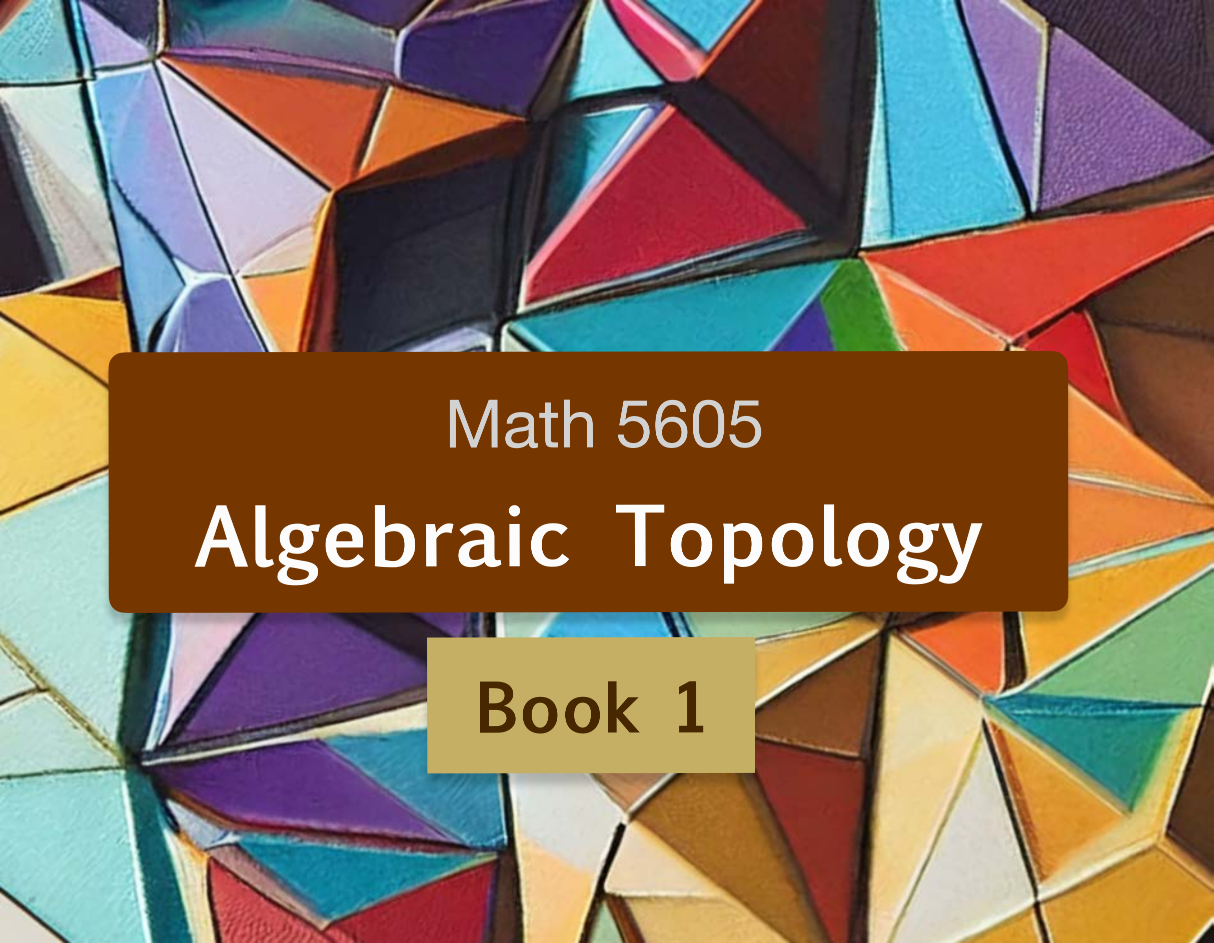


Math 5605

Algebraic Topology

Book 1

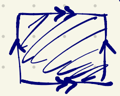
If X, Y are top. spaces, $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a <u>homeomorphism</u> if f is bijective and f, f^{-1} are continuous.

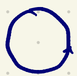
$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

$\mathbb{R}^2 \not\cong S^2$ since S^2 is compact; $\mathbb{R}^2$ is not.

2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\} =$ 

$S^2 \not\cong T^2 = S^1 \times S^1 =$  $=$ 

S^2, T^2 are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$T^1 = S^1 =$  $=$ circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \not\cong T^2$ because S^2 is simply connected whereas T^2 is not.

In S^2, every closed path can be "continuously shrunk" to a point (homotopic to a point, i.e. null-homotopic)

X  $\not\cong$ $T = T^2$  although both surfaces are compact, connected, not simply connected

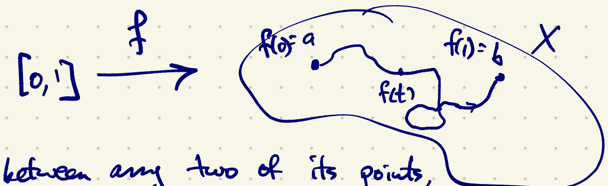
These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of <u>homotopy equivalence</u>.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$.

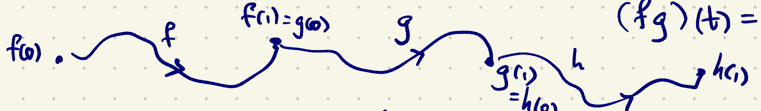
Given points $a, b \in X$ (a topological space), a <u>path</u> from a to b is a (continuous) function $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.



All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0)=0$, $\varphi(1)=1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:



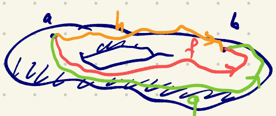
$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \tfrac{1}{2} \\ g(2t-1), & \tfrac{1}{2} \le t \le 1. \end{cases}$$

$(fg)h$ is the same path as $f(gh)$ after reparameterization:

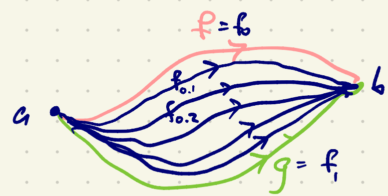
$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \tfrac{1}{4}] \\ g(4t-1), & t \in [\tfrac{1}{4}, \tfrac{1}{2}] \\ h(2t-1), & t \in [\tfrac{1}{2}, 1] \end{cases} \qquad (f(gh))(t) = \begin{cases} f(2t), & t \in [0, \tfrac{1}{2}] \\ g(4t-2), & t \in [\tfrac{1}{2}, \tfrac{3}{4}] \\ h(4t-3), & t \in [\tfrac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$
$(s,t) \mapsto f(s,t) = f_s(t)$
such that $f_0 = f$ i.e. $f_0(t) = f(t)$
$f_1 = g$ i.e. $f_1(t) = g(t)$
$f_s(0) = a$
$f_s(1) = b$ for all $s \in [0,1]$

This is a homotopy from f to g.



f, g are homotopic
but h is not homotopic to f, g



We say f, g are homotopic if there is a continuous family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi: [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f\circ\varphi : [0,1] \to X$ is homotopic to f. A homotopy from f to $f\circ\varphi$ is

$$[0,1]^2 \longrightarrow X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{\in [0,1]}) = f_s(t)$$

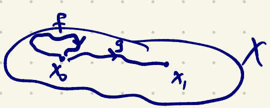
$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
$$f_s(0) = f(\cancel{(1-s)\cdot 0} + \cancel{s\cdot\varphi(0)}) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot \underline{\varphi(1)}) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)



Fix g path in X from x_0 to x_1

An isomorphism $\phi: \pi_1(X, x_0) \to \pi_1(X, x_1)$

$$f \xmapsto{\ \phi\ } \bar{g} f g$$
$$g f \bar{g} \xleftarrow{\ \phi^{-1}\ } h$$

$$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$
$f_1, f_2 \in \pi_1(X, x_0)$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f\gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is

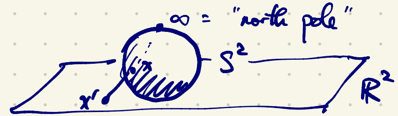
$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$

(same path in the reverse direction)

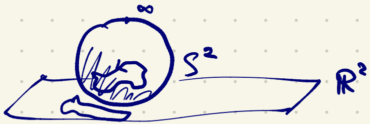
$f\bar{f} = \bar{f} f = \gamma$ = null path

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)



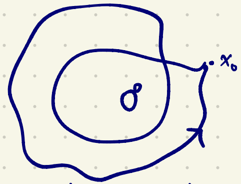
$x \longmapsto x'$ is stereographic projection from the north pole ∞



See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
punctured plane



follows from the fact that

$\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S^1$

$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).

Hatcher writes $X \approx Y$ for homeomorphic.

- retraction — "def. retraction in the weak sense"
- deformation retraction
- strong deformation retraction — "def. retraction".
- homotopy
- relative homotopy
- homotopy equivalence

Let $A \subseteq X$ (subspace of a top. space).

A <u>retraction</u> $f: X \to A$ is a (continuous) map such that $f|_A = id_A = \mathbb{1}_A$ i.e. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a <u>retract</u> of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \to \{a\}$, $x \mapsto a$ is a retraction.

$\mathbb{R}^2 \to x\text{-axis}$, $(x,y) \mapsto (x,0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \to S^1$.

(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to $\underset{X \to A}{\text{a retraction}}$), $A \subseteq X$.

i.e. $f : [0,1] \times X \longrightarrow X$

$$f(t, x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = id_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

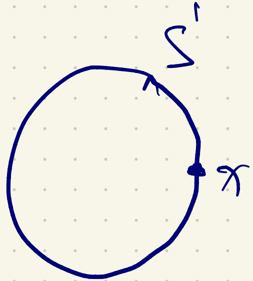
$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$

$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.



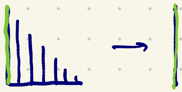
Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f : [0,1] \times X \to X$:

$f_0(x) = x$ i.e. $f_0 = id_X$

f_1 is a retraction $X \to A$

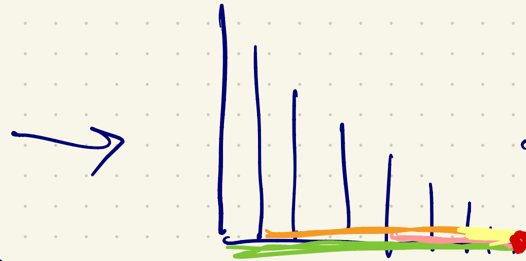
$f_t|_A = id_A$ for all $t \in [0,1]$



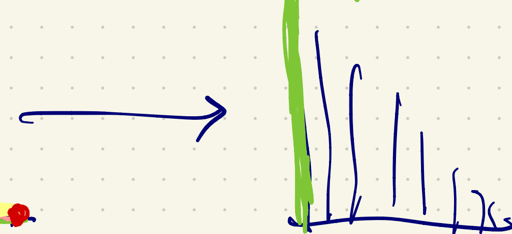
def. retract
but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $X = ([0,1] \times \{0\}) \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} (\{r\} \times [0, 1-r])$

X has a def. ret. to $A := \{0\} \times [0,1]$.



$0 \le t \le \frac{1}{2}$



$\frac{1}{2} \le t \le 1$

This is a def. retract but not a strong def. retract.

There is no strong def. retract $X \to A$.

Let $f_0, f_1 : X \to Y$ be maps. A <u>homotopy</u> from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0,x) = f_0(x)$, $f(1,x) = f_1(x)$.

$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 $(f_0, f_1 : X \to Y)$ is a homotopy $f : [0,1] \times X \to Y$ such that $f_t(a)$ is independent (constant) of $t \in [0,1]$ for all $a \in A$.

A path from a to b in X is a homotopy from a to b.

Given two paths f_0, f_1 in X from a to b $(a, b \in X)$ is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $f\circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$ $f(x) = 0$ for all $x \in \mathbb{R}^n$
$g(0) = 0 \in \mathbb{R}^n$

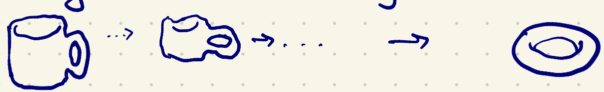
$g\circ f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ $f \circ g : \{0\} \to \{0\}$, $0 \mapsto 0$

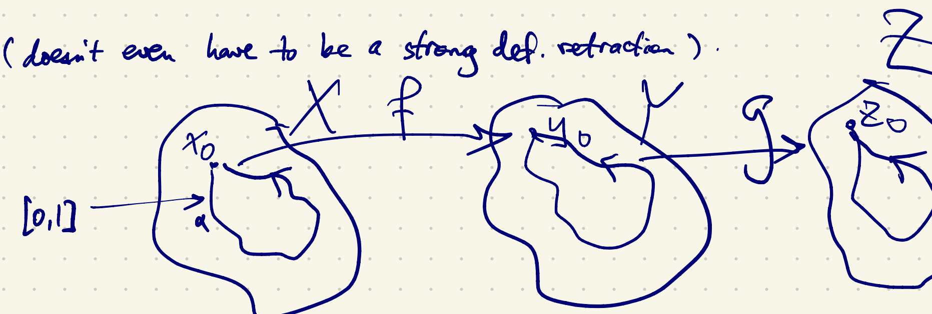
A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ to $id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$

Not relative to any subspace necessarily.

$h_0 = g\circ f$
$h_1 = id_{\mathbb{R}^n}$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).





S^1 is not homotopic to a point.
(S^1 is not null homotopic ; not contractible)

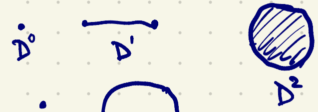
If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

$\alpha : [0,1] \to X$ gives $f_*\alpha = f \circ \alpha : [0,1] \to Y$ $(g\circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

D^n = closed ball in $\mathbb{R}^n$
"disk"

$S^n = \partial D^{n+1}$ = n-sphere



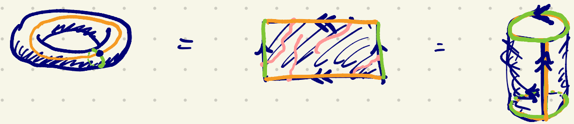
D^0 D^1 D^2

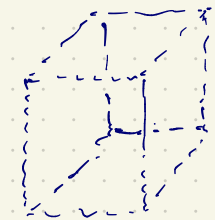
$S^0 = D^0 \sqcup D^0$ S^1

A CW complex is formed from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \dots$

X^n is a union of $\underbrace{\text{copies of } D^n}_{D^n_\alpha}$ with the boundaries of D^n_α attached to X^{n-1} via attaching maps.

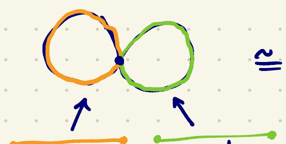
Eg. Torus $T^2 = S^1 \times S^1 =$





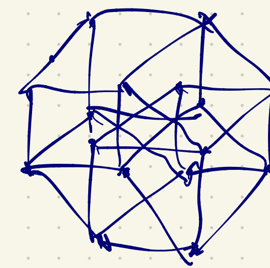
$X^0 = \bullet = D^0$

$X^1 =$



$\cong S^1 \vee S^1$

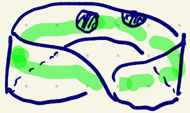
$D^1_0 \sqcup D^1_1$



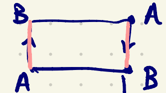
X^2

$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

Möbius strip  not homeomorphic to cylinder $S^1 \times [0,1]$ =  = 



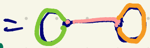
$X^0 = \cdot \quad \cdot$

$X^1 =$  

$X^2 =$

not orientable

Cylinder: $X^0 = \cdot \quad \cdot \quad D^0 \sqcup D^0$

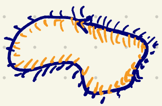
$X^1 =$ 

$X^2 =$ 

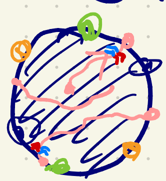
orientable

Both are homotopy equivalent to S^1 (def. retract to S^1)

Both have $\mathbb{Z}$ as fund. gp.



$P^2\mathbb{R}$ (or $\mathbb{RP}^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified

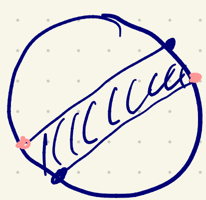


D^2

(non-orientable surface)

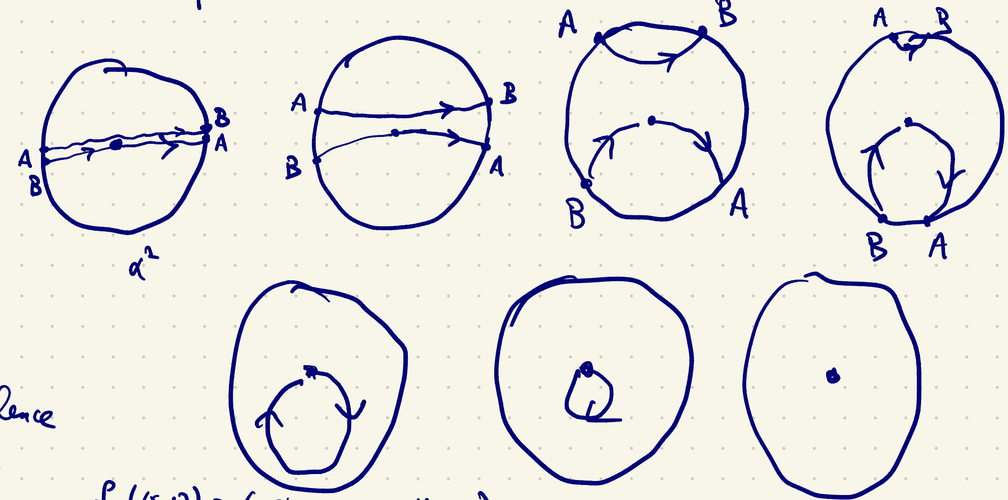
$P^2\mathbb{R} = D^2$ glued to a Möbius strip

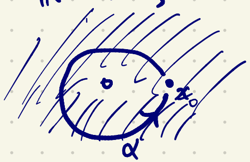


$\pi_1(P^2\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

α is not homotopic to the null path γ. α^2 is homotopic to γ



$\mathbb{R}^2 - \{0\} \simeq S^1$



A homotopy equivalence

$\mathbb{R}^2 - \{0\} \xrightarrow{f_1} S^1$ $\qquad f_1((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$f_t(v) = (1-t)v + t\frac{v}{|v|}$ $\qquad f_1(v) = \frac{v}{|v|}$

$f_t : \mathbb{R}^2 - \{0\} \longrightarrow \mathbb{R}^2 - \{0\}$ $\qquad$ strong def. retraction since $f_t|_{S^1} = id_{S^1}$ for all $t \in [0,1]$.

$f_0 = id_{\mathbb{R}^2 - \{0\}}$

f_1 is a retraction to S^1

$\{\alpha^n : n \in \mathbb{Z}\}$

$\parallel$

$\langle\alpha\rangle = \pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$ $\qquad$ free group on one generator

$\pi_1(S^1) \cong \mathbb{Z}$

Given a closed path α in S^1 with base point $1 \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$

define $w(\beta) = \frac{1}{2\pi i}\int_\beta \frac{dz}{z} = \frac{1}{2\pi i}\int_{\beta(0)}^{\beta(1)} \frac{dz}{z}$

$\beta : [0,1] \to S^1$

$\beta(0) = 1$

$\beta(e^{2\pi i\theta}) = \theta + n$

$w(\alpha) = 1$

$w(\alpha^n) = n$

w is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$.

$\int \frac{dz}{z} = \ln|z| + 2\pi i \arg z$

In $\mathbb{C} - \{0\}$ the same argument works

$\mathbb{R}^2 - \{O\}$ $\quad O = (0,0)$

$\pi_1(\mathbb{R}^2 - \{O\}) = \langle \alpha \rangle \cong \mathbb{Z}$

k-punctured plane $X = \mathbb{R}^2 - \{\overbrace{A_1, \cdots, A_k}^{\text{distinct}}\}$

$\pi_1(X) = F_k = \mathrm{Free}(\{\alpha_1, \cdots, \alpha_k\})$

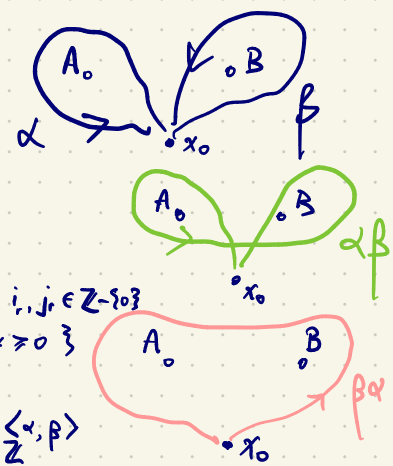
$X \simeq \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{k}$

$X = \mathbb{R}^2 - \{A, B\}$

$\pi_1(X) = \{\alpha^{i_0}\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{j_k},$
$\alpha^{i_0}\beta^{j_0}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{j_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k} \;:\; i_r, j_r \in \mathbb{Z} - \{0\}$
$k \geq 0\}$

$\pi_1(X)$ is the free group on two generators i.e.

$F_2 = \mathrm{Free}(\{\alpha, \beta\}) = \langle \alpha, \beta \rangle$
$= \langle \alpha \rangle * \langle \beta \rangle = \mathbb{Z} * \mathbb{Z}$



<u>The Van Kampen Theorem</u> gives a presentation for $\pi_1(X)$ when X is suitably described in terms of smaller pieces.

A <u>presentation</u> for a group G expresses G as a homomorphic image of a free group F i.e. $G \cong F/N$, $N \trianglelefteq F$.

Let X be a set of generators of G ($X \subseteq G$, $\langle X \rangle = G$).

$\mathrm{Free}(X) \longrightarrow G$ is a surjective homomorphism; N is its kernel.

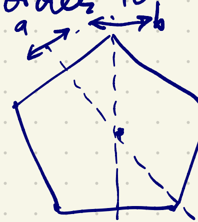
$G = \langle x_1, \dots, x_k : \overbrace{r_1, \dots, r_m}^{\in F} \rangle$ is a <u>presentation</u> for G if $X = \{x_1, \dots, x_k\}$ is a set of k symbols, $F = \mathrm{Free}(X)$ (the free group on $x_1, \dots, x_k$).

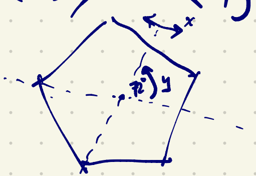
Let N be the smallest normal subgp of F containing $r_1, \dots, r_m$ i.e. the normal closure of $\langle r_1, \dots, r_m \rangle \le F$

i.e. the subgp. of F generated by $r_1, \dots, r_m$ <u>and</u> their conjugates in F

$N = \langle h r_i h^{-1} : i = 1, \dots, m;\ h \in F \rangle$. (When there are k generators and m relators, we say G is <u>finitely presented</u>.)

eg. the dihedral group of order 10 is $D_{10} \cong \langle a, b : a^2, b^2, (ab)^5 \rangle \cong \langle x, y : x^2, y^5, xyx^{-1}y \rangle$





$xyx^{-1}y = 1$

$\Longleftrightarrow xy = y^{-1}x$

$\Longleftrightarrow xyx^{-1} = y^{-1}$

$$D_{10} \cong \langle a,b : a^2, b^2, (ab)^5 \rangle \underset{\phi}{\longrightarrow} \langle x,y : x^2, y^5, xyx^{-1}y \rangle$$

$x = a$

$y = ab$

$x^2 = a^2 = 1$

$y^5 = (ab)^5 = 1$

$xyx^{-1} = a \cdot ab \cdot a^{-1} = ba$

whereas $y^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba$

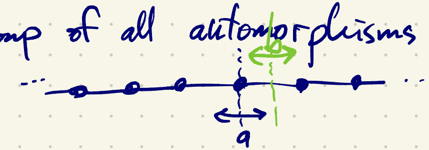
$\longleftarrow$ check.

A free product $G * H$ is the set of all words from elements in G and H having no relations between elements in G and elements in H (except if one of these elements is the identity).

$$\mathbb{Z} * \mathbb{Z} = \underbrace{\langle a \rangle}_{\text{infinite cyclic}} * \underbrace{\langle b \rangle}_{\text{infinite cyclic}} = \text{Free}\{a,b\} = \langle a,b \rangle = \{a^{i_1}b^{j_1}a^{i_2}b^{j_2}\cdots a^{i_k}b^{j_k}, \ldots\}$$

$$\langle a : a^2 \rangle * \langle b : b^2 \rangle = \langle a,b : a^2, b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \ldots\} = D_\infty$$

$$\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle x,y : x^2, \underbrace{xyx^{-1} = y^{-1}}_{xyx^{-1}y = 1} \rangle = \text{group of all automorphisms of}$$



ie. $\langle X : R \rangle = \langle x_1, \ldots, x_m : r_1, \ldots, r_k \rangle$

$\langle Y : S \rangle = \langle y_1, \ldots, y_n : s_1, \ldots, s_l \rangle$

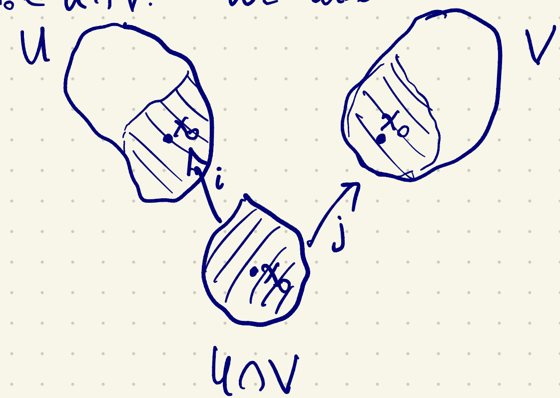
$$\langle X : R \rangle * \langle Y : S \rangle = \langle X \cup Y : R \cup S \rangle = \langle x_1, \ldots, x_m, y_1, \ldots, y_n : r_1, \ldots, r_k, s_1, \ldots, s_l \rangle$$

Free products with amalgamation: add more relations involving x_i's and y_j's e.g.

$$D_{10} = \langle a, b : a^2, b^2, (ab)^5 \rangle = \underbrace{\langle a : a^2 \rangle}_{\text{cyclic of order 2}} *_{(ab)^5} \underbrace{\langle b : b^2 \rangle}_{\text{cyclic of order 2}} = D_\infty \big/ \text{Normal closure of } \langle (ab)^5 \rangle$$

Let X be a path-connected top. space covered by two open sets U, V. Since X is connected, $U \cap V \neq \emptyset$. Pick $x_0 \in U \cap V$. We also assume $U \cap V$ is path-connected.

X:

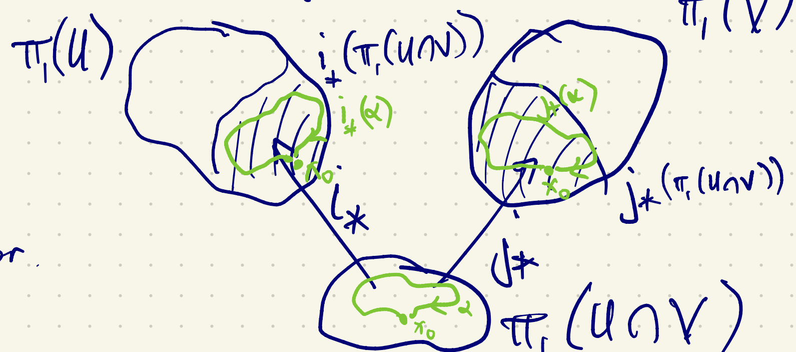


i, j inclusion maps as shown (injective, continuous)

Theorem (Van Kampen, special case)

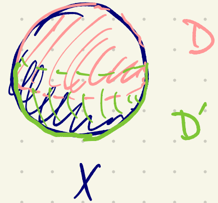
$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

where the amalgamation over $\pi_1(U \cap V)$ is given by; for all $\alpha \in \pi_1(U \cap V)$, identify $i_*(\alpha)$ with $j_*(\alpha)$ i.e. $i_*(\alpha)^{-1} j_*(\alpha)$ is a new relator.



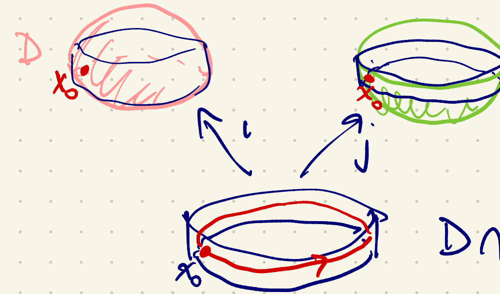
This induces group homomorphisms i_*, j_* as shown.

Eg. $X = S^2 = D \cup D'$



$\pi_1(S^2) = \pi_1(D) \underset{\square}{*} \pi_1(D') = 1$

$= \underbrace{\pi_1(D) * \pi_1(D')}_{\text{amalgamation}} = 1.$ trivial



$D \cap D' \simeq S^1$

$\pi_1(D) \cong \pi_1(D') = 1$ trivial

$\pi_1(D \cap D') = \langle \alpha \rangle$

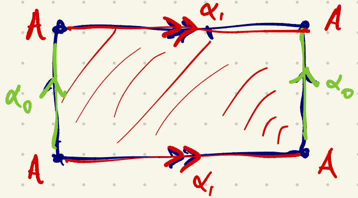
$$\begin{tikzcd}
\pi_1(D) & & \pi_1(D') \\
& \pi_1(D \cap D') \arrow[ul, "i_*"] \arrow[ur, "j_*"'] &
\end{tikzcd}$$

i_*, j_* not one-to-one

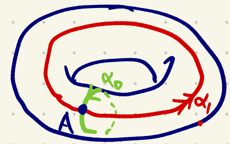
$i_*(\alpha) = 1$

$j_*(\alpha) = 1.$

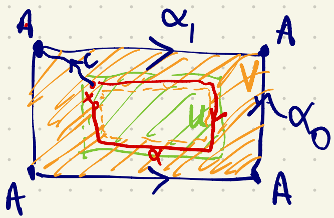
$T^2 = S^1 \times S^1 =$

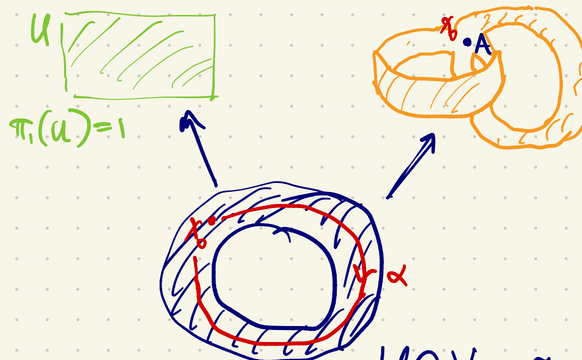


$=$



$T^2 =$





$V \simeq S^1 \vee S^1$

$\pi_1(V) = \langle a, b \rangle$

Free group on two generators

$\pi_1(U) * \pi_1(V)$

$= 1 * \langle a, b \rangle$

$= \langle a, b \rangle$

$U \cap V \cong S_1 \times (-\varepsilon, \varepsilon)$

$\pi_1(U \cap V) = \langle \alpha \rangle$

Then identify α with 1 due to the inclusion $U \cap V \subset U$. In V, α is homotopic to $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$

Then identify $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$ with α.

So we identify $c\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0 c^{-1}$ with 1.

So identify $\alpha_1\alpha_0^{-1}\alpha_1^{-1}\alpha_0$ with 1.

identify $\alpha_1\alpha_0^{-1}$ with $\alpha_0^{-1}\alpha_1$; $\alpha_0\alpha_1$ with $\alpha_1\alpha_0$.

So identify ab with ba.

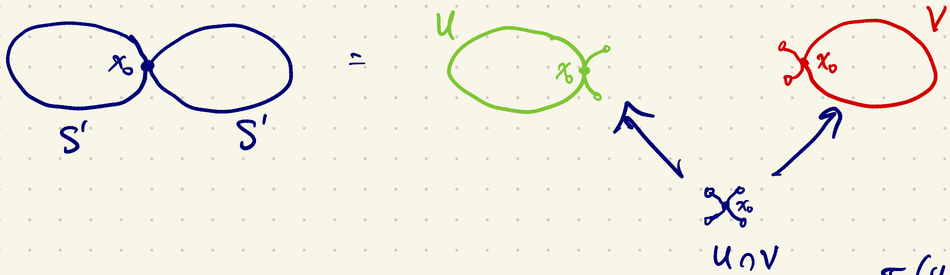
$\pi_1(T^2) = \langle a, b \rangle / \text{"}ab = ba\text{"} \cong \mathbb{Z} \times \mathbb{Z}$

$= \langle a, b \rangle /$ Normal closure of $aba^{-1}b^{-1}$

$=$ the abelianization of $\langle a, b \rangle$

$\pi_1(S^1 \vee S^1) = \pi_1(S^1) * \pi_1(S^1)$ also follows from Van Kampen's Theorem.

If X and Y are path-connected top. spaces then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.



$U \simeq V \simeq S^1$

$\pi_1(U \cap V) = 1$

$$\pi_1(S_1 \vee S_1) = \pi_1(U) \underset{\square}{*} \pi_1(V) = \mathbb{Z} * \mathbb{Z} = \text{free group on two generators.}$$

In Tp, the category of top. spaces, the coproduct of X and Y is $X \sqcup Y$ = disjoint union; product is $X \times Y$.

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f_1"] \arrow[r, "i"'] & X \sqcup Y \arrow[u, dashed] & Y \arrow[ul, "f_2"'] \arrow[l, "j"]
\end{tikzcd}$$

The category of pointed top. spaces i.e. pairs (X, x_0), $x_0 \in X$, X nonempty top. space

The coproduct $(X, x_0) \sqcup (Y, y_0)$ = disjoint union $X \sqcup Y$ with x_0, y_0 identified

denoted as $(X, x_0) \vee (Y, y_0) = (X \sqcup Y,\ x_0 = y_0)$

A morphism $(X, x_0) \longrightarrow (Y, y_0)$ is a (continuous) map $X \to Y$ satisfying $x_0 \longmapsto y_0$.

We have a functor $\{\text{pointed top. spaces}\} \longrightarrow Grp = \{\text{groups}\}$

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

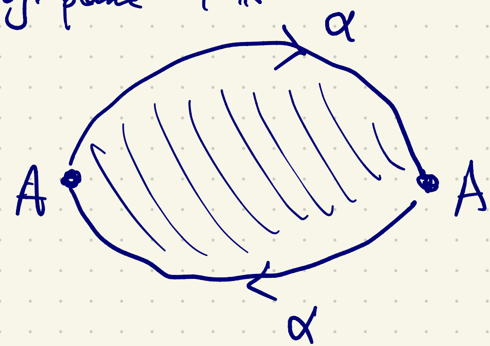
$$\begin{tikzcd}
(X, x_0) \arrow[r, "f"] \arrow[d, "F"'] \arrow[rr, bend left, "g \circ f"] & (Y, y_0) \arrow[r, "g"] \arrow[d, "F"] & (Z, z_0) \arrow[d] \\
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g \circ f)_* = g_* \circ f_*"'] & \pi_1(Y, y_0) \arrow[r, "g_*"] & \pi_1(Z, z_0)
\end{tikzcd}$$

$$\begin{tikzcd}
(X, x_0) \vee (Y, y_0) \arrow[d, "F"] \\
F((X, x_0) \vee (Y, y_0)) = F((X, x_0)) * F((Y, y_0))
\end{tikzcd}$$

coproduct in pointed top. spaces is wedge sum.

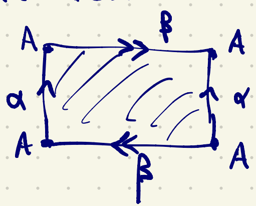
coproduct in Grp is just free product

Proj. plane $P^2\mathbb{R}$



$$\pi_1(P^2\mathbb{R}) = \langle \alpha : \alpha^2 \rangle = \langle\alpha\rangle / \langle\alpha^2\rangle = \text{group of order } 2$$
$$= \mathbb{Z}/2\mathbb{Z}$$

Klein Bottle K



$\pi_1(K) = \langle \alpha, \beta : \alpha\beta\alpha^{-1}\beta \rangle = \langle\alpha,\beta\rangle / N$ where N is the subgp of $\langle\alpha,\beta\rangle$ generated by $\alpha\beta\alpha^{-1}\beta$ and its conjugates

Force $\alpha\beta\alpha^{-1}\beta = 1$ in the quotient gp.

ie. $\alpha\beta\alpha^{-1} = \beta^{-1}$

So $\pi_1(K) = \mathbb{Z} \rtimes \mathbb{Z}$

$\langle\beta\rangle$ $\langle\alpha\rangle$ where α inverts β.

$(k,l)(r,s) = \begin{cases} (k+r,\ l+s) & \text{if } r \text{ is even} \\ (k+r,\ -l+s) & \text{if } r \text{ is odd} \end{cases}$

Every element can be uniquely expressed as $\alpha^k\beta^l$, $k, l \in \mathbb{Z}$

$$(\alpha^k\beta^l)(\alpha^r\beta^s) = \begin{cases} \alpha^{r+k}\beta^{l+s} & \text{if } r \text{ is even;} \\ \alpha^{r+k}\beta^{-l+s} & \text{if } r \text{ is odd.} \end{cases}$$

$\alpha\beta^3\alpha^{-1} = \alpha\beta\beta\beta\alpha^{-1} = (\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1})(\alpha\beta\alpha^{-1}) = \beta^{-1}\beta^{-1}\beta^{-1} = \beta^{-3}$ so $\alpha\beta^3 = \beta^{-3}\alpha$

$\alpha\beta^k\alpha^{-1} = \beta^{-k}$

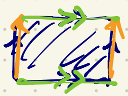
$\beta^k = \alpha^{-1}\beta^{-k}\alpha$

$\{\alpha^k\beta^l : k,l\in\mathbb{Z}\} = \pi_1(K) = \mathbb{Z}\rtimes\mathbb{Z}$

The elements with k even form an abelian subgp $\langle\alpha^2,\beta\rangle \cong \mathbb{Z}\times\mathbb{Z}$ (free abelian of rank 2).
and the quotient $\pi_1(K)/\langle\alpha^2,\beta\rangle$ is order 2.

Leading up to trefoil:

$T^2 \hookrightarrow \cancel{\mathbb{R}^3}\ S^3$



This surface partitions $\cancel{\mathbb{R}^3}\ S^3$ into two solid tori