

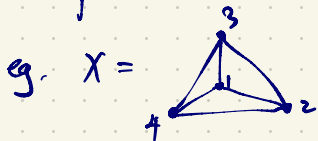


Math 5605

Algebraic Topology

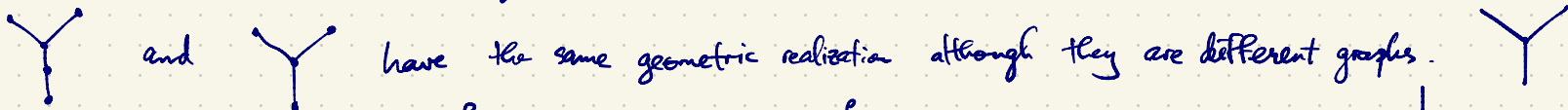
Book 2

When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

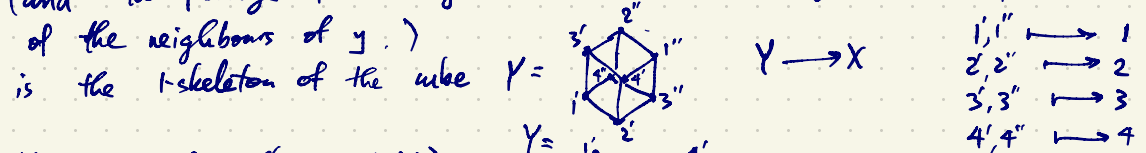
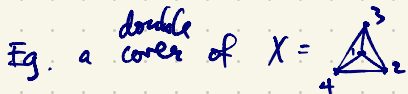


Graph i.e. combinatorial graph with vertices $\{1, 2, 3, 4\}$ and edges $\{1, 2\}, \{1, 3\}, \dots, \{3, 4\}$.

X is the geometric realization of this graph formed as a disjoint union of copies of $[0, 1]$ with endpoints identified as required by the picture.

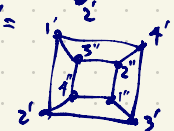
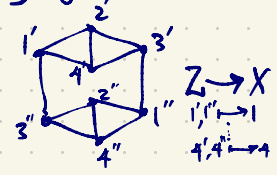


A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in $\Gamma \Rightarrow f(x) \sim f(y)$ in Γ' . A covering map of graphs is a homomorphism $(x, y \in V(\Gamma), \{x, y\} \in E(\Gamma))$ inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbours of y).



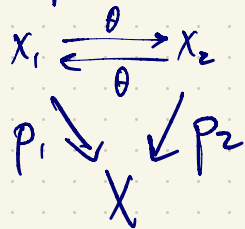
The covering space is $Y \rightarrow X$ (or informally just Y). Some other double covers of X :

Trivial double cover

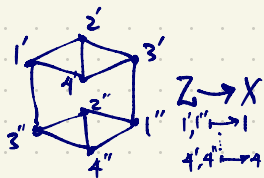


When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

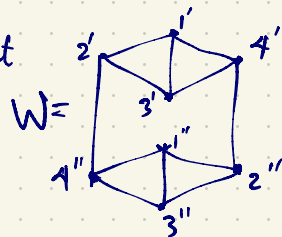
Let $p_1: X_1 \rightarrow X$, $p_2: X_2 \rightarrow X$ be covering spaces of X . We say $\theta: X_1 \rightarrow X_2$ is an equivalence or isomorphism of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e.



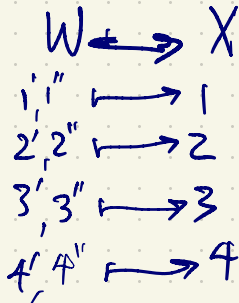
Ex.



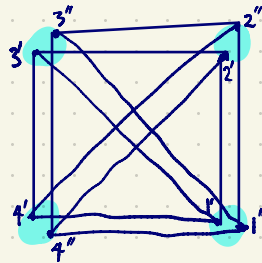
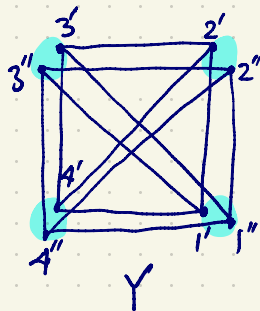
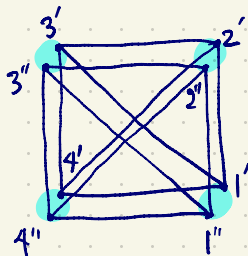
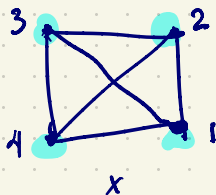
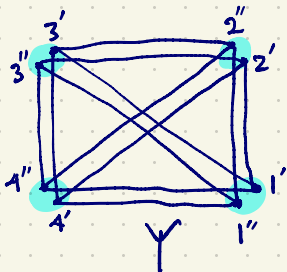
is not equivalent to $Y \rightarrow X$. But what about



Is this equivalent to $Z \rightarrow X$? No...



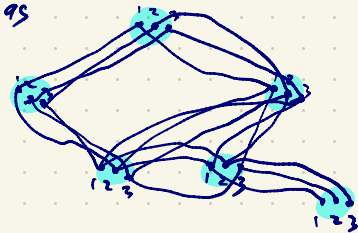
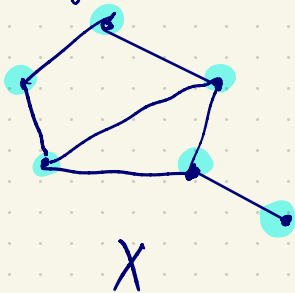
Another picture of these covers:



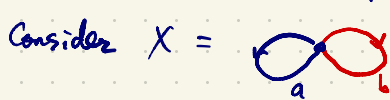
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To construct an r -fold cover of X , create one copy of $[r] = \{1, 2, \dots, r\}$ for each vertex of X . Then for each edge of X , match up the corresponding fibres in the cover using a chosen permutation.

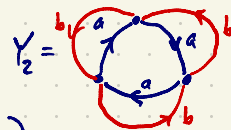
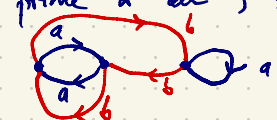
A triple cover $Y \rightarrow X$ is constructed as



Why is 2 more special than other positive integers (the oddest prime of all)?



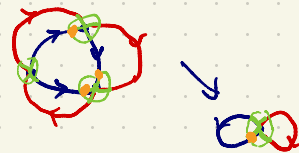
Consider $X =$ has many triple covers including



The covering maps $Y_1 \rightarrow X$ and $Y_2 \rightarrow X$ are not equivalent.

An equivalence between $Y \rightarrow X$ and itself (automorphism of the cover) is a deck transformation. This is the same as a homeomorphism $Y \rightarrow Y$ which preserves fibres.

In the example above, $Y_2 \rightarrow X$ has 3 automorphisms (deck transformations) but $Y_1 \rightarrow X$ has only one (trivial) deck transformation.



In a connected r -fold cover, there are at most r deck transformations. If equality holds, the covering space is normal or Galois. (not the same as normal space in point set topology). Double covers are always normal.

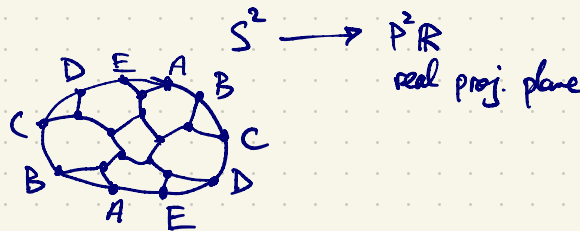
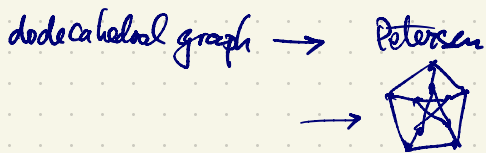
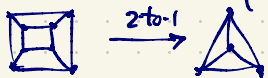


In group theory, subgroups of index 2 are normal.

In the case of ^(separable) extensions of fields, the extension is normal.

For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$ = dimension of E as a vector space over F . The number of F -automorphisms of E (i.e. $\sigma: E \rightarrow E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers: examples



S^n is not a top. group unless $n \in \{1, 3\}$.

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

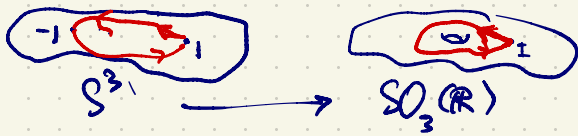
$$S^3 = \{z \in \mathbb{H} : |z| = 1\} \quad \mathbb{H} = \{a+bi+cj+dk : a, b, c, d \in \mathbb{R}\} \quad i^2 = j^2 = k^2 = ijk = -1$$

$$\cong SU_2(\mathbb{C}) = \left\{ A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C}, AA^* = A^*A = I, \det A = 1 \right\}$$

$$SO_3(\mathbb{R}) = \{A \in \mathbb{R}^{3 \times 3} : AA^T = A^T A = I, \det A = 1\}$$

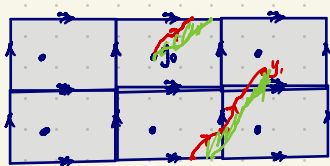
$$O_3(\mathbb{R}) = \{A \in \mathbb{R}^{3 \times 3} : AA^T = A^T A = I\} \text{ has two connected components}$$

Fact: $S^3 \cong SU_2(\mathbb{C}) \rightarrow SO_3(\mathbb{R})$ is a double cover. $Z(S^3) = \{\pm 1\}$ homeomorphism
 $PSU_2(\mathbb{C}) = S^3 / Z(S^3) \cong SO_3(\mathbb{R}) \cong P^3\mathbb{R}$

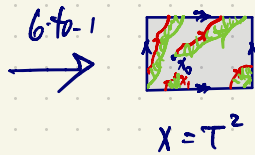


In general for $n \geq 3$, $\pi_1(SO_n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$.

$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R})$ is its universal cover; a double cover constructed from Clifford Algebras (generalizing #1). simply connected



$Y = T^2$



$X = T^2$

$f_0: [0,1] \rightarrow X$

$f_1: [0,1] \rightarrow X$

Assuming X is path-connected and $p: Y \rightarrow X$ is a path-connected covering space,

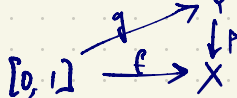
$X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff

$p(y_0) = p(y_1)$.

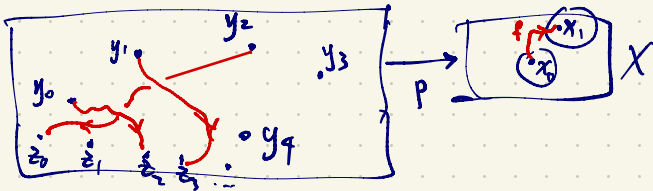
$f_0: [0,1] \rightarrow X$
path in X
from $f_0(0) = x_0$
to $f_0(1) = x_1$

$f_1: [0,1] \rightarrow X$
is another path in X
from x_0 to x_1
homotopic to f_0

In any covering space $p: Y \rightarrow X$ and given any path $f: [0,1] \rightarrow X$ starting at $f(0) = x_0$, the path f can be lifted to Y i.e. there is a path $g: [0,1] \rightarrow Y$ such that $f = p \circ g$ i.e.



and this lift is unique if we say which of the points in $f(x_0)$ to take as the starting point for g .



$$p^{-1}(x_0) = \{y_0, y_1, y_2, \dots\}$$

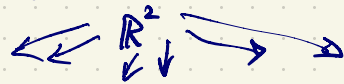
$$p^{-1}(x_1) = \{z_0, z_1, z_2, \dots\}$$

Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \rightarrow p^{-1}(x_1)$.

In particular if p is k -to-1 at x_0 i.e.

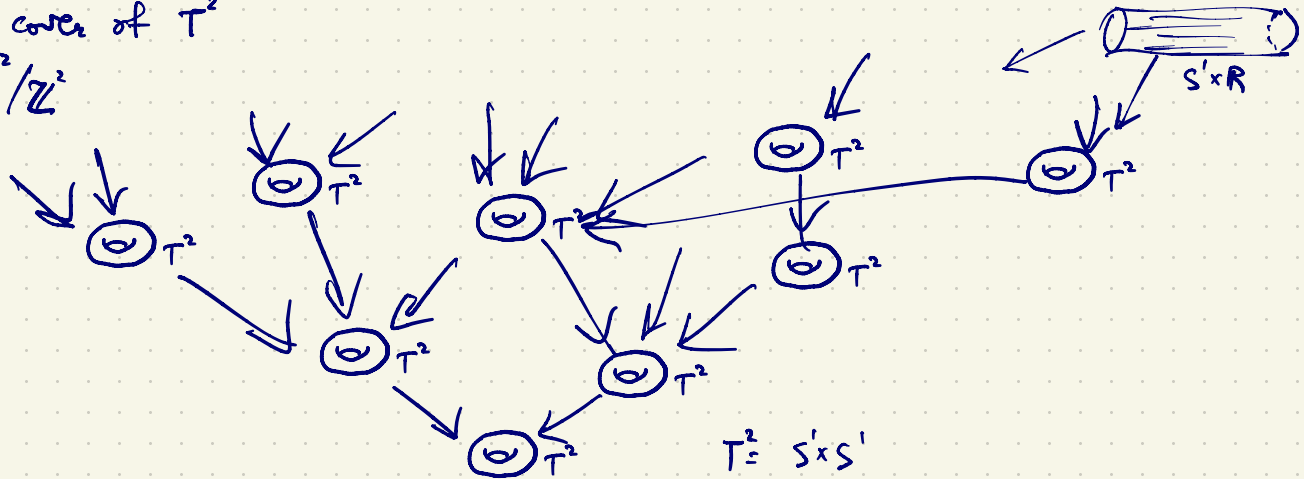
$|p^{-1}(x_0)| = k$ then it is k -to-1 everywhere i.e. $|p^{-1}(x)| = k$ for all $x \in X$.

More generally, if f_t is a homotopy in X and we are given f_0 , then every lifting of f_0 to Y extends to a lifting of f_t to Y .



\mathbb{R}^2 is the universal cover of T^2

$$\mathbb{R}^2 \rightarrow T^2 = \mathbb{R}^2 / \mathbb{Z}^2$$



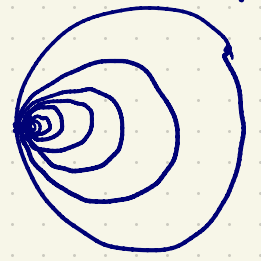
Which top. spaces have a universal cover? (equivalently a simply connected cover)

Let X be a path-connected space. Then X has a path-connected and universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected



Example of a top. space without a universal cover: Hawaiian earring $\subset \mathbb{R}^2$



(not a CW complex)

$$\neq \underbrace{S^1 \vee S^1 \vee S^1 \vee \dots}_{\text{countable wedge sum (CW complex)}}$$

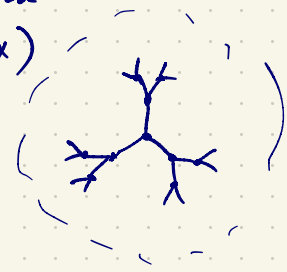
countable wedge sum (CW complex)



Universal cover of K_4

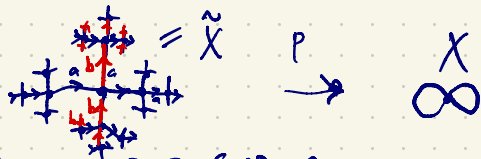


trivalent tree



(also the universal cover of any trivalent connected graph)
i.e. regular of degree 3 connected

Universal cover of any connected regular graph of degree $\neq 1$ is

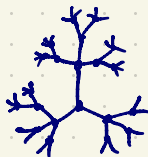


Cayley graph of $\text{Free}\{a,b\} = G$
 Vertices correspond to elements of G .

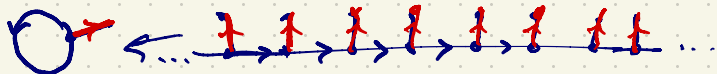
Every vertex $w \in G$ has
 edges to wa, wa^{-1}, wb, wb^{-1} .



Universal cover of $K_{3,4}$



$$\tilde{X} = X/G$$



$P^2\mathbb{R}$ has S^2 as its universal cover.

$$S^2 \longrightarrow P^2\mathbb{R} = S^2/G$$

quotient of S^2
 by the antipodal
 relation.

$G = \{1, -1\}$ acts on S^2

$$1x = x$$

$$(-1)x = -x \quad (\text{antipode of } x)$$

X/\sim = partition of X into equivalence classes of the equiv. relation " \sim "

X/G = partition of X into the orbits of G
($x \sim xg$ or $g(x)$)
for all $g \in G$.

$$X \rightarrow X/\sim$$

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

$$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 = S^1 \times S^1$$



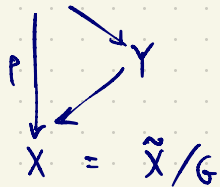
A non-discrete action of \mathbb{Z} on \mathbb{R} eg. $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\}$

G acts discretely on X if for every $x \in X$ there is an open nbhd U of x such that the only $g \in G$ mapping $x \mapsto x^g \in U$ is $g = 1$.

$G = \{x \mapsto 2^k x + l : k, l \in \mathbb{Z}\}$ is non-discrete

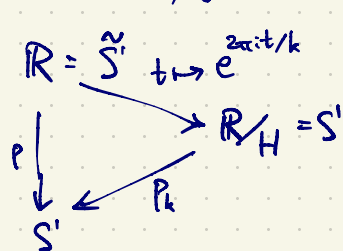
If X is "nice" (path-connected, locally path-connected, SLSC) then X has a simply connected (and path-connected) cover which is a universal cover. It is unique up to isomorphism of covering spaces.

\tilde{X} universal cover. Fix $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$.
 $G \cong \pi_1(X, x_0)$.



Every other covering space $Y \rightarrow X$ (path-connected) has the form $Y = \tilde{X}/H$, $H \leq G$.

eg.

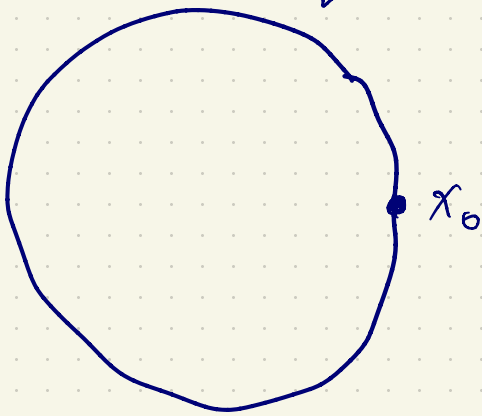
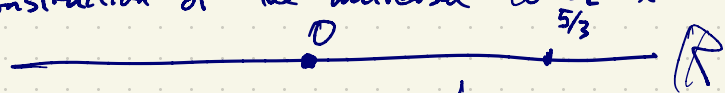


$$p: t \mapsto e^{2\pi i t}$$

$$p_k: z \mapsto z^k$$

$G = \mathbb{Z}$
 $H \leq G$ has the form $H = k\mathbb{Z}$, $k \in \mathbb{Z}$.

Construction of the universal cover $\tilde{X} \rightarrow X$?



$\tilde{X} = \{ \text{paths in } X \text{ starting at the chosen base point } x_0 \} / \sim$
 i.e. paths up to homotopy with fixed starting and ending point

If $f \in \tilde{X}$ then
 $p(f) = f(1) \in X$ (the endpoint of f)

Cohomology Consider a sequence of vector spaces over F given by

$$\dots \xrightarrow{d_4} V_3 \xrightarrow{d_3} V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0$$

or

$$0 \xrightarrow{d^0} V^0 \xrightarrow{d^1} V^1 \xrightarrow{d^2} V^2 \xrightarrow{d^3} V^3 \xrightarrow{\dots}$$

V^i or V_i has i just an index for purposes of reference.

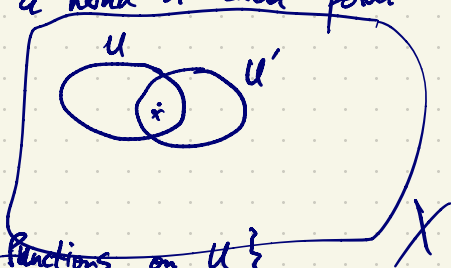
If $d_i \circ d_{i+1} = 0$ then d is a boundary map and the sequence of V_i 's is a complex.

(similarly if $d^{i+1} \circ d^i = 0$, d_i is a coboundary map.)

d_i linear transformations
(more generally V_i ,
 V^i are modules over
a ring R and d_i, d^i
are R -homomorphisms
i.e. $d(av+bw) = adv + bdw$
 $a, b \in F; v, w \in V$)

Notable example: differential forms

Let X be a real n -manifold. In a neighborhood of each point $x \in X$, $x \in U \subseteq X$, we have local coordinates $(x_1, \dots, x_n) = x$.



$$R = C^\infty(U) = \{ \text{smooth real-valued functions on } U \}$$

$$V^0 = R, \quad d: V^0 \rightarrow V^1 = \{ \text{differential 1-forms on } U \} = \{ f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + \dots + f_n dx_n : f_i \in R \}$$

V^1 is a vector space over R (∞ -dimensional)
but n -dimensional as module over R

Eg. $X = \mathbb{R}^2 - \{(0,0)\}$.

$$0 \xrightarrow{0} V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} 0$$

$V^0 = \{ \text{smooth functions } X \rightarrow \mathbb{R} \} = \mathbb{R} = \text{"0-forms"}$

$V^1 = \{ \text{1-forms on } X \}$ i.e. smooth differential 1-forms

$V^2 = \{ \text{2-forms on } X \}$

$V^1 = \{ f dx + g dy : f, g \in \mathbb{R} \}$

ω is closed
but not exact

Eg. $\omega = \frac{x dy - y dx}{x^2 + y^2}$ $w = \omega$

Integrate ω over the path $\gamma(t) = (\cos t, \sin t)$ $t \in [0, 2\pi]$

$$\int_{\gamma} \omega = \int_{\gamma} \frac{x dy - y dx}{x^2 + y^2} = \int_0^{2\pi} \frac{\cos^2 t dt + \sin^2 t dt}{1} = \int_0^{2\pi} dt = 2\pi$$

$x = \cos t$
 $dx = -\sin t dt$

$y = \sin t$
 $dy = \cos t dt$



x, y : global coordinates in X

r, θ : local coordinates (not global)

$$\int_{\gamma} \omega = \int \dots = 2\pi$$

$x, y, r, \theta \in \mathbb{R} = V^0$
coordinate functions $X \rightarrow \mathbb{R}$
or on $U \subset X$

$\omega = \cos \theta$

$dx = \cos \theta dr - r \sin \theta d\theta$
 $dy = \sin \theta dr + r \cos \theta d\theta$

$x = r \cos \theta$
 $y = r \sin \theta$

$\gamma(t): r(t) = 1$
 $\theta(t) = t$

$$\begin{array}{l}
 V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \\
 f \longmapsto df \\
 x \longmapsto dx \\
 y \longmapsto dy \\
 r \longmapsto dr \\
 \text{etc.}
 \end{array}$$

If X is an n -manifold then $V^k = \{k\text{-forms on } X\}$ is an \mathbb{R} -module of dimension $\binom{n}{k}$.
 We need X to be orientable

d is \mathbb{R} -linear but not R -linear

$$d(fg) \neq f dg$$

$$V^2 = \{f dx \wedge dy : f \in \mathbb{R}\}$$

If X has local coordinates x_1, \dots, x_n then $V^1 = \{f_1 dx_1 + \dots + f_n dx_n : f_i \in \mathbb{R}\}$
 $V^2 = \{f_{12} dx_1 \wedge dx_2 + f_{13} dx_1 \wedge dx_3 + \dots : f_{ij} \in \mathbb{R}\}$

$$dx_i \wedge dx_i = 0$$

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

Wedge products are \mathbb{R} -multilinear eg. $(f\omega + g\omega') \wedge \eta = f\omega \wedge \eta + g\omega' \wedge \eta$ $f, g \in \mathbb{R}$

$$dx \wedge (dy \wedge dz) = (dx \wedge dy) \wedge dz = dx \wedge dy \wedge dz$$

$$= (-dy \wedge dx) \wedge dz = -dy \wedge (dx \wedge dz) = -dy \wedge (-dz \wedge dx) = dy \wedge dz \wedge dx$$

$$dx \wedge (dy \wedge dz) = (dy \wedge dz) \wedge dx$$

If ω is an i -form and η is a j -form then $\omega \wedge \eta = (-1)^{ij} \eta \wedge \omega$ is an $(i+j)$ -form.

ω, ω', η
diff. forms

V^k is spanned by terms like $f dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} =: \omega \in V^k$ wlog $1 \leq i_1 < i_2 < \dots < i_k \leq n$

$V^k \xrightarrow{d} V^{k+1}$

$d\omega = d(f dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \underbrace{df}_{\in V^1} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \in V^{k+1}$

$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$

In \mathbb{R}^3 with (global) coordinates x, y, z

$R = V^0 = \{ \text{smooth functions } \mathbb{R}^3 \rightarrow \mathbb{R} \}$

$d: V^k \rightarrow V^{k+1}$ is
 \mathbb{R} -linear but not \mathbb{R} -linear

Pick $f \in V^0$ i.e. $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function

$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in V^1$ This is a very special 1-form
because it is exact. ($\in dV^0$)

$$d(df) = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right)$$

$$= d\left(\frac{\partial f}{\partial x}\right) \wedge dx + d\left(\frac{\partial f}{\partial y}\right) \wedge dy + d\left(\frac{\partial f}{\partial z}\right) \wedge dz$$

$$= \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial x} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial x} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial x} dz\right) \wedge dx + \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial y} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial y} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial y} dz\right) \wedge dy$$

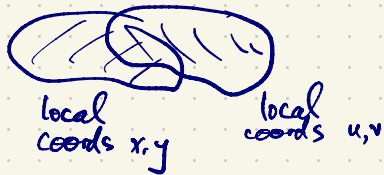
$$+ \left(\frac{\partial}{\partial x} \frac{\partial f}{\partial z} dx + \frac{\partial}{\partial y} \frac{\partial f}{\partial z} dy + \frac{\partial}{\partial z} \frac{\partial f}{\partial z} dz\right) \wedge dz = 0$$

$$\boxed{d^2 = 0}$$

i.e. $d^2\omega = d(d\omega)$
for all $\omega = 0$

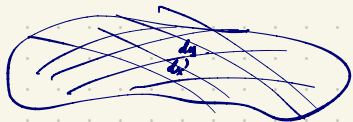
Imagine a surface $S \subset \mathbb{R}^3$. We integrate an arbitrary 2-form $\omega \in V^2$ over S .
 If $\omega = f(x,y,z) dx \wedge dy + g(x,y,z) dx \wedge dz + h(x,y,z) dy \wedge dz \in V^2$ then $\int_S \omega$
 $= \int_S f(x,y,z) dx \wedge dy + \dots$

If $x = x(u,v)$
 $y = y(u,v)$



then $f(x,y) dx \wedge dy$

$$= f(x(u,v), y(u,v)) \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv$$



$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge$$

$$\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right)$$

$$= \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv$$

For a region $X \subset \mathbb{R}^2$,

γ path in X from P to Q ,

$\omega \in V^1$, we define the path integral

$$\int_{\gamma} \omega$$

If $\omega = df$ (an exact 1-form) then

$$\int_{\gamma} \omega = \int_{\gamma} df = f(Q) - f(P)$$

by the Fundamental Theorem of calculus
 But for

But if $\gamma' \sim \gamma$ in X then $\int_{\gamma'} \omega = \int_{\gamma} \omega = f(Q) - f(P)$ whenever $\omega = df$.

Stokes' Theorem (general Fundamental Theorem of Calculus)

Let X be an orientable n -manifold with boundary ∂X which is also orientable $(n-1)$ -manifold. Let $\omega \in \Lambda^{n-1}$, so that $d\omega \in \Lambda^n$. Then

$$\int_{\partial X} \omega = \int_X d\omega$$

Special case: $X = [a, b] \subseteq \mathbb{R}$, $\partial X = \{a, b\}$, $\omega = f \in \mathcal{F}$ (smooth function $X \rightarrow \mathbb{R}$)
 $\int_a^b f' = \int_a^b f'(t) dt = f(b) - f(a)$ $d\omega = f'(x) dx$



$\gamma \sim \gamma'$ in X

$$\partial A = \gamma - \gamma'$$

If $\omega \in \Lambda^1(X)$ then $d\omega \in \Lambda^2(X)$

$$\int_{\gamma} \omega - \int_{\gamma'} \omega = \int_{\partial A} \omega = \int_A d\omega$$

If in particular $d\omega = 0$ (ω a closed 1-form) then RHS = 0 giving $\int_{\gamma} \omega = \int_{\gamma'}$.

Exact forms are automatically closed (if $\omega = df$ then $d\omega = d^2f = 0$).

Not conversely! unless X is simply connected.

The "gap" between {closed forms} and {exact forms} is measured by cohomology.

$$\begin{array}{ccccc} V^{n-1} & \xrightarrow{d^{n-1}} & V^n & \xrightarrow{d^n} & V^{n+1} \\ \text{(n-1)-forms} & & \text{n-forms} & & \text{(n+1)-forms} \end{array}$$

image of $d^{n-1}: V^{n-1} \rightarrow V^n$ is $B^n = \{\text{exact n-forms}\}$

kernel of $d^n: V^n \rightarrow V^{n+1}$ is $Z^n = \{\text{closed n-forms}\}$

$H^n = Z^n / B^n = n^{\text{th}}$ cohomology group (or vector space over \mathbb{R})

$\dim H^n$ is the number of "n-dimensional holes" in X .