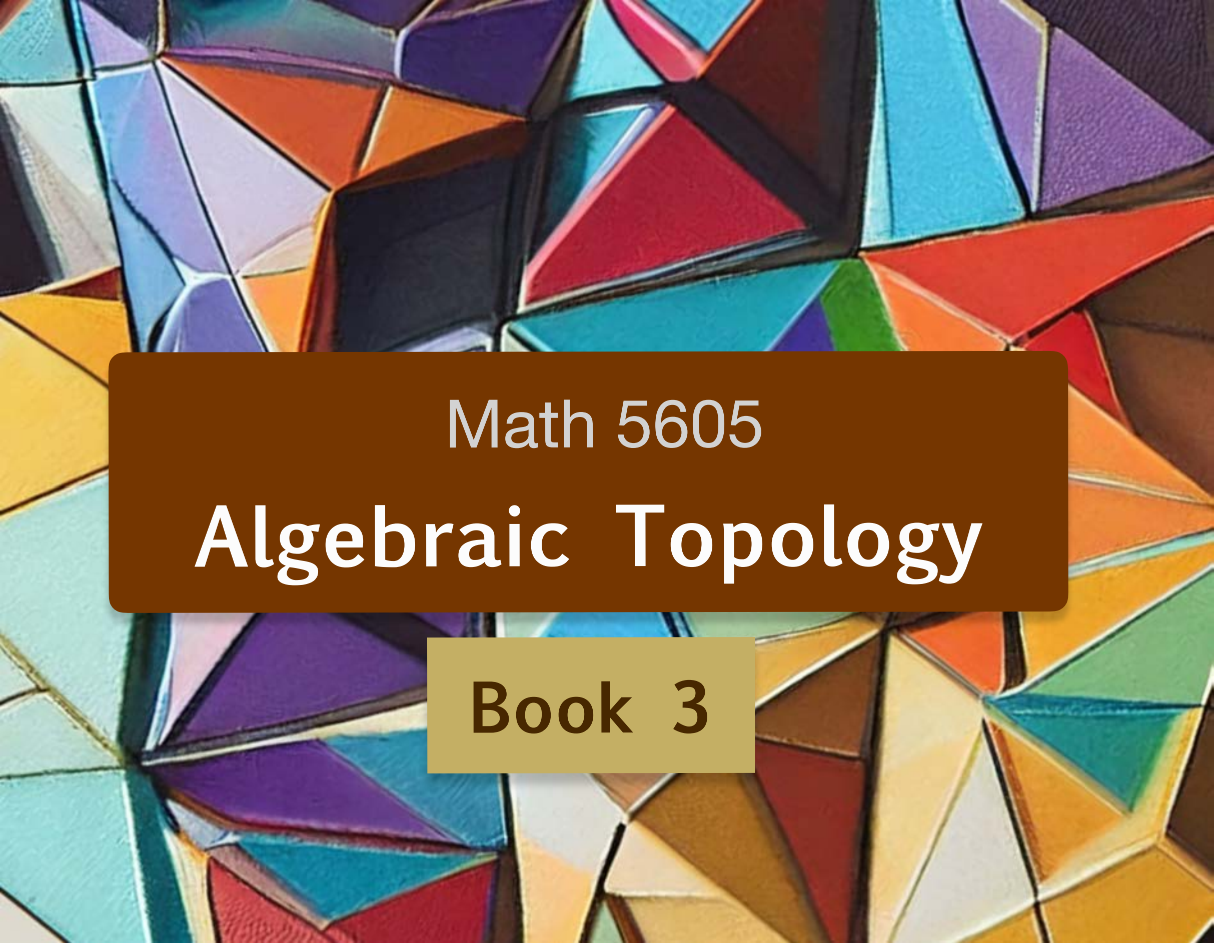


Math 5605

Algebraic Topology

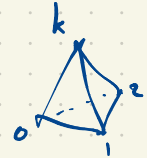
Book 3

Cup product for simplicial cohomology $\quad H^k \times H^l \xrightarrow{\cup} H^{k+l}$
makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains : $(k=0,1,2,3,\dots)$ ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex $\{$all subsets of $\{0,1,2,\dots,k\}\}$.

This has a geometric realization 

$\xrightarrow{\sigma} X$

$$\Delta = \Delta^n = \{\underbrace{(v_0, v_1, \dots, v_n)}_{\substack{\text{barycentric}\\ \text{coordinates}}} : v_i \geq 0,\ \Sigma v_i = 1\} \subset R^{n+1}$$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \longrightarrow X$.

$C_n = \{$n-chains in $X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{$n-cochains in $X\} = \mathrm{Hom}(C_n, R) = \{R$-homomorphisms $C_n \longrightarrow R\}$

$\partial : C_n \longrightarrow C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]$ $\qquad \partial^2 = 0$, $(\partial^*)^2 = 0$

$\partial^* : C^{n-1} \longrightarrow C^n$

If $\phi \in C^k$ k-cochain, $\psi \in C^l$ l-cochain, then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$, $[v_0, \dots, v_{k+l}] \mapsto \sigma(v_0, \dots, v_{k+l})$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\, \psi(\sigma|[v_k, \dots, v_{k+l}])$$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)
making $H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R)$ into a graded ring.

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{\text{1-dim'l subspaces of } \mathbb{R}^{n+1}\} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$

$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2\}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$

Proof is by contradiction

Suppose $f : S^n \to S^{n-1}$ is antipodal. $\quad (f(-x) = -f(x))$

Then f induces a well-defined map $P^n\mathbb{R} \xrightarrow{f} P^{n-1}\mathbb{R}$

$$\pm x \mapsto \pm f(x) \qquad (x \in S^n)$$

$S^n \to S^{n-1}$ 

$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces $f^* : H^*(P^{n-1}\mathbb{R} ; \mathbb{F}_2) \to H^*(P^n\mathbb{R} ; \mathbb{F}_2)$ mapping $x \longmapsto x$

$$H^*(P^{n-1}\mathbb{R} ; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^n), \qquad H^*(P^n\mathbb{R} ; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$$

$$x^n \longmapsto x^{n+1} \quad ; \text{ contradiction.}$$

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A) \text{ canonically}} \oplus T(A)$ where $T(A) =$ torsion subgp of $A = \{$elements of A of finite order$\}$.

$k = \operatorname{rank} A = \dim A$.

For any chain complex $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$ (over $\mathbb{Q}$ or $\mathbb{R}$)

we have homology groups $H_n = \ker \partial_{n-1} / \operatorname{im} \partial_n$ with well-defined $\operatorname{rank} H_n(X;\mathbb{Z}) = \operatorname{rank} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{R})$

and Euler characteristic

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i .$$

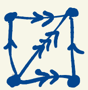
$C_n \xrightarrow{\partial_n} C_{n-1} \qquad \dim C_n = \dim \ker \partial_n + \dim \operatorname{im} \partial_n$

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1} \qquad \dim H_n = \dim \ker \partial_n - \dim \operatorname{im} \partial_{n+1}$

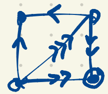
eg. $\chi(S^2) = 4 - 6 + 4 = 2$

Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

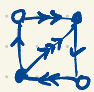
	S^2	T^2	$P^2\mathbb{R}$	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0

T^2 

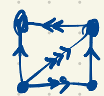
$1-3+2=0$

S^2 

$3-3+2=2$

$P^2\mathbb{R}$ 

$2-3+2=1$

K^2 Klein bottle 

$1-3+2=0$

T^2 T^2 T^2 T^2

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ for any two closed surfaces S_1, S_2

$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0+0-2 = -2$

$\underbrace{T^2 \# \cdots \# T^2}_{g}$ = 

$\chi(T^2 \# \cdots \# T^2) = 2-2g$ g = genus of orientable surface

$\chi(\underbrace{P^2\mathbb{R} \# P^2\mathbb{R}}_{K^2}) = 1+1-2 = 0$