \cap V$. We also assume $U \cap V$ is path-connected.

X:

i, j inclusion maps as shown (injective, continuous)

Theorem (Van Kampen, special case)

$$\pi_1(X) = \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$$

where the amalgamation over $\pi_1(U\cap V)$ is given by:
for all $\alpha \in \pi_1(U\cap V)$, identity $i_*(\alpha)$ with $j_*(\alpha)$ i.e.
$i_*(\alpha)^{-1} j_*(\alpha)$ is a new relator.

$U \cap V$

$\pi_1(U)$ $i_*(\pi_1(U\cap V))$ $\pi_1(V)$

$i_*(\alpha)$ $j_*(\pi_1(U\cap V))$

i_* j_*

$\pi_1(U\cap V)$
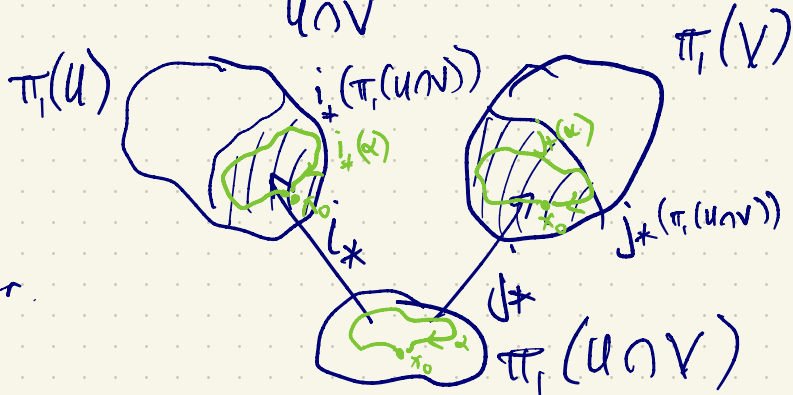
This induces group homomorphisms i_*, j_* as shown.

Eg. $X = S^2 = D \cup D'$

X

$\pi_1(S^2) = \pi_1(D) \underset{\square}{*} \pi_1(D') = 1$

trivial

$= \overbrace{\pi_1(D) * \pi_1(D')}^{} \quad = 1.$

amalgamation
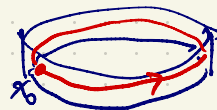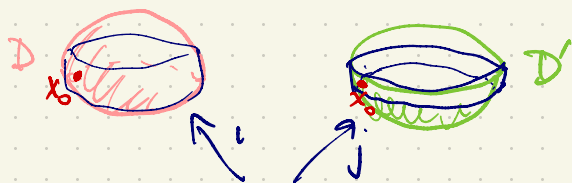
$D \cap D' \simeq S^1$

$\pi_1(D) \cong \pi_1(D') = 1 \quad$ trivial

$\pi_1(D \cap D') = \langle \alpha \rangle$

$\pi_1(D) \quad\quad \pi_1(D')$

$i_* \quad\quad\quad j_*$

$\pi_1(D \cap D')$

i_*, j_*

not one-to-one

$i_*(\alpha) = 1$

$j_*(\alpha) = 1.$

$T^2 = S^1 \times S^1 = $

$=$
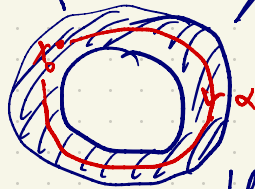

$T^2 = $

U

$\pi_1(U) = 1$

$V \approx S^1 \vee S^1$

$\pi_1(V) = \langle a, b \rangle$

free group on two generators

$\pi_1(U) * \pi_1(V)$

$= 1 * \langle a, b \rangle$

$= \langle a, b \rangle$

$U \cap V \cong S_1 \times (-\varepsilon, \varepsilon)$

$\pi_1(U \cap V) = \langle \alpha \rangle$

Then identify α with 1
due to the inclusion $U \cap V \subset U$. In V, α is homotopic to $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$

Then identify $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$ with α.

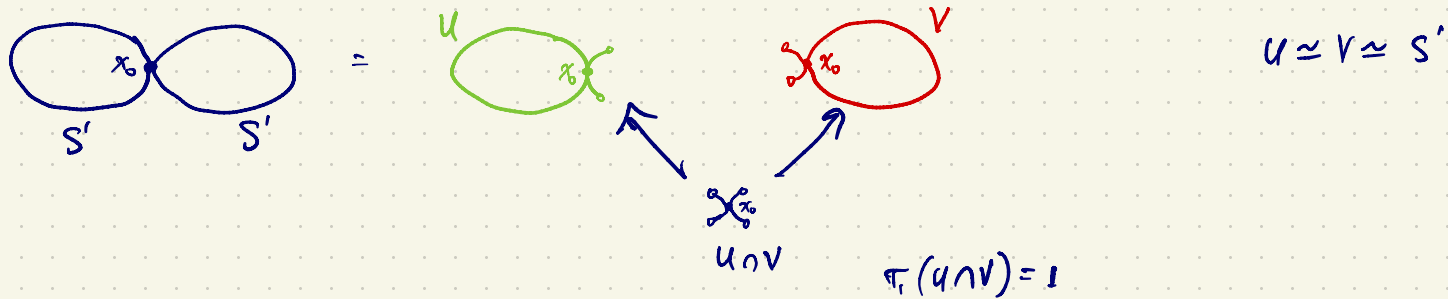So we identify $c \alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0 c^{-1}$ with 1.

So identify $\alpha_1 \alpha_0^{-1} \alpha_1^{-1} \alpha_0$ with 1.

identify $\alpha_1 \alpha_0^{-1}$ with $\alpha_0^{-1} \alpha_1$; $\alpha_0 \alpha_1$ with $\alpha_1 \alpha_0$.

So identify ab with ba.

$\pi_1(T^2) = \langle a, b \rangle / \text{``} ab = ba \text{''} \cong \mathbb{Z} \times \mathbb{Z}$

$= \langle a, b \rangle / \text{Normal closure of } aba^{-1}b^{-1}$

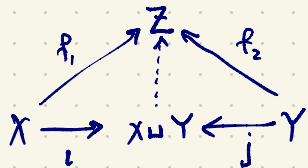$= $ the abelianization of $\langle a, b \rangle$

$$\pi_1(S' \vee S') = \pi_1(S') * \pi_1(S'') \qquad \text{also follows from Van Kampen's Theorem.}$$

If X and Y are path-connected top. spaces then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

$U \simeq V \simeq S'$

$$\pi_1(U \cap V) = 1$$

$$\pi_1(S_1 \vee S_1) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z} = \text{free group on two generators.}$$

In T_p, the category of top. spaces, the coproduct of X and Y is $X \sqcup Y$ = disjoint union; product is $X \times Y$.
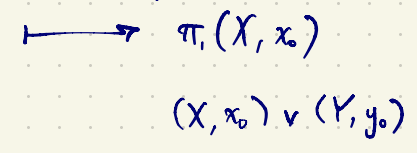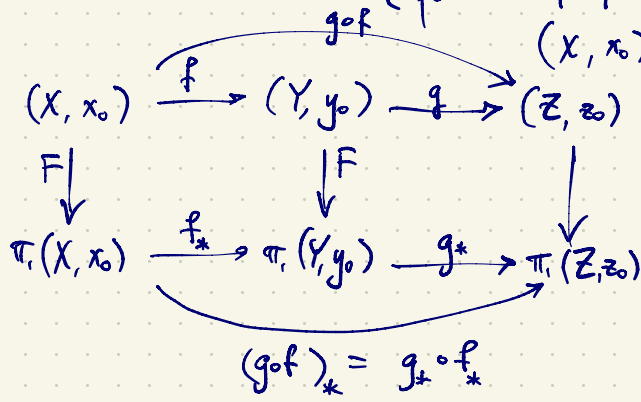
The category of pointed top. spaces i.e. pairs (X, x_0), $x_0 \in X$, X nonempty top. space

The coproduct $(X, x_0) \sqcup (Y, y_0) = $ disjoint union $X \sqcup Y$ with x_0, y_0 identified

denoted as $(X, x_0) \vee (Y, y_0) = (X \sqcup Y, x_0 = y_0)$

A morphism $(X, x_0) \longrightarrow (Y, y_0)$ is a (continuous) map $X \to Y$ satisfying $x_0 \longmapsto y_0$.

We have a functor $\{$ pointed top. spaces $\} \longrightarrow$ Grp $= \{$ groups $\}$

$$\quad (X, x_0) \longmapsto \pi_1(X, x_0)$$

$$
\begin{array}{ccc}
(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0) & \overset{g \circ f}{\frown} \\
F \downarrow \qquad F \downarrow \qquad \downarrow \\
\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(Z, z_0)
\end{array}
$$

$$(g \circ f)_* = g_* \circ f_*$$

$(X, x_0) \vee (Y, y_0)$

$$\downarrow F$$

$$F\big((X, x_0) \vee (Y, y_0)\big) = F\big((X, x_0)\big) * F\big((Y, y_0)\big)$$

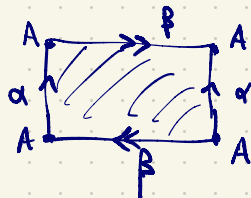$\uparrow$ coproduct in pointed top. spaces is wedge sum.

$\uparrow$ coproduct in Grp is just free product

Proj. plane $P^2\mathbb{R}$
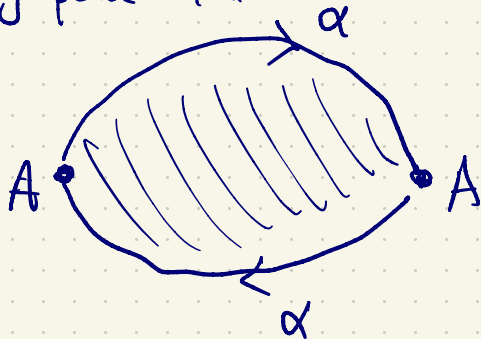
$$\pi_1(P^2\mathbb{R}) = \langle \alpha : \alpha^2 \rangle = \langle \alpha \rangle / \langle \alpha^2 \rangle = \text{group of order 2}$$
$$= \mathbb{Z}/2\mathbb{Z}$$

Klein Bottle K

$$\pi_1(K) = \langle \alpha, \beta : \alpha\beta\alpha^{-1}\beta \rangle = \langle \alpha, \beta \rangle / N$$

where N is the subgp. of $\langle \alpha, \beta \rangle$ generated by $\alpha\beta\alpha^{-1}\beta$ and its conjugates

Force $\alpha\beta\alpha^{-1}\beta = 1$ in the quotient gp.

ie. $\alpha\beta\alpha^{-1} = \beta^{-1}$

So $\pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

$\langle \beta \rangle \qquad \langle \alpha \rangle$ where α inverts β.

$(k,l)(r,s) = \begin{cases} (k+r, l+s) & \text{if } r \text{ is even} \\ (k+r, -l+s) & \text{if } r \text{ is odd} \end{cases}$

Every element can be uniquely expressed as $\alpha^k \beta^l$, $\quad k, l \in \mathbb{Z}$

$(\alpha^k \beta^l)(\alpha^r \beta^s) = \begin{cases} \alpha^{r+k} \beta^{l+s} & \text{if } r \text{ is even;} \\ \alpha^{r+k} \beta^{-l+s} & \text{if } r \text{ is odd.} \end{cases}$

$\alpha\beta^3\alpha^{-1} = \alpha\beta\beta\beta\alpha^{-1} = (\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1}) = \beta^{-1}\beta^{-1}\beta^{-1} = \beta^{-3}$ so $\alpha\beta^3 = \beta^{-3}\alpha$
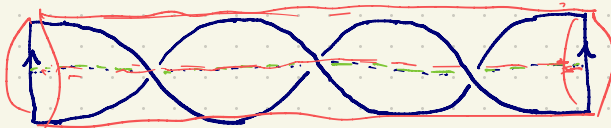
$\alpha\beta^k\alpha^{-1} = \beta^{-k}$
$\beta^k = \alpha^{-1}\beta^{-k}\alpha$

$\{\alpha^k \beta^l : k, l \in \mathbb{Z}\} = \pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

The elements with K even form an abelian subgp $\langle \alpha^2, \beta \rangle \cong \mathbb{Z} \times \mathbb{Z}$ (free abelian of rank 2).

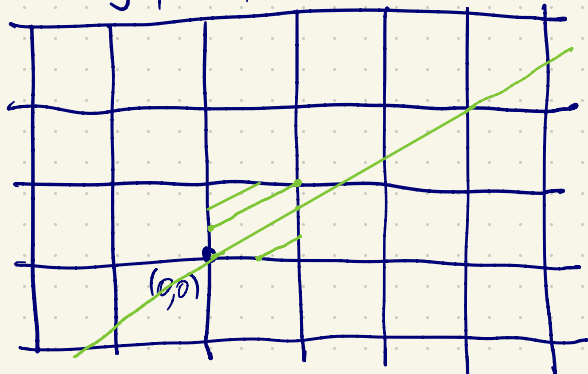and the quotient $\pi_1(K) / \langle \alpha^2, \beta \rangle$ is order 2.

Leading up to trefoil:

$T^2 \longrightarrow \mathbb{R}^3 \ S^3$

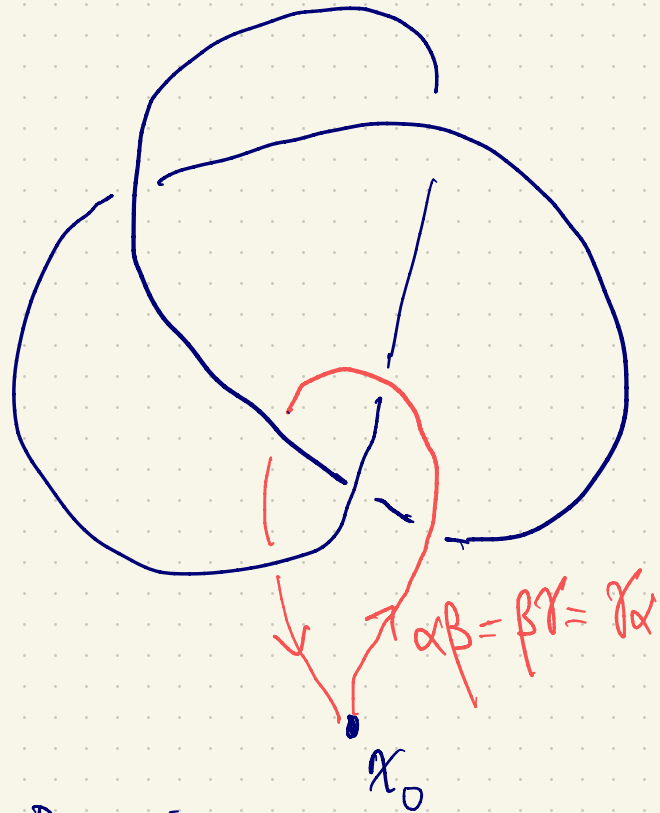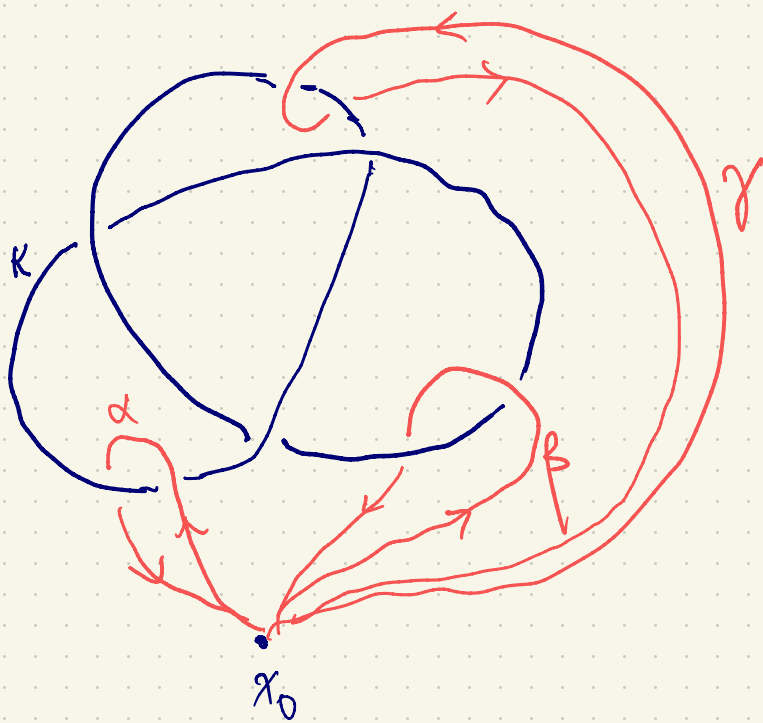This surface partitions $\mathbb{R}^3 \ S^3$ into two solid tori

for any $m, n > 1$ relatively prime, we have an (m, n) torus knot embedded in the torus

Take the line $y = \frac{n}{m} x$

$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$

$(0,0)$

The trefoil is the $(3,2)$ torus knot.

or $(2,3)$

$$\pi_1(\mathbb{R}^3 \smallsetminus K) = \langle \alpha, \beta, \gamma : \alpha\beta = \beta\gamma = \gamma\alpha \rangle = \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle \quad ?$$
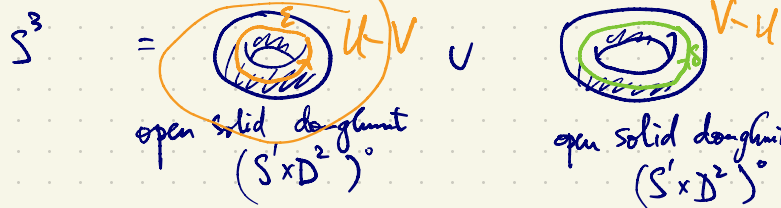
or S^3

$$\gamma = \bar{\beta}^{-1}\alpha\beta = \alpha\beta\alpha^{-1}$$

Prove using
Van Kampen's Theorem

$S^3 \smallsetminus K = U \cup V$

↑
trefoil knot

U, V open path-connected
$U \cap V$ ·· ···

$S^3 = $ $U \smallsetminus V \cup$ $V \smallsetminus U$

open solid doughnut
$(S' \times D^2)°$

open solid doughnut
$(S' \times D^2)°$

$\pi_1(U) = \langle \varepsilon \rangle \cong \mathbb{Z}$

$\pi_1(V) = \langle \delta \rangle \cong \mathbb{Z}$

$\varepsilon^2 = \gamma = \delta^3$

U open nbhd of torus

(exterior)

$S^3 \smallsetminus K = $ open solid doughnut $\cup$ open solid doughnut

(interior)

$\pi_1(S^3 \smallsetminus K) = \underbrace{\pi_1(U) * \pi(V)}_{U \cup V} = \langle \varepsilon \rangle * \langle \delta \rangle \Big/ \begin{array}{c}\text{Normal closure} \\ \text{of } \langle \varepsilon^2 \delta^3 \rangle\end{array}$

U open torus $- K$

 $= \gamma$

$= U \cap V$

$\pi_1(U \cap V) = \langle \gamma \rangle \cong \mathbb{Z}$

$= \langle \varepsilon, \delta : \varepsilon^2 = \delta^3 \rangle$

$\cong \langle \alpha, \beta : \alpha\beta\alpha = \beta\alpha\beta \rangle$

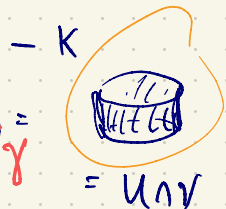$\varepsilon = \alpha\beta\alpha, \quad \delta = \alpha\beta \quad \Rightarrow \quad \delta^3 = \underbrace{\alpha\beta\alpha}_{\varepsilon}\underbrace{\beta\alpha\beta}_{\varepsilon} = \varepsilon^2$