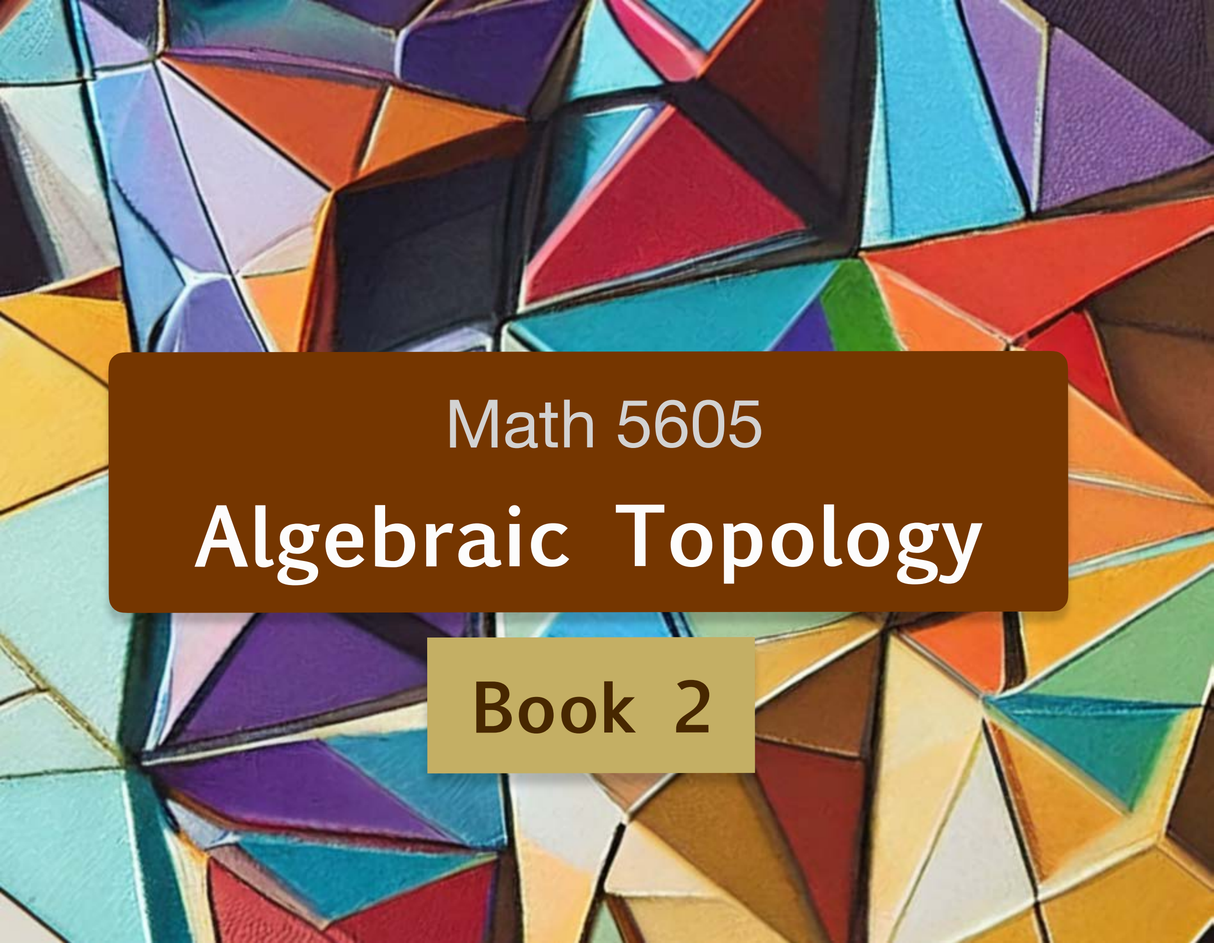
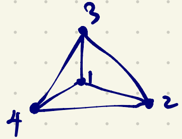


Math 5605

Algebraic Topology

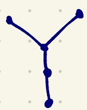
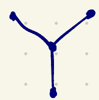
Book 2

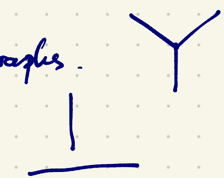
When are two covering maps of X equivalent? Say $Y \xrightarrow{f} X$, $Y' \xrightarrow{f'} X$ are covering maps.

e.g. $X =$ 

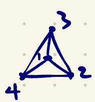
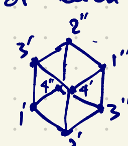
Graph i.e. combinatorial graph with vertices $\{1,2,3,4\}$ and edges $\{\{1,2\}, \{1,3\}, \ldots, \{3,4\}\}$.

X is the geometric realization of this graph formed as a disjoint union of copies of $[0,1]$ with endpoints identified as required by the picture.

 and  have the same geometric realization although they are different graphs.

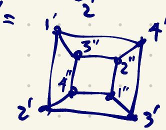


A homomorphism of graphs $\Gamma \xrightarrow{f} \Gamma'$ is a map $V(\Gamma) \xrightarrow{f} V(\Gamma')$ preserving adjacency i.e. $x \sim y$ in Γ $(x, y \in V(\Gamma)$, $\{x,y\} \in E(\Gamma))$ $\Rightarrow$ $f(x) \sim f(y)$ in Γ'. A covering map of graphs is a homomorphism inducing a bijection on the neighbours of each vertex of Γ (and the preimage of the neighbours of each vertex $y \in \Gamma'$ are copies of the neighbors of y.)

Eg. a double cover of $X =$  is the 1-skeleton of the cube $Y =$ 

$Y \longrightarrow X$

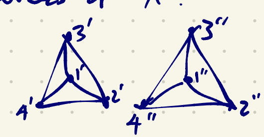
$1', 1'' \longmapsto 1$
$2', 2'' \longmapsto 2$
$3', 3'' \longmapsto 3$
$4', 4'' \longmapsto 4$

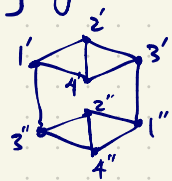
$Y =$ 

The covering space is $Y \to X$ (or informally just Y).

Some other double covers of X:

Trivial double cover

$X \sqcup X$ 



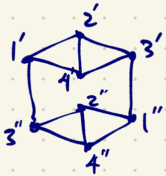
$Z \to X$

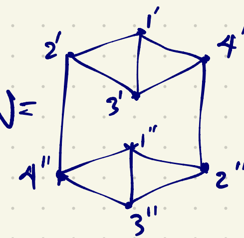
$1', 1'' \mapsto 1$
$\vdots$
$4', 4'' \mapsto 4$

When are two covers of X equivalent (isomorphic, i.e. essentially the same)?

Let $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ be covering spaces of X. We say $\theta : X_1 \to X_2$ is an <u>equivalence</u> or <u>isomorphism</u> of the two covers if θ is a homeomorphism and $p_2 \circ \theta = p_1$, i.e. this diagram commutes.

$$\begin{tikzcd} X_1 \arrow[rr, shift left, "\theta"] \arrow[dr, "p_1"'] & & X_2 \arrow[ll, shift left, "\theta"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Eg.  $Z \to X$, $1', 1'' \mapsto 1$, ⋮ $4', 4'' \mapsto 4$ is not equivalent to $Y \to X$. But what about

$W =$ 

$W \to X$

$1', 1'' \mapsto 1$

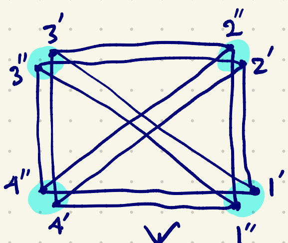
$2', 2'' \mapsto 2$

$3', 3'' \mapsto 3$

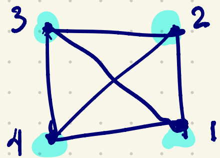
$4', 4'' \mapsto 4$

Is this equivalent to $Z \to X$? No...

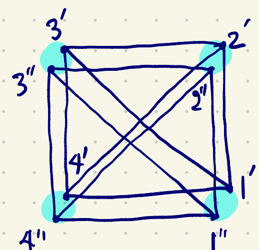
Another picture of these covers:



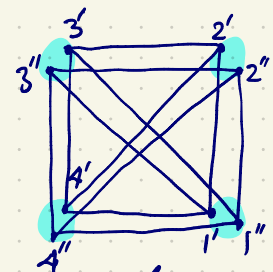
Y



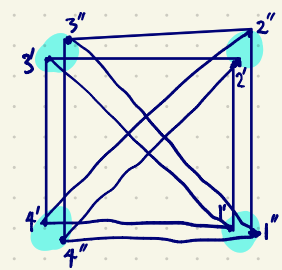
X



$X \sqcup X$

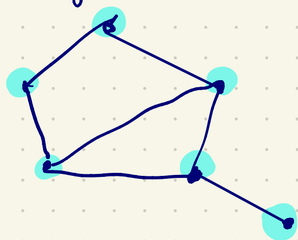


Y



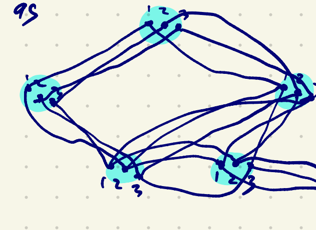
W

To construct an r-fold cover of X, created one copy of $[r] = \{1,2,\cdots,r\}$ for each vertex of X. Then for each edge of X, match up the corresponding fibres in the cover using a chosen permutation.

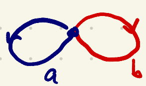
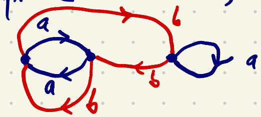


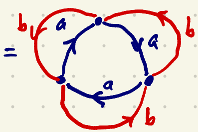
X

A triple cover $Y \to X$ is constructed as



Why is 2 more special than other positive integers (the oddest prime of all)?

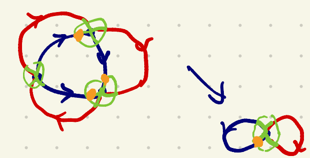
Consider $X =$  has many triple covers including $Y_1 =$ 

$Y_2 =$ 

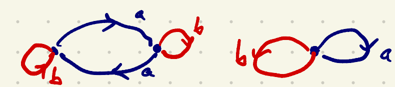
The covering maps $Y_1 \to X$ and $Y_2 \to X$ are not equivalent.

An equivalence between $Y \to X$ and itself (automorphism of the cover) is a <u>deck transformation</u>. This is the same as a homeomorphism $Y \to Y$ which preserves fibres.

In the example above, $Y_2 \to X$ has 3 automorphisms (deck transformations)
But $Y_1 \to X$ has only one (trivial) deck transformation.



In a connected r-fold cover, there are at most r deck transformations.
If equality holds, the covering space is <u>normal</u> or <u>Galois</u>.
(not the same as normal space in point set topology).
Double covers are always normal.

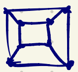
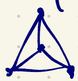
$Y_3 =$ 

In group theory, subgroups of index 2 are normal.

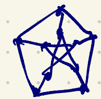
In the case of (separable) extensions of fields, the extension is normal.

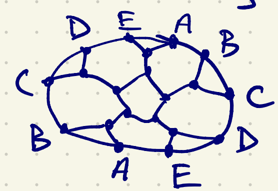
For a field extension $E \supseteq F$, the degree of the extension is $[E:F]$: dimension of E as a vector space over F. The number of F-automorphisms of E (i.e. $\sigma: E \to E$ automorphism fixing $\sigma(a) = a$ for all $a \in F$) is at most $[E:F]$. If this number is equal, it's a normal or Galois extension. Extensions of degree 2 (quadratic extensions) are always normal.

Double covers: examples

 $\xrightarrow{\text{2-to-1}}$ 

dodecahedral graph $\longrightarrow$ Petersen

$\longrightarrow$ 



$S^2 \longrightarrow P^2\mathbb{R}$ real proj. plane

S^n is not a top. group unless $n \in \{1,3\}$.

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

$S^3 = \{z \in \mathbb{H} : |z| = 1\}$ $\qquad \mathbb{H} = \{a+bi+cj+dk : a,b,c,d \in \mathbb{R}\}$ $\qquad i^2 = j^2 = k^2 = ijk = -1$

$\cong SU_2(\mathbb{C}) = \left\{A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} : \alpha,\beta,\gamma,\delta \in \mathbb{C},\ AA^* = A^*A = I,\ \det A = 1\right\}$

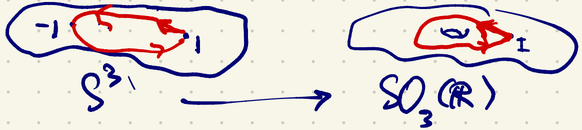
$SO_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I,\ \det A = 1\}$

$O_3(\mathbb{R}) = \{A \in \mathbb{R}^{3\times 3} : AA^T = A^TA = I\}$ has two connected components

Fact: $S^3 \cong SU_2(\mathbb{C}) \longrightarrow SO_3(\mathbb{R})$ is a double cover.

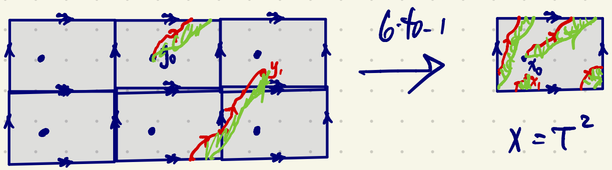
$Z(S^3) = \{\pm 1\}$

$PSU_2(\mathbb{C}) = S^3/Z(S^3) \cong SO_3(\mathbb{R}) \overset{\text{homeomorphism}}{\cong} P^3\mathbb{R}$.



In general for $n \geq 3$, $\pi_1(\underbrace{SO_n(\mathbb{R})}) \cong \mathbb{Z}/2\mathbb{Z}$.

$Spin_n(\mathbb{R}) \longrightarrow SO_n(\mathbb{R})$ is its universal cover: a simply connected double cover constructed from Clifford Algebras (generalizing $\mathbb{H}$).



$Y = T^2$ $\qquad$ $X = T^2$

$f_0 : [0,1] \to X$ path in X from $f(0) = x_0$ to $f(1) = x_1$

$f_1 : [0,1] \to X$ is another path in X from x_0 to x_1 homotopic to f_0

$f_t : [0,1] \to X$

$f : [0,1]^2 \to X$

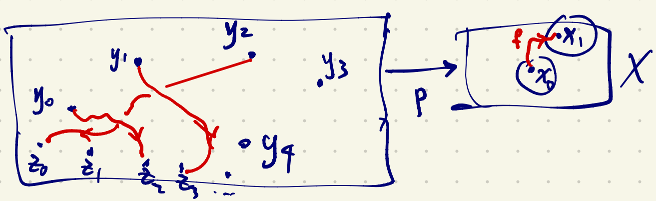
In any covering space $p : Y \to X$ and given any path $f : [0,1] \to X$ starting at $f(0) = x_0$, the path f can be lifted to Y

i.e. there is a path $g : [0,1] \to Y$ such that $f = p \circ g$ i.e.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

and this lift is unique if we say which of the points in $f^{-1}(x_0)$ to take as the starting point for g.

Assuming X is path-connected and $p : Y \to X$ is a path-connected covering space, $X = Y/\sim$ where two points $y_0, y_1 \in Y$ satisfy $y_0 \sim y_1$ iff $p(y_0) = p(y_1)$.



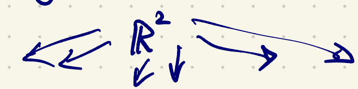
$p^{-1}(x_0) = \{y_0, y_1, y_2, \cdots\}$

$p^{-1}(x_1) = \{z_0, z_1, z_2, \cdots\}$

Every path f in X from x_0 to x_1 gives a bijection between fibres $p^{-1}(x_0) \longrightarrow p^{-1}(x_1)$.

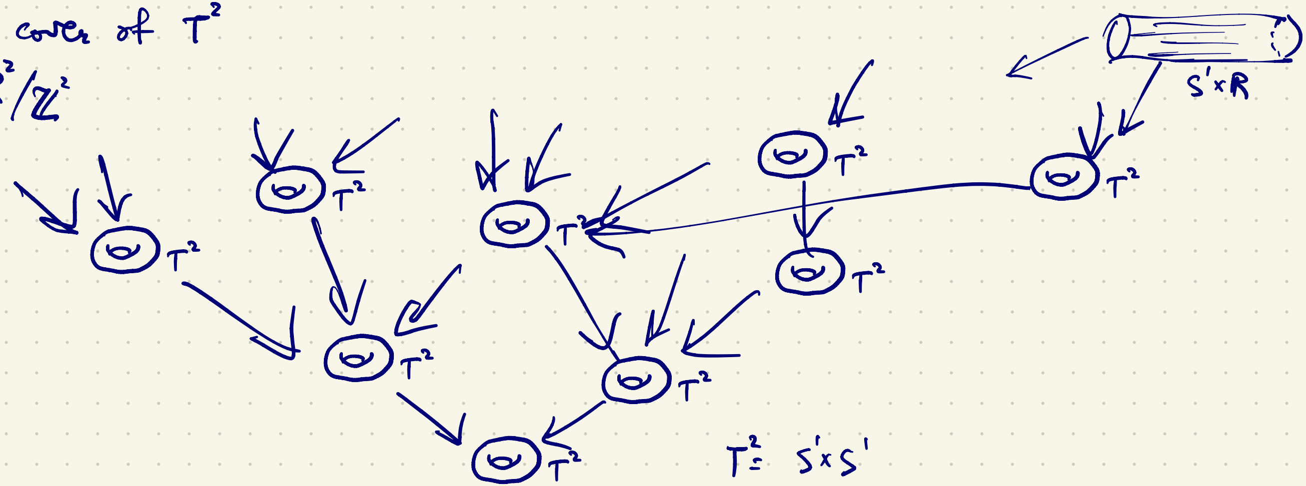
In particular if p is k-to-1 at x_0 i.e. $|p^{-1}(x_0)| = k$ then it is k-to-1 everywhere i.e. $|p^{-1}(x)| = k$ for all $x \in X$.

More generally, if f_t is a homotopy in X and we are given f_0, then every lifting of f_0 to Y extends to a lifting of f_t to Y.



$\mathbb{R}^2$ is the universal cover of T^2

$\mathbb{R}^2 \xrightarrow{p} T^2 = \mathbb{R}^2/\mathbb{Z}^2$

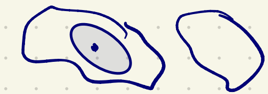


$T^2 = S^1 \times S^1$

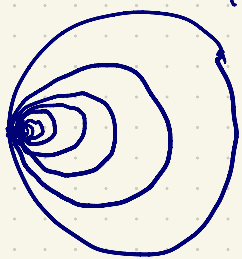
Which top. spaces have a universal cover? (equivalently a simply connected cover)

Let X be a path-connected space. Then X has a path-connected and universal cover iff X is

- path-connected
- locally path-connected
- semi-locally simply connected



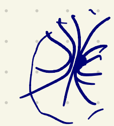
Example of a top. space without a universal cover: Hawaiian earring $\subset \mathbb{R}^2$

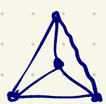


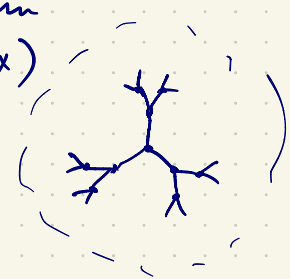
(not a CW complex)

$\not\cong \underbrace{S^1 \vee S^1 \vee S^1 \vee \dots}$

Countable wedge sum (CW complex)

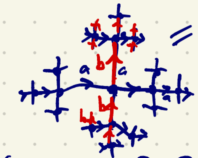


Universal cover of K_4  : trivalent tree



(also the universal cover of any trivalent connected graph)
i.e. regular of degree 3 connected

Universal cover of any connected regular graph of degree 4 is

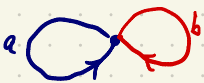


$= \tilde{X} \xrightarrow{p} X$

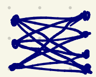
Cayley graph of Free$\{a,b\} = G$

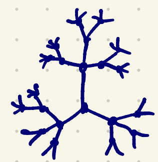
Vertices correspond to elements of G.

Every vertex $w \in G$ has edges to wa, wa^{-1}, wb, wb^{-1}.

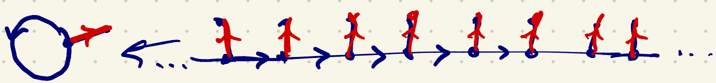


Universal cover of $K_{3,4}$





$\tilde{X} = X/G$



$P^2\mathbb{R}$ has S^2 as its universal cover.

$S^2 \longrightarrow P^2\mathbb{R} = S^2/G$

quotient of S^2 by the antipodal relation.

$G = \{1, -1\}$ acts on S^2

$1x = x$

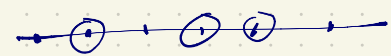
$(-1)x = -x$ (antipode of x)

$X/_\sim$ = partition of X into equivalence classes of the equiv. relation "$\sim$"

X/G = partition of X into the orbits of G
($x \sim xg$ or $g(x)$)
for all $g \in G$.

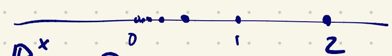
$X \longrightarrow X/_\sim$

$\mathbb{R}/\mathbb{Z} \cong S^1$



$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2 = S^1 \times S^1$

A non-discrete action of $\mathbb{Z}$ on $\mathbb{R}$ eg. $\langle 2 \rangle = \{2^k : k \in \mathbb{Z}\} < \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$



G acts <u>discretely</u> on X if for every $x \in X$ there is an open nbhd U of x such that the only $g \in G$ mapping $x \longmapsto x^g \in U$ is $g = 1$.

$G = \{x \longmapsto 2^k x + l \ : \ k, l \in \mathbb{Z}\}$ is non-discrete

---

If X is "nice" (path-connected, locally path-connected, SLSC) then X has a simply connected (and path-connected) cover which is a universal cover. It is unique up to isomorphism of covering spaces.

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"'] \arrow[r] & Y \arrow[dl] \\ X = \tilde{X}/G & \end{tikzcd}$$

$\tilde{X}$ universal cover. Fix $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \in \tilde{X}$.

$G \cong \pi_1(X, x_0)$.

Every other covering space $Y \to X$ (path-connected) has the form $Y = \tilde{X}/H$, $H \leqslant G$.

eg.

$$\begin{tikzcd} \mathbb{R} = \tilde{S^1} \arrow[d, "p"'] \arrow[r] & \mathbb{R}/H = S^1 \arrow[dl, "p_k"] \\ S^1 & \end{tikzcd}$$

$t \mapsto e^{2\pi i t/k}$

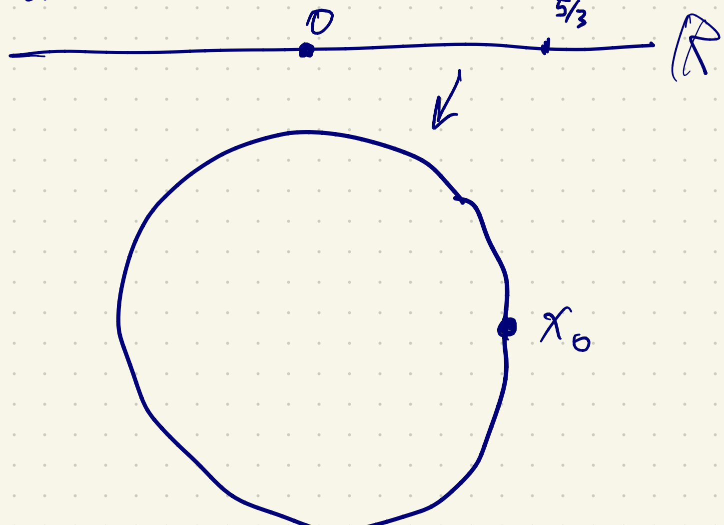
$p : t \longmapsto e^{2\pi i t}$

$p_k : z \longmapsto z^k$

$G = \mathbb{Z}$

$H \leq G$ has the form $H = k\mathbb{Z}$, $k \in \mathbb{Z}$.

Construction of the universal cover $\tilde{X} \to X$?



$\tilde{X} = \{$paths in X starting at the chosen base point $x_0\}/\sim$

i.e. paths up to homotopy with fixed starting and ending point

If $f \in \tilde{X}$ then

$p(f) = f(1) \in X$ (the endpoint of f)

Cohomology. Consider a sequence of vector spaces over F given by

$$\cdots \xrightarrow{\partial_4} V_3 \xrightarrow{\partial_3} V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} 0$$

or

$$0 \xrightarrow{0} V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} V^3 \xrightarrow{d^3} \cdots$$

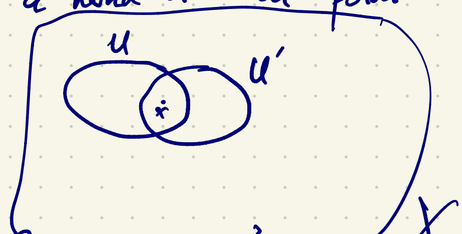
∂_i, d^i linear transformations (more generally V_i, V^i are modules over a ring R and ∂_i, d^i are R-homomorphisms i.e. $d(av+bw) = a\,dv + b\,dw$, $a,b \in F$; $v,w \in V$)

V^i or V_i has i just an index for purposes of reference.

If $\partial_i \circ \partial_{i+1} = 0$ then ∂ is a boundary map and the sequence of V_i's is a complex.
(similarly if $d^{i+1} \circ d^i = 0$, d is a coboundary map.)

Notable example: differential forms

Let X be a real (smooth) n-manifold. In a nbhd of each point $x \in X$, $x \in U \subseteq X$, we have local coordinates $(x_1, \dots, x_n) = x$.



$R = C^\infty(U) = \{$smooth real-valued functions on $U\}$.

$V^0 = R$. $\quad d: V^0 \to V^1 = \{$differential 1-forms on $U\} = \{f_1 dx_1 + f_2 dx_2 + f_3 dx_3 + \cdots + f_n dx_n : f_i \in R\}$

V^1 is a vector space over $\mathbb{R}$ (∞-dimensional) but n-dimensional as module over R