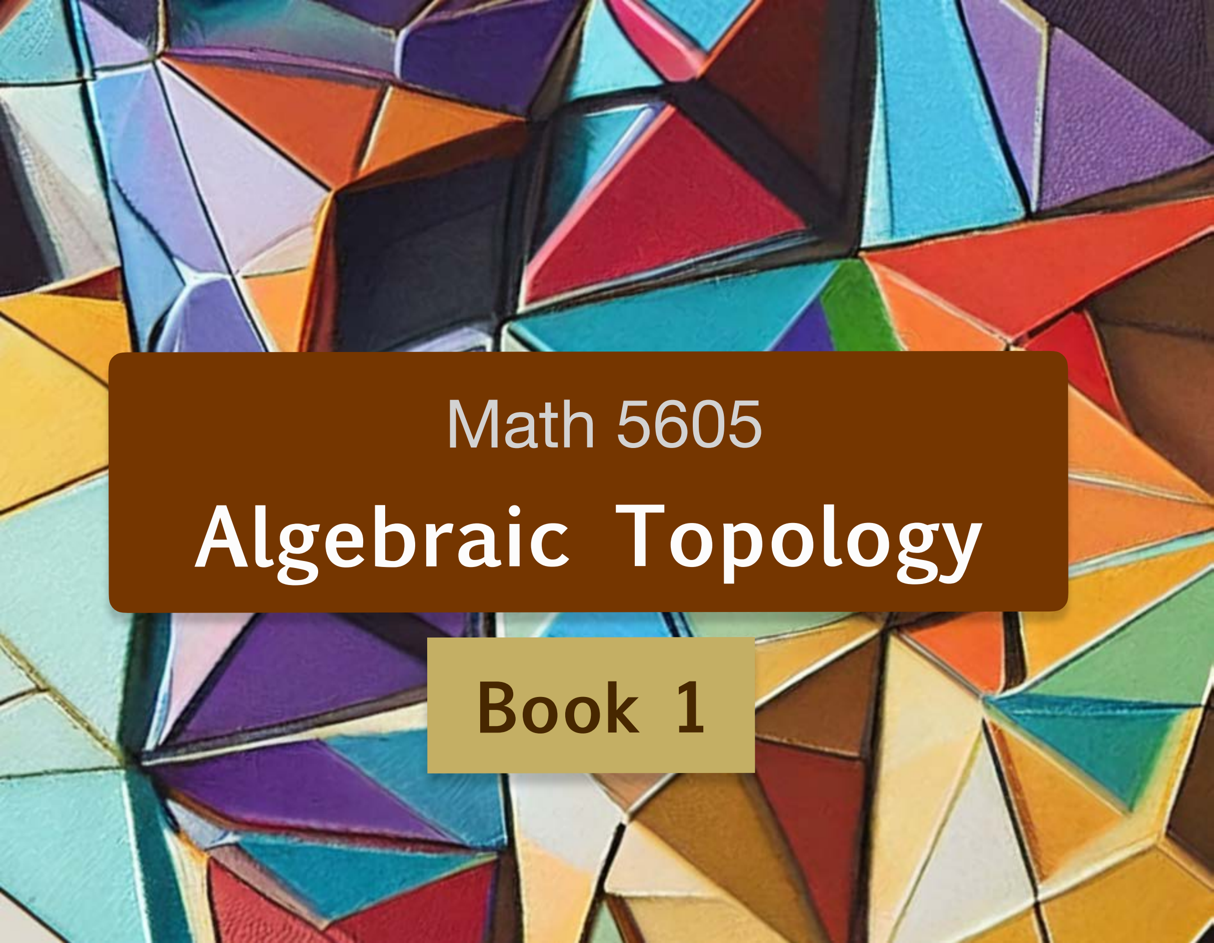


Math 5605

Algebraic Topology

Book 1

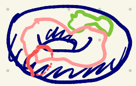
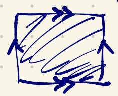
If X, Y are top. spaces, $f: X \to Y$ is <u>continuous</u> if $f^{-1}(U) \subseteq X$ is open whenever $U \subseteq Y$ is open.

$f: X \to Y$ is a <u>homeomorphism</u> if f is bijective and f, f^{-1} are continuous.

$X \cong Y$ are <u>homeomorphic</u> if there exists a homeomorphism $X \xrightarrow{\cong} Y$.

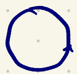
$\mathbb{R}^2 \ncong S^2$ since S^2 is compact; $\mathbb{R}^2$ is not.

2-sphere $S^2 \cong \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ = 

$S^2 \ncong T^2 = S^1 \times S^1$ =  = 

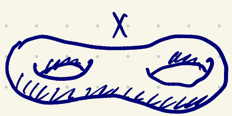
$S^1 = S$

S^2, T^2 are compact surfaces. They are locally homeomorphic but not globally homeomorphic.

$T^1 = S^1 =$  = circle $\cong \{z \in \mathbb{C} : |z| = 1\}$

$S^2 \ncong T^2$ because S^2 is simply connected whereas T^2 is not.

In S^2, every closed path can be "continuously shrunk" to a point (homotopic to a point, i.e. null-homotopic)



$\ncong$

although both surfaces are compact, connected, not simply connected

These two surfaces have different fundamental group: $\pi_1(X)$ is nonabelian, $\pi_1(T) \cong \mathbb{Z}^2$ is a (nontrivial) abelian group.

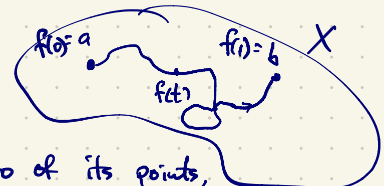
If $X \cong Y$ (homeomorphic) then $\pi_1(X) \cong \pi_1(Y)$.

For much of alg. top., the algebraic invariants that we define are actually invariant under the weaker equivalence relation of <u>homotopy equivalence</u>.

Eg. For every $n \geq 0$, $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\bullet\}$.

Given points $a, b \in X$ (a topological space), a *path* from a to b is a (continuous) function $f : [0,1] \to X$ such that $f(0) = a$, $f(1) = b$.

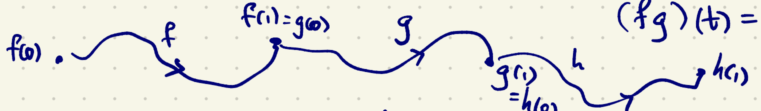
$$[0,1] \xrightarrow{f}$$



All maps (unless indicated otherwise) are assumed to be continuous.

If X has a path between any two of its points, then X is path-connected. For the time being, we'll assume X is path-connected. (In general, we instead define the fundamental groupoid of X.) If $\varphi : [0,1] \to [0,1]$ (recall: continuous) such that $\varphi(0) = 0$, $\varphi(1) = 1$ then $f \circ \varphi : [0,1] \to X$ is just a reparameterization of the same path and we don't distinguish it from f.

If $f, g : [0,1] \to X$ are paths such that $f(1) = g(0)$ then we can concatenate them to form a new path from $f(0)$ to $g(1)$:



$$(fg)(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

$(fg)h$ is the same path as $f(gh)$ after reparameterization:

$$((fg)h)(t) = \begin{cases} f(4t), & t \in [0, \frac{1}{4}] \\ g(4t-1), & t \in [\frac{1}{4}, \frac{1}{2}] \\ h(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

$$(f(gh))(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(4t-2), & t \in [\frac{1}{2}, \frac{3}{4}] \\ h(4t-3), & t \in [\frac{3}{4}, 1] \end{cases}$$

More precisely, we require a map $[0,1]^2 \to X$

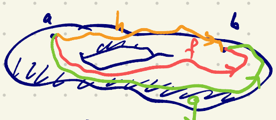
$$(s,t) \mapsto f(s,t) = f_s(t)$$

such that $f_0 = f$ i.e. $f_0(t) = f(t)$

$f_1 = g$ i.e. $f_1(t) = g(t)$

$f_s(0) = a$, $f_s(1) = b$ for all $s \in [0,1]$

This is a homotopy from f to g.



f, g are homotopic but h is not homotopic to f, g

We say f, g are homotopic if there is a 'continuous' family of paths from a to b in X, f_s ($s \in [0,1]$) with $f_0 = f$, $f_1 = g$.

If $\varphi : [0,1] \to [0,1]$ is a map with $\varphi(0)=0$, $\varphi(1)=1$ then the reparameterized path $f\circ\varphi : [0,1] \to X$ is homotopic to f. A homotopy from f to $f\circ\varphi$ is

$$[0,1]^2 \longrightarrow X$$
$$(s,t) \longmapsto f(\underbrace{(1-s)t + s\,\varphi(t)}_{\Uparrow\ [0,1]}) = f_s(t)$$

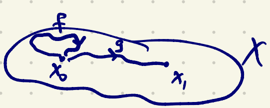
$$f_0(t) = f(t)$$
$$f_1(t) = f(\varphi(t))$$
$$f_s(0) = f(\cancel{(1-s)\cdot 0} + \cancel{s\cdot\varphi(0)}) = f(0)$$
$$f_s(1) = f((1-s)\cdot 1 + s\cdot\underline{\varphi(1)}) = f(1)$$

Fix $x_0 \in X$. Assume X is path-connected. $\pi_1(X, x_0)$ is the group of all homotopy classes of paths from x_0 to x_0 in X under concatenation. It turns out $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

This gives the fundamental group $\pi_1(X)$.

$\pi_1(\mathbb{R}^n) = 1$ (trivial group).

$\pi_1(S^1) \cong \mathbb{Z}$ (free group on one generator)



Fix g path in X from x_0 to x_1

An isomorphism $\phi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$$f \xmapsto{\ \phi\ } \bar{g} f g$$
$$g f \bar{g} \xleftarrow{\ \phi^{-1}\ } h$$

$$\phi(f_1 f_2) = \bar{g} f_1 f_2 g = (\bar{g} f_1 g)(\bar{g} f_2 g)$$
$f_1, f_2 \in \pi_1(X, x_0)$

γ: Identity in $\pi_1(X, x_0)$

$\gamma(t) = x_0$ for $t \in [0,1]$

$\gamma f = f\gamma = f$ for all $f \in \pi_1(X, x_0)$

The inverse of $f \in \pi_1(X, x_0)$ is

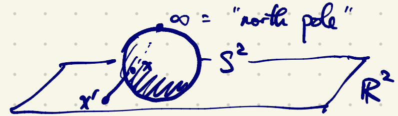
$$\bar{f}(t) = f(1-t), \quad t \in [0,1]$$

(same path in the reverse direction)

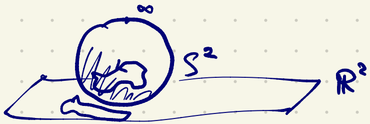
$f\bar{f} = \bar{f}f = \gamma$ = null path

$\pi_1(S^2) = 1$ (trivial group: all closed paths in S^2 are null-homotopic)

$S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ (one-point compactification)



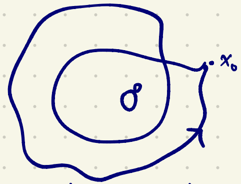
$x \longmapsto x'$ is stereographic projection from the north pole ∞



See Hatcher for general case including possibly space-filling curves.

$\pi_1(\mathbb{R}^2) = 1$

$\pi_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{Z}$
punctured plane



follows from the fact that
$\mathbb{R}^2 - \{0\}$ and S^1 have the same homotopy type

$\mathbb{R}^3 - (x\text{-axis}) \simeq \mathbb{R}^2 - \{0\} \simeq S^1$

$\mathbb{R}^3 - \{0\} \simeq S^2$

$X \simeq Y$: X, Y are homotopic / have the same homotopy type / are homotopy equivalent

Note: this is weaker than $X \cong Y$ (homeomorphic).
Hatcher writes $X \approx Y$ for homeomorphic.

- retraction
- deformation retraction "def. retraction in the weak sense"
- strong deformation retraction "def. retraction".
- homotopy
- relative homotopy
- homotopy equivalence

Let $A \subseteq X$ (subspace of a top. space).

A <u>retraction</u> $f: X \to A$ is a (continuous) map such that $f|_A = \mathrm{id}_A = \mathbb{1}_A$ i.e. $f(a) = a$ for all $a \in A$.

If such a map exists then A is a <u>retract</u> of X.

Eg. $\mathbb{R}^n$ has a retraction to any one of its points. If $a \in \mathbb{R}^n$ then the constant map $\mathbb{R}^n \to \{a\}$, $x \mapsto a$ is a retraction.

$\mathbb{R}^2 \to x\text{-axis}$, $(x,y) \mapsto (x,0)$.

If $S^1 \subset \mathbb{R}^2$ is the unit circle, then there is no retraction $\mathbb{R}^2 \to S^1$.
(But this may not be obvious.)

A deformation retraction is (a homotopy from id_X to $\underset{X\to A}{\text{a retraction}}$), $A \subseteq X$.

i.e. $f : [0,1] \times X \longrightarrow X$

$$f(t,x) = f_t(x)$$

$f_0(x) = x$ i.e. $f_0 = \mathrm{id}_X$

$f_1(x) \in A$ f_1 is a retraction $X \to A$

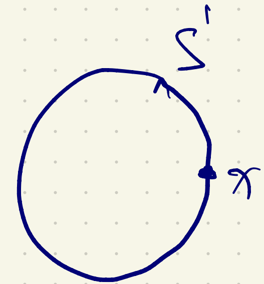
$f_1(a) = a$ for all $a \in A$.

If a def. retraction exists from X to $A \subseteq X$, we say A is a deformation retract of X.

(This is stronger than retract)

Eg. $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$

$f_t(x) = (1-t)x$ is a def. retraction to $\{0\}$.



Eg. $x \in S^1$ x is a retract of S^1 but not a def. retract of S^1.

A strong def. retract $f : [0,1] \times X \to X$:

$f_0(x) = x$ i.e. $f_0 = \mathrm{id}_X$

f_1 is a retraction $X \to A$

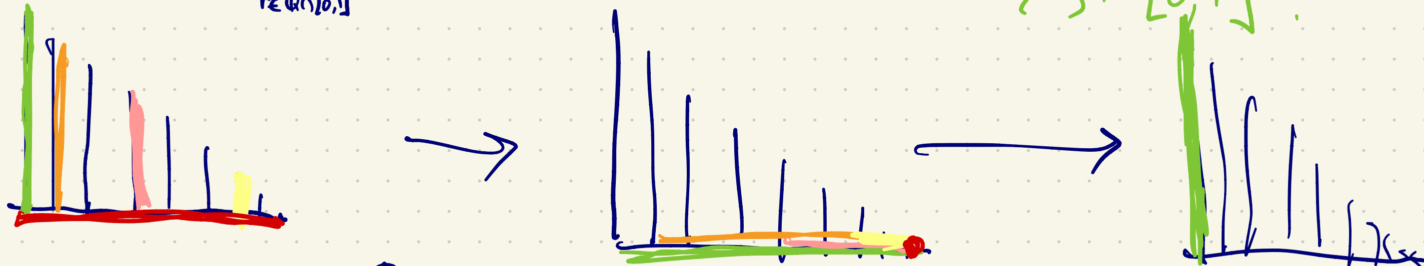
$f_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$

X

def. retract
but not strong def. retract.

$X \subseteq \mathbb{R}^2$ $\qquad X = ([0,1] \times \{0\}) \cup \bigcup_{r \in \mathbb{Q}\cap[0,1]} (\{r\} \times [0,1-r])$

X has a def. ret. to $A = \{0\} \times [0,1]$.



$0 \leq t \leq \frac{1}{2}$ $\qquad \frac{1}{2} \leq t \leq 1$

This is a def. retract but not a strong def. retract.

There is no strong def. retract $X \to A$.

Let $f_0, f_1 : X \to Y$ be maps. A homotopy from f_0 to f_1 is a map ("continuous deformation") $f : [0,1] \times X \to Y$ such that $f(0,x) = f_0(x)$, $f(1,x) = f_1(x)$.

$f(t,x) = f_t(x)$

If $A \subseteq X$ is any subspace, a homotopy relative to A from f_0 to f_1 ($f_0, f_1 : X \to Y$) is a homotopy $f : [0,1] \times X \to Y$ such that $f_t(a)$ is independent (constant) of $t \in [0,1]$ for all $a \in A$.

A path from a to b in X is a homotopy from a to b.

Given two paths f_0, f_1 in X from a to b ($a, b \in X$) is a homotopy relative to $\{0,1\}$.

A homotopy equivalence from X to Y is a pair of maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $f \circ g : X \to X$ and $g \circ f : Y \to Y$ are homotopic to id_X and id_Y respectively.

Eg. $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0 = \{\cdot\}$ (or $\mathbb{R}^n \simeq \{\cdot\}$)

$$\mathbb{R}^n \underset{g}{\overset{f}{\rightleftarrows}} \{0\}$$

$f(x) = 0$ for all $x \in \mathbb{R}^n$
$g(0) = 0 \in \mathbb{R}^n$

$$g \circ f : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto 0$$

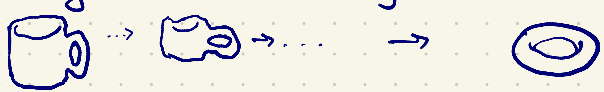
$$f \circ g : \{0\} \to \{0\}, \quad 0 \mapsto 0$$

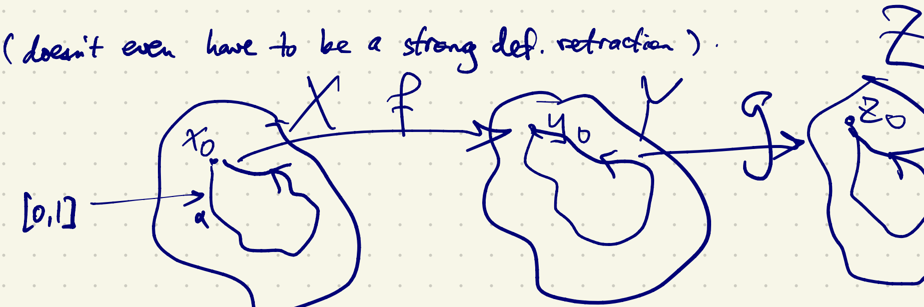
A homotopy from $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ to $id_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x$ is $h_t(x) = tx$, $0 \le t \le 1$, $x \in \mathbb{R}^n$

Not relative to any subspace necessarily.

$h_0 = g \circ f$
$h_1 = id_{\mathbb{R}^n}$

The same argument works for any def. retraction (doesn't even have to be a strong def. retraction).





S^1 is not homotopic to a point.
(S^1 is not null homotopic; not contractible)

If $f : X \to Y$ where both X, Y are path-connected then f induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$

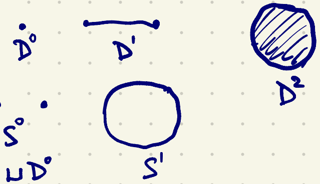
$\alpha : [0,1] \to X$ gives $f_* \alpha = f \circ \alpha : [0,1] \to Y$

$(g \circ f)_* = g_* \circ f_*$

If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

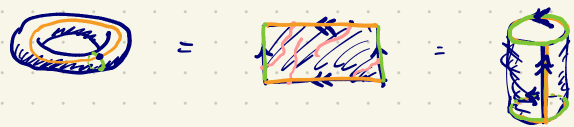
D^n = closed ball in $\mathbb{R}^n$
"disk"

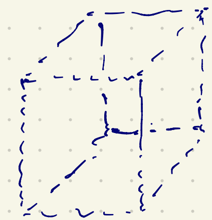
$S^n = \partial D^{n+1}$ = n-sphere



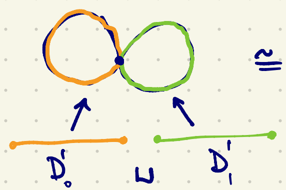
A CW complex is formed from $X = X^0 \cup X^1 \cup X^2 \cup X^3 \cup \cdots$

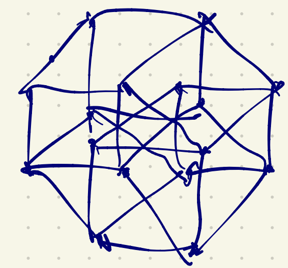
X^n is a union of $\underbrace{\text{copies of } D^n}_{D^n_\alpha}$ with the boundaries of D^n_α attached to X^{n-1} via attaching maps.

Eg. Torus $T^2 = S^1 \times S^1 =$ 



$X^0 = \bullet = D^0$

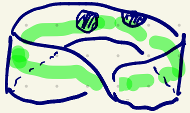
$X^1 =$  $\cong S^1 \vee S^1$



X^2

$\pi_1(T^2) \cong \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$

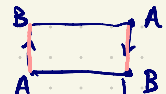
Möbius strip  not homeomorphic to cylinder $S^1 \times [0,1]$ =  = 



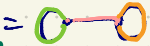
not orientable

$X^0 =$ $\cdot \quad \cdot$

$X^1 =$ 

$X^2 =$ 

Cylinder: $X^0 =$ $\cdot \quad \cdot$ $D^0 \sqcup D^0$

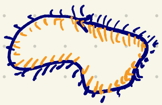
$X^1 =$ 

$X^2 =$ 

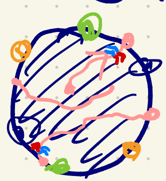
orientable

Both are homotopy equivalent to S^1 (def. retract to S^1)

Both have $\mathbb{Z}$ as fund. gp.



$P^2\mathbb{R}$ (or $\mathbb{RP}^2$) is the real projective plane is obtained from a disk D^2 with opposite boundary points identified

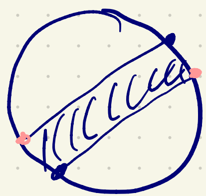


D^2

(non-orientable surface)

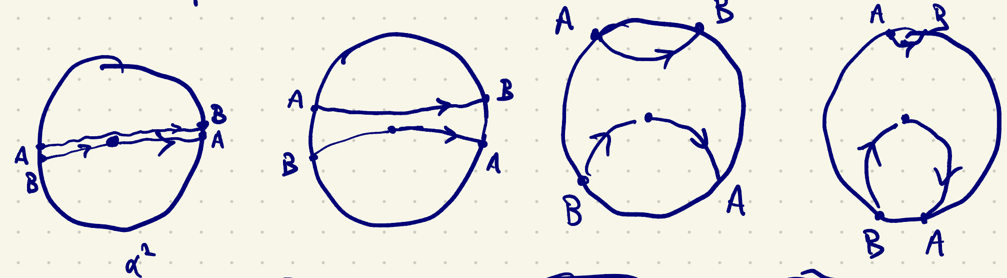
$P^2\mathbb{R} = D^2$ glued to a Möbius strip

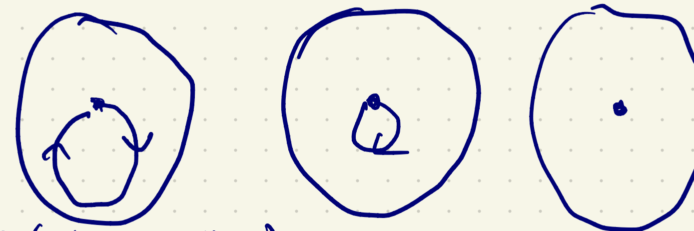
 



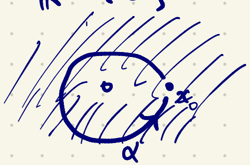
$\pi_1(P^2\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

α is not homotopic to the null path γ. α^2 is homotopic to γ





$\mathbb{R}^2-\{0\} \simeq S^1$



A homotopy equivalence

$$\mathbb{R}^2-\{0\} \xrightarrow{f_1} S^1$$

$$f_1((x,y)) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$f_t(v) = (1-t)v + t\frac{v}{|v|} \qquad f_1(v) = \frac{v}{|v|}$$

$f_t : \mathbb{R}^2-\{0\} \longrightarrow \mathbb{R}^2-\{0\}$ strong def. retraction since $f_t|_{S^1} = id_{S^1}$ for all $t \in [0,1]$.

$f_0 = id_{\mathbb{R}^2-\{0\}}$

f_1 is a retraction to S^1

$$\{\alpha^n : n \in \mathbb{Z}\} = \langle\alpha\rangle = \pi_1(\mathbb{R}^2-\{0\}) \cong \mathbb{Z}$$

free group on one generator

$\pi_1(S^1) \cong \mathbb{Z}$

Given a closed path α in S^1 with base point $1 \in S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

define $w(\beta) = \frac{1}{2\pi i} \int_\beta \frac{dz}{z} = \frac{1}{2\pi i} \int_{\beta(0)}^{\beta(1)} \frac{dz}{z}$

$\beta : [0,1] \to S^1$

$\beta(0) = 1$

$\beta(e^{2\pi i \theta}) = \theta + n$

$w(\alpha) = 1$

$w(\alpha^n) = n$

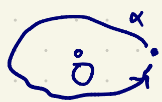
w is an isomorphism from $\pi_1(S^1)$ to $\mathbb{Z}$.

$\int \frac{dz}{z} = \ln|z| + 2\pi i \arg z$

In $\mathbb{C} - \{0\}$ the same argument works

$\mathbb{R}^2 - \{O\}$ $\quad O = (0,0)$

$\pi_1(\mathbb{R}^2 - \{O\}) = \langle \alpha \rangle \cong \mathbb{Z}$



k-punctured plane $X = \mathbb{R}^2 - \{ \overbrace{A_1, \dots, A_k}^{\text{distinct}} \}$

$\pi_1(X) = F_k = \mathrm{Free}(\{\alpha_1, \dots, \alpha_k\})$

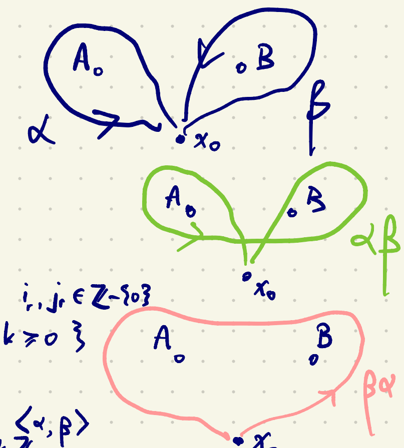
$X \simeq \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{k}$

$X = \mathbb{R}^2 - \{A, B\}$

$\pi_1(X) = \{ \alpha^{i_0}\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{j_k},$
$\alpha^{i_0}\beta^{j_0}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_k}\beta^{i_k},$
$\beta^{j_0}\alpha^{i_1}\beta^{j_1}\cdots\alpha^{i_{k-1}}\beta^{j_{k-1}}\alpha^{i_k} \;:\; i_r, j_r \in \mathbb{Z} - \{0\},\ k \geq 0 \}$

$\pi_1(X)$ is the free group on two generators i.e.

$F_2 = \mathrm{Free}(\{\alpha, \beta\}) = \langle \alpha, \beta \rangle$
$= \langle \alpha \rangle * \langle \beta \rangle = \mathbb{Z} * \mathbb{Z}$



The Van Kampen Theorem gives a presentation for $\pi_1(X)$ when X is suitably described in terms of smaller pieces.

A presentation for a group G expresses G as a homomorphic image of a free group F i.e. $G \cong F/N$, $N \trianglelefteq F$.

Let X be a set of generators of G ($X \subseteq G$, $\langle X \rangle = G$).

$\mathrm{Free}(X) \longrightarrow G$ is a surjective homomorphism; N is its kernel.

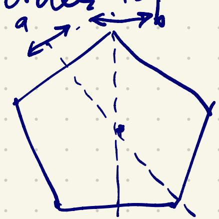
$G = \langle x_1, \dots, x_k : \overbrace{r_1, \dots, r_m}^{\in F} \rangle$ is a presentation for G if $X = \{x_1, \dots, x_k\}$ is a set of k symbols, $F = \mathrm{Free}(X)$ (the free group on $x_1, \dots, x_k$).

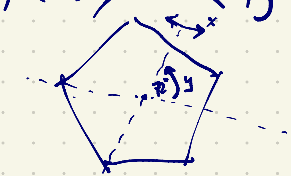
Let N be the smallest normal subgp of F containing $r_1, \dots r_m$ i.e. the normal closure of $\langle r_1, \dots, r_m \rangle \leq F$

i.e. the subgp. of F generated by $r_1, \dots, r_m$ and their conjugates in F

$N = \langle h r_i h^{-1} : i = 1, \dots, m;\ h \in F \rangle$. (When there are k generators and m relators, we say G is finitely presented.)

eg. the dihedral group of order 10 is $D_{10} \cong \langle a, b : a^2, b^2, (ab)^5 \rangle \cong \langle x, y : x^2, y^5, xyx^{-1}y \rangle$





$xyx^{-1}y = 1$
$\Longleftrightarrow xy = y^{-1}x$
$\Longleftrightarrow xyx^{-1} = y^{-1}$

$D_{10} \cong \langle a,b : a^2, b^2, (ab)^5 \rangle \xrightarrow[\phi]{} \cong \langle x,y : x^2, y^5, xyx^{-1}y \rangle$

$x^2 = a^2 = 1$

$y^5 = (ab)^5 = 1$

$x = a$

$y = ab$

$xyx^{-1} = a \cdot ab \cdot a^{-1} = ba$

whereas $y^{-1} = (ab)^{-1} = b^{-1}a^{-1} = ba$

$\longleftarrow$ check.

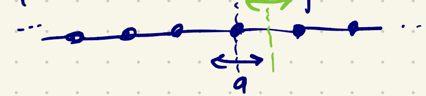
A free product $G * H$ is the set of all words from elements in G and H having no relations between elements in G and elements in H (except if one of these elements is the identity).

$\mathbb{Z} * \mathbb{Z} = \underbrace{\langle a \rangle}_{\text{infinite cyclic}} * \underbrace{\langle b \rangle}_{\text{infinite cyclic}} = \text{Free}\{a,b\} = \langle a,b \rangle = \{a^{i_1}b^{j_1}a^{i_2}b^{j_2} \ldots a^{i_k}b^{j_k}, \ldots \}$

$= \langle a,b : a^2, b^2 \rangle$

$\langle a : a^2 \rangle * \langle b : b^2 \rangle = \{1, a, b, ab, ba, aba, bab, abab, baba, ababa, babab, \ldots\} = D_\infty$

$\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle x,y : x^2, \underbrace{xyx^{-1} = y^{-1}}_{xyx^{-1}y = 1} \rangle$ = group of all automorphisms of



ie. $\langle X : R \rangle = \langle x_1, \ldots, x_m : r_1, \ldots, r_k \rangle$

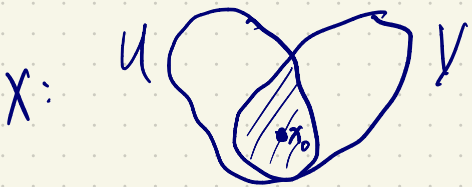
$\langle Y : S \rangle = \langle y_1, \ldots, y_n : s_1, \ldots, s_l \rangle$

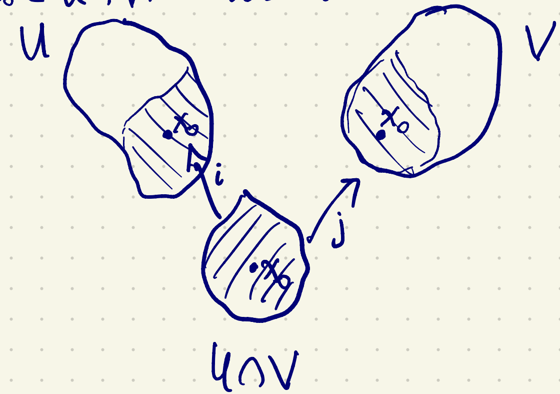
$\langle X : R \rangle * \langle Y : S \rangle = \langle X \cup Y : R \cup S \rangle = \langle x_1, \ldots, x_m, y_1, \ldots, y_n : r_1, \ldots, r_k, s_1, \ldots, s_l \rangle$

Free products with amalgamation: add more relations involving x_i's and y_j's e.g.

$$D_{10} = \langle a, b : a^2, b^2, (ab)^5 \rangle = \underbrace{\langle a : a^2 \rangle}_{\substack{\text{cyclic of}\\ \text{order 2}}} *_{(ab)^5} \underbrace{\langle b : b^2 \rangle}_{\substack{\text{cyclic of}\\ \text{order 2}}} = D_\infty \big/ \text{Normal closure of } \langle (ab)^5 \rangle$$

Let X be a path-connected top. space covered by two open sets U, V. Since X is connected, $U \cap V \neq \emptyset$. Pick $x_0 \in U \cap V$. We also assume $U \cap V$ is path-connected.



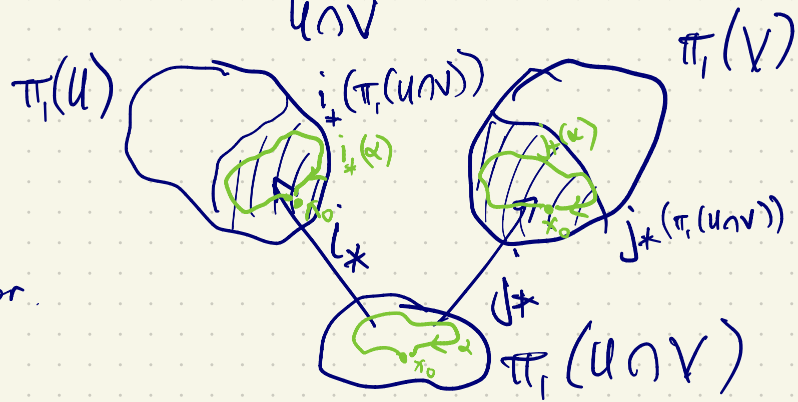


i, j inclusion maps as shown (injective, continuous)

Theorem (Van Kampen, special case)

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

where the amalgamation over $\pi_1(U \cap V)$ is given by: for all $\alpha \in \pi_1(U \cap V)$, identify $i_*(\alpha)$ with $j_*(\alpha)$ i.e. $i_*(\alpha)^{-1} j_*(\alpha)$ is a new relator.



This induces group homomorphisms i_*, j_* as shown.