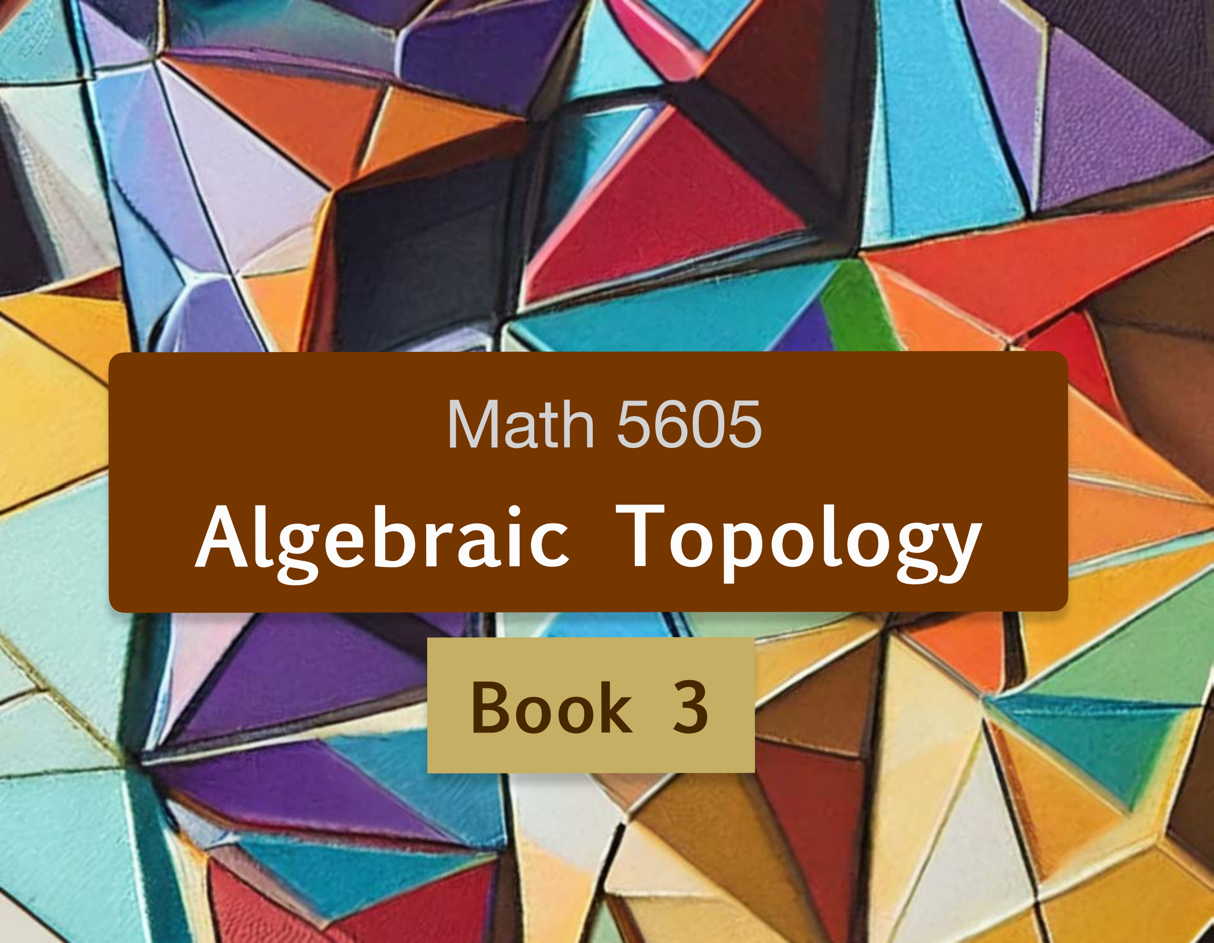


Math 5605

Algebraic Topology

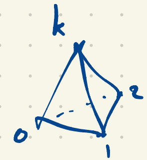
Book 3

Cup product for simplicial cohomology $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains : $(k = 0, 1, 2, 3, \dots)$ ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex {all subsets of $\{0,1,2,\dots,k\}$}.

This has a geometric realization  $\xrightarrow{\sigma} X$

$\Delta = \Delta^n = \{\underbrace{(v_0, v_1, \dots, v_n)}_{\text{barycentric coordinates}} : v_i \geq 0,\ \Sigma v_i = 1\} \subset \mathbb{R}^{n+1}$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \longrightarrow X$.

$C_n = \{$n-chains in $X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{$n-cochains in $X\} = \mathrm{Hom}(C_n, R) = \{R\text{-homomorphisms } C_n \longrightarrow R\}$

$\partial : C_n \longrightarrow C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]$ $\qquad \partial^2 = 0,\ (\partial^*)^2 = 0$

$\partial^* : C^{n-1} \longrightarrow C^n$

If $\phi \in C^k$ k-cochain then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$
$\psi \in C^l$ l-cochain

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\,\psi(\sigma|[v_k, \dots, v_{k+l}])$$

$[v_0, \dots, v_{k+l}] \longmapsto \sigma(v_0, \dots, v_{k+l})$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)
making $H^*(X;R)$ into a graded ring
$$H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R).$$

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{\text{1-dim'l subspaces of } \mathbb{R}^{n+1}\} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$ 

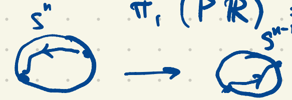
$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2\}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$
Proof is by contradiction

Suppose $f: S^n \to S^{n-1}$ is antipodal. $\quad (f(-x) = -f(x))$

Then f induces a well-defined map $P^n\mathbb{R} \xrightarrow{f} P^{n-1}\mathbb{R}$

$$\pm x \mapsto \pm f(x) \qquad (x \in S^n)$$



$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces $f^*: H^*(P^{n-1}\mathbb{R}; \mathbb{F}_2) \to H^*(P^n\mathbb{R}; \mathbb{F}_2)$ mapping $x \longmapsto x$

$$H^*(P^{n-1}\mathbb{R}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n) \qquad H^*(P^n\mathbb{R}; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$$

$$x^n \longmapsto x^{n+1}$$; contradiction.

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A)} \oplus T(A)$ canonically. where $T(A) =$ torsion subgp of $A = \{$elements of A of finite order$\}$

$k = \operatorname{rank} A = \dim A$.

For any chain complex $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$ (over $\mathbb{Q}$ or $\mathbb{R}$)

we have homology groups $H_n = \ker \partial_{n-1} / \operatorname{im} \partial_n$ with well-defined rank $H_n(X;\mathbb{Z}) = \operatorname{rank} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{R})$

and Euler characteristic

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i .$$

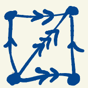
$C_n \xrightarrow{\partial_n} C_{n-1} \qquad \dim C_n = \dim \ker \partial_n + \dim \operatorname{im} \partial_n$

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1} \qquad \dim H_n = \dim \ker \partial_n - \dim \operatorname{im} \partial_{n+1}$

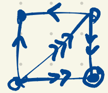
eg. $\chi(S^2) = 4 - 6 + 4 = 2$

Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

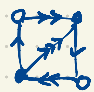
	S^2	T^2	$P^2\mathbb{R}$	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0

T^2 

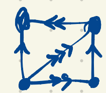
$1-3+2=0$

S^2 

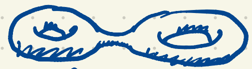
$3-3+2=2$

$P^2\mathbb{R}$ 

$2-3+2=1$

K^2 Klein bottle 

$1-3+2=0$

 $\#$  $=$ 

T^2 $\quad$ T^2 $\quad$ T^2 $\quad$ T^2

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ for any two closed surfaces S_1, S_2

$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0+0-2 = -2$

$\underbrace{T^2 \# \cdots \# T^2}_{g} =$  $\quad \chi(T^2 \# \cdots \# T^2) = 2-2g \qquad g =$ genus of orientable surface

$\chi(\underbrace{P^2\mathbb{R} \# P^2\mathbb{R}}_{K^2}) = 1+1-2 = 0$

Exact sequences $\quad \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \qquad \ker \partial_n = \operatorname{im} \partial_{n+1}$

$0 \to C \to 0$ is exact iff $C = 0$

$0 \to A \to B \to 0$ is exact iff $A \cong B$

$0 \to A \to B \to C \to 0$ is exact (short exact) iff $C \cong B/A$

If $f: X \to X$ is an endomorphism of an abel. gp. X (or vector space) (at least in an abelian category) Some important short exact sequences are

$$0 \to \ker f \to X \to f(X) \to 0$$

$$0 \leftarrow \operatorname{coker} f \leftarrow X \leftarrow f(X) \leftarrow 0$$

If $f: X \to Y$ then $\operatorname{coker} f = Y/\underbrace{f(X)}_{\operatorname{im} f}$

If $\cdots \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to \cdots$ are exact then we get an exact seq.

$$\to A \to B \to D \to E \to \cdots$$

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & A \arrow[dr] & & & & E \arrow[r] & \cdots \\
& & B \arrow[dr] \arrow[rr, dashed] & & D \arrow[ur] & & \\
& & & C \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$0 \to \ker f \to X \xrightarrow{f} X \to \operatorname{coker} f \to 0 \quad \text{is exact.}$$

If X is a fin. dim'l vector space over F then the Euler char. of this sequence is

$$\dim \operatorname{coker} f - \dim X + \dim X - \dim \ker f = 0$$

If $T: X \to X$ is an operator (endomorphism) (don't worry about boundedness) the index of T is $\operatorname{ind} T = \dim \operatorname{coker} T - \dim \ker T$ when both of these terms are finite (i.e. T is Fredholm).

Theorem: Let $S, T : X \to X$ be operators (lin. transf). Of the three operators S, T, ST, whenever two are Fredholm then so is the third and in this case $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

(or abel. gps)

In general (i.e. for any lin. transf. $S, T : X \to X$ we have an exact sequence

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0$$

So its Euler characteristic is zero. i.e. $\operatorname{ind} S + \operatorname{ind} T - \operatorname{ind} ST = 0$.

<u>Snake Lemma</u> In an abel. category we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

then we have a six-term exact seq.

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[ddl, bend left, in=0, out=0, to path={-- (\tikztotarget)}, phantom] \arrow[dddll, bend left=80] & \\
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} a \arrow[r] \arrow[d] & \operatorname{coker} b \arrow[r] \arrow[d] & \operatorname{coker} c \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Group Cohomology : used in the study of group extensions

If G and H are groups then an extension of H by G is a group X giving an exact sequence
$$1 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 1$$

Note: Groups are not necessarily abelian. We are asking for a new group X having a normal subgp $\cong H$ s.t. $X/H \cong G$. G on top, H on the bottom.

Trivial: $X = G \times H$. (split extension)

G is now an arbitrary group and A is an abelian group (G multiplicative; A additive notation) on which G acts (each $g \in G$ gives $g \in GL(A)$ (automorphisms of A as an abel. gp or $\mathbb{Z}$-module).

$(g_1 g_2)(a) = g_1(g_2(a))$; $g(a+b) = ga + gb$; $\underset{\hat{G}}{1}a = a$. $G \xrightarrow[\text{(fixed)}]{\text{homo.}} \operatorname{Aut} A = GL(A)$

We construct an extension of A by G i.e. an exact sequence of gps

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \hat{G} & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \\ 1 & \longrightarrow & A & \longrightarrow & \hat{\hat{G}} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

i.e. $\hat{G}$ is a gp with normal subgp. iso. to A with $\hat{G}/A \cong G$

Two extensions $\hat{G}$ $\hat{\hat{G}}$ are equivalent if we have a commutative diagram as shown with α, β, γ isomorphisms of groups (with exact rows). Note that the action of G on A is fixed throughout.

Cohomology of groups is the tool for this.

$\cdots \xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} 0$ is an exact sequence of additive abel. gps where $C^k = C^k(G;A)$ is the set of all maps $\underbrace{G^k}_{G\times G\times\cdots\times G} \to A$ as an additive abel. gp. i.e. $\mathbb{Z}$-module

i.e. k-tuples of G

$C^0 = A$ (maps $\{1\} \to A$)

$C^1 = A^G$ = maps $G \to A$ i.e. $f: G \to A$

$C^2 = A^{G\times G}$ = maps $G\times G \to A$ etc.

Given $a \in C^0$ i.e. $a \in A$, $\delta a \in C^1$ is $\delta a : G \to A$

$g \mapsto ga - a$

Given $f \in C^1$ i.e. $f: G \to A$

Construct $\delta f \in C^2$ i.e. $(\delta f): G\times G \to A$

$$(\delta f)(g,h) = g\underbrace{f(h)}_{\in A} - \underbrace{f(gh)}_{\in A} + \underbrace{f(g)}_{\in A} \in A.$$

Given $f \in C^2$ i.e. $f: G\times G \to A$

Construct $(\delta f): G\times G\times G \to A$

$$(\delta f)(g,h,k) = g f(h,k) - f(gh,k) + f(g,hk) - f(g,h)$$

See p. 2 bottom of handout for $\delta: C^k \to C^{k+1}$ in general

Check: $C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0$ $\delta^2 = 0$?

Take $a \in C^0 = A$.

$(\delta a): G \to A$

$(\delta a)(g) = ga - a$.

$\delta^2 a : G\times G \to A$

$$(\delta^2 a)(f,g) = f(\delta a)(g) - (\delta a)(fg) + \delta a(f)$$

$$= f(ga-a) - ((fg)(a) - a) + (fa - a)$$

$$= \cancel{fga} - \cancel{fa} - \cancel{fga} + \cancel{a} + \cancel{fa} - \cancel{a}$$

$= 0$

Classify extensions $1 \to A \to \hat{G} \to G \to 1$ where G is a group acting on an abelian gp A i.e. $\hat{G}$ is a group with normal subgp A with $\hat{G}/A \cong G$, using cohomology. Start with a split extension i.e. A has a complementary subgp in $\hat{G}$. So $\hat{G}$ acts on the subgps complementary to A by conjugation.

$H^1(G;A)$ classifies the complementary subgps up to conjugacy.

Fix an action of G on A. $\qquad a(g_1g_2) = (ag_1)g_2$ $\qquad$ here A is a right G-module.

G: $\underset{\sim}{\text{mult.}}$ $\quad A$: $\underset{\sim}{\text{additively}}$

$$a1 = a \quad (1 \text{ id. of } G)$$

$$(a+a')g = ag + a'g \qquad \text{for } a, a' \in A; \quad 1, g, g_1, g_2 \in G.$$

$\hat{G}$ is isomorphic to the semidirect product $A \rtimes G = \{(a,g) : a \in A, \ g \in A\}$ $\qquad$ $G \ltimes A$ for left action

$$(a_1, g_1)(a_2, g_2) = (a_1g_2 + a_2, \ g_1g_2) \qquad \text{identity } (\underset{\text{in } A}{0}, \underset{\text{in } G}{1})$$

Alternative notation: $A \rtimes G = \left\{ \begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} : a \in A, g \in G \right\}$

$$\begin{bmatrix} g_1 & 0 \\ a_1 & 1 \end{bmatrix}\begin{bmatrix} g_2 & 0 \\ a_2 & 1 \end{bmatrix} = \begin{bmatrix} g_1g_2 & 0 \\ a_1g_2 + a_2 & 1 \end{bmatrix}$$

$$C^2 \xleftarrow{\ \delta^1\ } C^1 \xleftarrow{\ \delta^0\ } \underset{A}{\overset{\shortparallel}{C^0}} \longleftarrow 0 \qquad \text{For } a \in \overset{C^0}{\overset{\shortparallel}{A}}, \ (\delta a)(g) = ag - a$$

How do we construct a subgp of $A \rtimes G$ complementary to A?

Any such subgp $H \leq A \rtimes G$ has the form $\{(t_g, g) : g \in G\} = \left\{ \begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix} : g \in G \right\}$

Here $g \overset{f}{\longmapsto} t_g$, $G \longrightarrow A$. This will automatically be a complement to A as long as it is a subgp. eg. $t_1 = 0$ but most importantly, closure.

$$\begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix}\begin{bmatrix} g' & 0 \\ t_{g'} & 1 \end{bmatrix} = \begin{bmatrix} gg' & 0 \\ t_{gg'} & 1 \end{bmatrix} \quad \text{i.e. } \underbrace{t_{gg'}}_{\in A} = \underbrace{t_g}_{\in A}\underbrace{g'}_{\in G} + \underbrace{t_{g'}}_{\in A} \quad \text{so } (\delta f)(g, g') = -f(gg') + f(g)g' + f(g') = 0.$$

Complements of A in $\hat{G}$ are given by 1-cocycles.

When are two complements of A conjugate in $\hat{G}$? $\hat{G} = A \rtimes G$

If A has complementary subgps $H_1, H_2 \le \hat{G}$ given by $H_i = \{ (f_i(g), g) : g \in G \}$, $f_i \in Z^1(G;A)$

$$(\delta f_i)(g,g') = f_i(g') - f_i(gg') + f_i(g)g' = 0$$

when are H_1, H_2 conjugate in $\hat{G}$?

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}^{-1} = \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix}$$

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use $\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \in \hat{G}$ (a, g fixed) to conjugate H_1 :

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ f_1(x) & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gx & 0 \\ ax + f_1(x) & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ axg^{-1} + f_1(x)g^{-1} - ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ f_2(gxg^{-1}) & 1 \end{bmatrix}$$

The 1-cycle defining this conjugate subgp would have to be f_2: so

$$f_2(gxg^{-1}) = axg^{-1} + f_1(x)g^{-1} - ag^{-1} = f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1}$$

$$\begin{aligned} ax + f_1(x) - a &= f_2(g)x + f_2(x) - f_2(g) \\ f_2(x) - f_1(x) &= (a - f_2(g))x - (a - f_2(g)) \\ &= \delta(a - f_2(g))(x) \end{aligned}$$

$$\begin{aligned} f_2(gxg^{-1}) &= f_2(g(xg^{-1})) \\ &= f_2(g)xg^{-1} + f_2(xg^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} + f_2(g^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1} \end{aligned}$$

f is a 1-cocycle: $f \in Z^1$

$(\delta f)(x,y) = f(x)y - f(xy) + f(y) = 0$

f is a crossed homomorphism or derivations

(If G acts trivially on A

i.e. $ag = a$ for all $a \in A, g \in G$)

then f is a homo. $G \to A$.

Extensions of A by G correspond to elements of $H^1 = Z^1/B^1$.

$f \in B^1$ (1-coboundary) iff

$f(x) = ax - a = (\delta a)(x)$, $a \in A$.

(principal crossed homomorphisms)

(inner derivations)

$f \in C^1$ i.e. $f: G \to A$

$$f(xy) = f(x)y + f(y)$$

$$f(1) = f(1 \cdot 1) = f(1)1 + f(1) \Rightarrow f(1) = 0$$

$$0 = f(1) = f(gg^{-1}) = f(g)g^{-1} + f(g^{-1}) \Rightarrow f(g^{-1}) = -f(g)g^{-1}$$

Eg. Classify extensions of $C_4 = \langle x : x^4 = 1\rangle$ by $C_2 = \langle y : y^2 = 1\rangle$

$x = (1234)$
$y = (14)(23)$
$x^2y = (12)(34)$
$xy = (13)$
$x^3y = (24)$

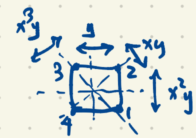
$$1 \longrightarrow C_4 \longrightarrow \hat{G} \longrightarrow C_2 \longrightarrow 1$$

Two cases depending on the action of C_2 on C_4

Case I: y inverts x i.e. $yxy^{-1} = x^{-1} = x^3$

$|H'| = 2$. = how many complementary subgps of C_4 up to conjugacy.

C_4 has four complementary subgps in $\hat{G} \cong$ dihedral gp of order 8.



$\langle y\rangle$, $\langle x^2y\rangle$ are conjugate to each other in $\hat{G}$ } Not conjugate.
$\langle xy\rangle, \langle x^3y\rangle$ are conjugate to each other in $\hat{G}$

Case II: y commutes with x. $xy = yx$ $\quad \hat{G} = C_4 \times C_2$

$\langle x\rangle$ has two complements in $\hat{G}$, namely $\langle y\rangle$, $\langle x^2y\rangle$. They are not conjugate. $|H'| = 2$.

How many extensions $1 \to C_4 \to \hat{G} \to C_2 \to 1$ are there up to equivalence, if we don't require the extension to be split? (Split $\Longleftrightarrow$ there is a complementary subgp for C_4)

eg. C_8 is a nonsplit extension of C_4 by C_2.

Case I: C_2 acts trivially on C_4. Here there are two extensions: C_8 (nonsplit) and $C_4 \times C_2$ (split).

Case II: C_2 acts nontrivially on C_4. Here there are two extensions: dihedral of order 8 (split), quaternion gp of order 8 (nonsplit).

The Schur-Zassenhaus Theorem

Given groups G, N with G acting N (the action of G on N is fixed) we consider exact sequences of groups

$$1 \longrightarrow N \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

i.e. extensions of N by G i.e. groups $\hat{G}$ having a normal subgp isomorphic to N.

If $|N|, |G|$ are relatively prime then $H^1 = 1$ and $H^2 = 1$.

This says that the extension splits i.e. $\hat{G}$ has a subgp complementary to N and any two complements of N are conjugate in $\hat{G}$.

This generalizes Sylow theory; N and its complements are <u>Hall</u> <u>subgps</u>.

Note: We do not require N to be abelian. If N is abelian then the formulas are simpler. Even simpler if $N \subset Z(G)$. (<u>central</u> extension of N by G)

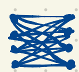
A <u>loop</u> is a set L with a binary operation $\begin{array}{c}(x,y) \longmapsto xy \\ L \times L \longrightarrow L\end{array}$ such that any two of x, y, xy uniquely determine the other.

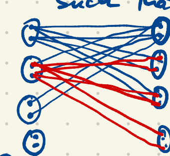
We also assume $\exists\, 1 \in L$ such that $1x = x1 = x$ for all $x \in L$.

A Bol loop satisfies $((xy)z)y = x((yz)y)$ for all $x, y, z \in L$.

A <u>Hadamard</u> <u>matrix</u> is an $n \times n$ matrix H with entries ± 1 such that $HH^T = nI = H^TH$

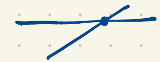
eg. $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$ gives a $\overset{\text{distance}}{(\text{regular})}$ double cover of $K_{4,4}$:





A <u>complex</u> <u>Hadamard</u> <u>matrix</u> is an $n \times n$ matrix H with entries in $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that $HH^* = nI = H^*H$

I classified complex Had. matrixes with an automorphism group which is doubly transitive on rows.

A proj. plane has points and lines satisfying  

Any $(n^2+n+1)\times(n^2+n+1)$ matrix with 0/1 entries, $n+1$ ones in any row/col, row $\cdot$ row $= 1 =$ col $\cdot$ col

n = The order of the plane

In all known cases, $n = p^k$, p prime, $k \geq 1$.

Some long exact sequences in homology

Take a top. space X having a closed subspace $A \subset X$ which is the deformation retract of an open nbhd in X.

X/A : collapse A to a single point, leaving points outside A untouched.

quotient space

X

Then we have a long exact sequence

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A)$$
$$\to \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(X/A) \to \cdots$$
$$\cdots \to \tilde{H}_0(X/A) \to 0$$

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$$\to C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

has $H_n(X)$

$$\to C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

has reduced homology $\tilde{H}_n(X)$.

Theorem $(k \geq 1)$
$$H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n \in \{0, k\} \\ 0 & \text{otherwise} \end{cases}$$

i.e. $$\tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{else} \end{cases}$$

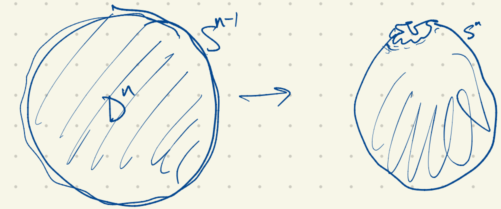
$$A \longrightarrow X \longrightarrow X/A$$

Proof: Take $X = D^n$ = closed n-ball $(n \geq 1)$

$A = \partial X = S^{n-1}$

$X/A \cong S^n$ = one-point compactification of $\mathbb{R}^n$

$\tilde{H}_k(D_n) = 0$ for all k



$$\cdots \to \underset{\overset{\|}{0}}{\tilde{H}_k(\underbrace{D^n}_{X})} \to \tilde{H}_k(\underbrace{S^n}_{X/A}) \to \tilde{H}_{k-1}(\underbrace{S^{n-1}}_{A}) \to \underset{\overset{\|}{0}}{\tilde{H}_{k-1}(\underbrace{D^n}_{X})} \to \cdots$$

$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ depends only on $k-n$.

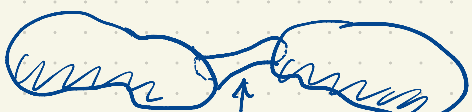
$\tilde{H}_n(S^n) \cong \cdots \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$

$$\tilde{H}_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k=1 \\ 0 & \text{else.} \end{cases}$$

Homotopy groups $\pi_n(X)$ "counts" ways to map $S^n \xrightarrow{f,g} X$

X is assumed to be path-connected.

In X:



Cut holes $\cong D^n$ in $f(S^n)$, $g(S^n)$ and join using $\underbrace{S^{n-1} \times [0,1]}_{\text{tube}}$

Identity element: constant.

Elements of $\pi_n(X)$ are up to homotopy.

$f_0, f_1 : S^n \to X$ are homotopic iff there exists $f: S^n \times [0,1] \to X$, $(x,t) \mapsto f_t(x)$

The group operation is associative.
It's commutative for $n \geq 2$.

If X, Y have the same homotopy type then $\pi_n(X) \cong \pi_n(Y)$ for all n. (Not conversely)
If $\pi_k(X) = 1$ for $k < n$ and $\pi_n(X) \neq 1$, then $H_n(X) =$ abelianization of $\pi_n(X)$.
(theorem of Hurewicz)

Homotopy groups are easier to define than homology groups
much harder to compute

$\pi_n(S^k)$ is not known in general

But some is known:

$\pi_k(S^n) = 0$ for $1 \leq k \leq n-1$

$\pi_n(S^n) \cong \mathbb{Z}$

e.g. $(1,2) \in P^1\mathbb{C}$
has preimage (fibre)
$\{(z, 2z) : |z|^2 + |2z|^2 = 1\}$
$|z|^2 = \frac{1}{5}$
$\left\{ e^{i\theta}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) : \theta \in [0, 2\pi) \right\}$

$\pi_3(S^2) \cong \mathbb{Z}$ is due to the Hopf fibration $1 \longrightarrow S^1 \longrightarrow S^3 \longrightarrow S^2 \longrightarrow 1$

This means we have a map $f: S^3 \longrightarrow S^2$ which is surjective and all fibres are circles. In other words we can partition S^3 into circles.

$S^3 =$ unit sphere in $\mathbb{R}^4 =$ unit sphere in $\mathbb{H}$.
$= \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$

$$f(z,w) = \begin{cases} \left(1, \frac{w}{z}\right), & \text{if } z \neq 0 \\ (0, 1), & \text{if } z = 0 \end{cases} \in S^2 = P^1\mathbb{C} = \text{Riemann sphere}$$

$1 \longrightarrow S^0 \longrightarrow S^1 \longrightarrow S^1 \longrightarrow 1$ $\quad \mathbb{R} \subset \mathbb{C}$

$1 \longrightarrow S^1 \longrightarrow S^3 \longrightarrow S^2 \longrightarrow 1$ $\quad \mathbb{C} \subset \mathbb{H}$

$1 \longrightarrow S^3 \longrightarrow S^7 \longrightarrow S^4 \longrightarrow 1$ $\quad \mathbb{H} \subset \mathbb{O}$

$1 \longrightarrow S^7 \longrightarrow S^{15} \longrightarrow S^8 \longrightarrow 1$ $\quad \mathbb{O} \subset \mathbb{S}$

$1 \longrightarrow S^1 \longrightarrow S^1 \times S^2 \longrightarrow S^2 \longrightarrow 1$ trivial bundle has (global) nonvanishing section

The Hopf fibration does not have such a section; it is a nontrivial fibre bundle.

A fibre bundle over B with fibres F (B = base space) is a way of continuously attaching a copy of F at each point of B.

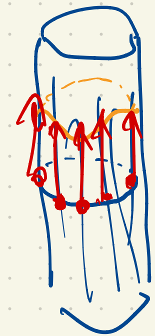
Trivial bundle: $E = F \times B$

Another way: eg. S^1 has more than one kind of line bundle (vector bundle over $\mathbb{R}$ of dimension 1) having fibre space $F = \mathbb{R}$

$\mathbb{R} \longrightarrow E \longrightarrow \boxed{S^1}$

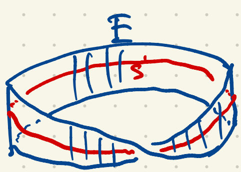
↑ "Total space"

Trivial bundle $\quad \mathbb{R} \longrightarrow \underbrace{S^1 \times \mathbb{R}}_{\text{"infinite cylinder"}} \longrightarrow S^1$



S^1

Nontrivial:



$\mathbb{R} \longrightarrow E \longrightarrow S^1$

Möbius strip (infinite)

A section of a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ is a right inverse for p

i.e. $g : B \longrightarrow E$ such that $p \circ g = id_B$.

This fibre bundle is nontrivial because it has no nonvanishing global sections.

For every fibre bundle $F \to E \to B$ we have a long exact sequence of homotopy groups

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

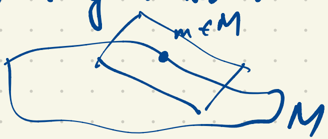
A line bundle is a bundle with fibre space given by some field (usually $\mathbb{R}$ or $\mathbb{C}$)

A vector bundle has fibre space F^n (e.g. $\mathbb{R}^n$, $\mathbb{C}^n$, ...)

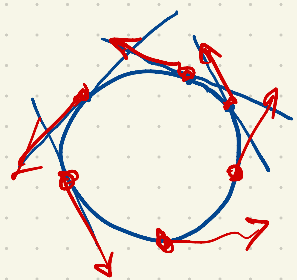
Sections of a vector bundle are the same thing as vector fields.

A circle bundle has fibre space $\cong S^1$.

A tangent bundle for a surface (hypersurface) $M \subset \mathbb{R}^n$ (M an $n-1$-manifold) has fibres $\cong \mathbb{R}^{n-1}$.

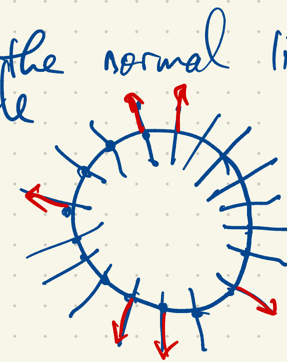


Eg. the tangent line bundle on S^1
Which one is it (trivial or Möbius strip)?

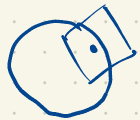


also trivial.

Eg. the normal line bundle



Eg. $S^2 \subset \mathbb{R}^3$ Tangent bundle E



$$\mathbb{R}^2 \longrightarrow E \longrightarrow S^2$$

2 dim'l 4-dim'l 2-dim'l

$(s, v) \longmapsto s$

$(s, v) \longleftarrow s$ v constant

Is this bundle trivial? No.

The trivial bundle $S^2 \times \mathbb{R}^2$ has a nonvanishing section.

$\cup$

$(s, v) \longrightarrow S^2$

$(s, v) \longmapsto s \in S^2$

Hairy Ball Theorem: The tangent bundle on S^2 (or S^{2k}) has no nonvanishing (global) section i.e. on S^2 there is no nonvanishing vector field. On the Earth at any instant in time, any tangent vector field (eg. wind) must be zero at some point.

Proof: Suppose $g: S^2 \to E$ is a nonvanishing section, WLOG $g(s) = (s, v)$ where $\|v\| = 1$. Such a map g defines a homotopy from id_{S^2} to the antipodal map on S^2

$f_t : S^2 \longrightarrow S^2$, $0 \le t \le 1$

$f_t(s) = f(s,t)$, $(s,t) \in S^2 \times [0,1]$

$f_0(s) = s$ identity on S^2

$f_1(s) = -s$ (antipodal point on S^2)

$f_t(s)$: start at $s \in S^2$ and go $t\pi$ radians on S^2 in the direction of $g(s)$.

Remark: If $g: S^2 \to S^2$ then g induces a map on $H_2(S^2) \cong \mathbb{Z}$ (take singular homology).

or $\pi_2(S^2) = \mathbb{Z}$

S^2

$g_* \in H_2(S^2)$ induced by g has a well-defined degree of g in $\mathbb{Z}$.

The identity $S^2 \to S^2$ has degree 1.

A constant map $S^2 \to S^2$ " " 0

The antipodal map $S^2 \to S^2$ has degree -1.

If $H \leq G$ then $G/H = \{gH : g \in G\}$ set of left cosets of H in G. View this as a homogeneous space for G in the case where G is a top. gp. (G acts transitively)

If $H \trianglelefteq G$ (normal subgp) then G/H is actually a group.

In general $H \leq G$ gives a fibre bundle $H \to G \to G/H$
$$g \longmapsto gH$$

In particular take $G = O_n(\mathbb{R}) = \{n \times n$ matrices A over $\mathbb{R}$ such that $AA^T = A^TA = I\}$
orthogonal group.

For $A \in O_n(\mathbb{R})$, $\det A = \pm 1$
$SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) : \det A = 1\}$
$= \{$rotations of $\mathbb{R}^n$ about $0\}$.

H = stabilizer of $(1,0,0,\dots,0)$ in G

$$= \left\{ \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix} : B \in O_{n-1}(\mathbb{R}) \right\} \cong O_{n-1}(\mathbb{R})$$

$G/H \cong S^{n-1}$

$O_{n-1} \to O_n \to S^{n-1}$ fibration

$\cdots \to \pi_k(O_{n-1}) \to \pi_k(O_n) \to \pi_k(S^{n-1}) \to \pi_{k-1}(O_{n-1}) \to \cdots$ long exact sequence

$\pi_k(O_n)$ are better understood than $\pi_k(S^n)$

Special cases: $O_2 \to O_3 \to S^2$

$SO_2 \to SO_3 \to S^2$

$S^1/\{\pm I\} \to S^3/\{\pm I\} \to S^2$

$S^1 \to S^3 \to S^2$ Hopf fibration

$SO_3 \cong S^3/\{\pm I\}$

$SO_2 \cong S^1$

$$\frac{S^3/\{\pm I\}}{S^1/\{\pm I\}} \cong S^3/S^1 \cong S^2$$

Stability of homotopy groups of spheres: $\pi_m(S^k) = \pi_{n+k}(S^k)$ is constant (indep. of k) if k is sufficiently large

$n+k = m \geq n$

$\pi_n^S(S^k) = \lim_{k\to\infty} \pi_{n+k}(S^k)$ = n^{th} stable homotopy gp for spheres

Similar for O_n: $\pi_{n+k}(O_k)$ has a limit as $k \to \infty$ (const. for $k \gg \infty$)

given by

$$\pi_n(O_\infty) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 0 \bmod 8 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 1 \bmod 8 \\ 0 & \text{if } n \equiv 2 \ \cdots \\ \mathbb{Z} & \cdots\ 3 \ \cdots \\ 0 & 4 \\ 0 & 5 \\ 0 & 6 \\ \mathbb{Z} & 7 \end{cases}$$

Bott Periodicity Theorem

$O_\infty = \bigcup O_n$

I have a nice 28×28 complex Hadamard matrix with entries complex 7th roots of unity.
It corresponds to a nice 7-fold cover of $K_{28,28}$.

Monodromy representations

Let $X = \mathbb{C} \setminus \{0,1\}$. For each $\lambda \in X$ consider the elliptic curve $E_\lambda : y^2 = x(x-1)(x-\lambda)$ $\quad (x,y) \in \mathbb{C}^2$

$E_\lambda \cong T^2$

$\pi_1(E_\lambda) \cong \mathbb{Z}^2$