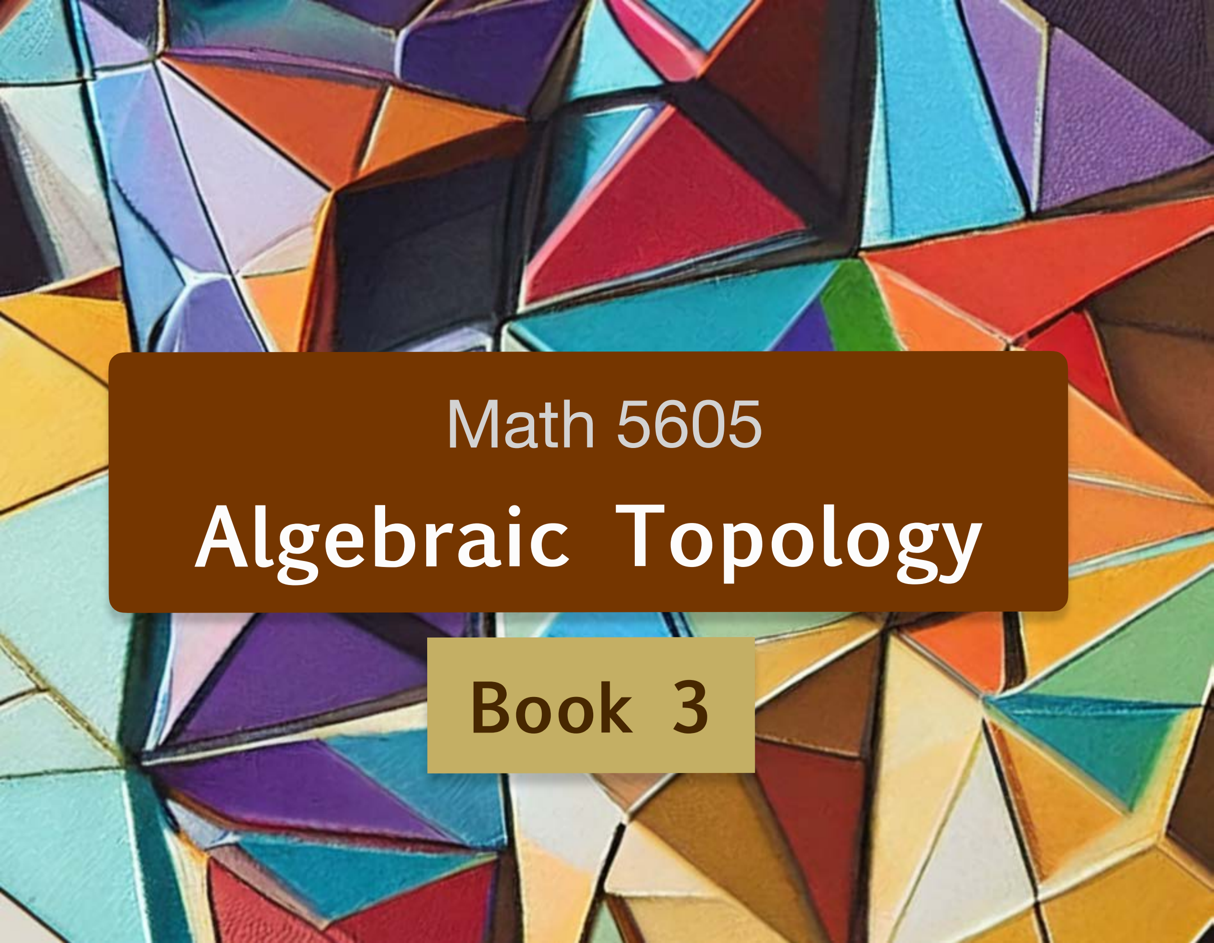


Math 5605

Algebraic Topology

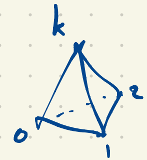
Book 3

Cup product for simplicial cohomology $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains: ($k = 0, 1, 2, 3, \dots$) ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex $\{$all subsets of $\{0,1,2,\dots,k\}\}$.

This has a geometric realization 

$\xrightarrow{\sigma} X$

$$\Delta = \Delta^n = \{\underbrace{(v_0, v_1, \dots, v_n)}_{\text{barycentric coordinates}} : v_i \geq 0,\ \sum v_i = 1\} \subset \mathbb{R}^{n+1}$$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \to X$.

$C_n = \{n\text{-chains in } X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{n\text{-cochains in } X\} = \mathrm{Hom}(C_n, R) = \{R\text{-homomorphisms } C_n \to R\}$

$\partial : C_n \to C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma | [v_0, \dots, \hat{v}_i, \dots, v_n]$ $\qquad \partial^2 = 0$, $(\partial^*)^2 = 0$

$\partial^* : C^{n-1} \to C^n$

If $\phi \in C^k$ k-cochain, $\psi \in C^l$ l-cochain, then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$

$[v_0, \dots, v_{k+l}] \longmapsto \sigma(v_0, \dots, v_{k+l})$

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\, \psi(\sigma|[v_k, \dots, v_{k+l}])$$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$ inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product) making $H^*(X;R)$ into a graded ring

$$H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R).$$

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{$ 1-dim'l subspaces of $\mathbb{R}^{n+1} \} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$ 

$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{ a_0 + a_1 x + \dots + a_n x^n : a_i \in \mathbb{F}_2 \}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$

Proof is by contradiction

Suppose $f : S^n \to S^{n-1}$ is antipodal. $(f(-x) = -f(x))$

Then f induces a well-defined map $P^n\mathbb{R} \xrightarrow{f} P^{n-1}\mathbb{R}$

$\pm x \mapsto \pm f(x)$ $(x \in S^n)$

$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$

f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces $f^* : H^*(P^{n-1}\mathbb{R}; \mathbb{F}_2) \to H^*(P^n\mathbb{R}; \mathbb{F}_2)$ mapping $x \longmapsto x$

$\cong \mathbb{F}_2[x]/(x^n)$, $\cong \mathbb{F}_2[x]/(x^{n+1})$

$x^n \longmapsto x^{n+1}$; contradiction.

If A is an additive abelian gp then $A \cong \underbrace{\mathbb{Z}^k}_{A/T(A)} \oplus T(A)$ canonically where $T(A) =$ torsion subgp of A = {elements of A of finite order}

$k = \operatorname{rank} A = \dim A$.

For any chain complex $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{0} 0$ (over $\mathbb{Q}$ or $\mathbb{R}$)

we have homology groups $H_n = \ker \partial_{n-1} / \operatorname{im} \partial_n$ with well-defined rank $H_n(X;\mathbb{Z}) = \operatorname{rank} H_n(X;\mathbb{Q}) = \operatorname{rank} H_n(X;\mathbb{R})$

and Euler characteristic

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} H_i(X) = \sum_{i=0}^{n} (-1)^i \operatorname{rank} C_i .$$

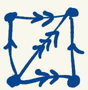
$C_n \xrightarrow{\partial_n} C_{n-1}$ $\dim C_n = \dim \ker \partial_n + \dim \operatorname{im} \partial_n$

$H_n = \ker \partial_n / \operatorname{im} \partial_{n+1}$ $\dim H_n = \dim \ker \partial_n - \dim \operatorname{im} \partial_{n+1}$

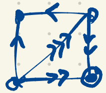
eg. $\chi(S^2) = 4 - 6 + 4 = 2$

Closed 2-manifolds i.e. connected compact 2-manifolds without boundary are completely classified using Euler characteristic and orientability (Yes/No)

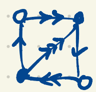
	S^2	T^2	$P^2\mathbb{R}$	K^2
$\dim H_2$	1	1	0	0
$\dim H_1$	0	2	0	1
$\dim H_0$	1	1	1	1
$\chi(X)$	2	0	1	0

T^2 

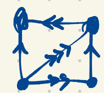
$1-3+2=0$

S^2 

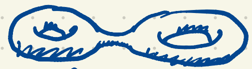
$3-3+2=2$

$P^2\mathbb{R}$ 

$2-3+2=1$

K^2 Klein bottle 

$1-3+2=0$

 #  = 

T^2 $\quad$ T^2 $\quad$ T^2 $\quad$ T^2

$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ for any two closed surfaces S_1, S_2

$\chi(T^2 \# T^2) = \chi(T^2) + \chi(T^2) - 2 = 0+0-2 = -2$

$\underbrace{T^2 \# \cdots \# T^2}_{g}$ = 

$\chi(T^2 \# \cdots \# T^2) = 2-2g$ $\qquad$ g = genus of orientable surface

$\chi(\underbrace{P^2\mathbb{R} \# P^2\mathbb{R}}_{K^2}) = 1+1-2 = 0$

Exact sequences $\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$ $\quad \ker \partial_n = \operatorname{im} \partial_{n+1}$

$0 \to C \to 0$ is exact iff $C = 0$

$0 \to A \to B \to 0$ is exact iff $A \cong B$

$0 \to A \to B \to C \to 0$ is exact (short exact) iff $C \cong B/A$

If $f: X \to X$ is an endomorphism of an abel. gp. X (or vector space) (at least in an abelian category) Some important short exact sequences are

$$0 \to \ker f \to X \to f(X) \to 0$$

$$0 \leftarrow \operatorname{coker} f \leftarrow X \leftarrow f(X) \leftarrow 0$$

If $f: X \to Y$ then $\operatorname{coker} f = Y/\underbrace{f(X)}_{\operatorname{im} f}$

If $\cdots \to A \to B \to C \to 0$ and $0 \to C \to D \to E \to \cdots$ are exact then we get an exact seq.

$$\to A \to B \to D \to E \to \cdots$$

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & A \arrow[dr] & & & E \arrow[r, dashed] & \cdots \\
& & B \arrow[rr, dashed] \arrow[dr] & & D \arrow[u] & \\
& & & C \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

$$0 \to \ker f \to X \xrightarrow{f} X \to \operatorname{coker} f \to 0 \quad \text{is exact.}$$

If X is a fin. dim'l vector space over F then the Euler char. of this sequence is

$$\dim \operatorname{coker} f - \dim X + \dim X - \dim \ker f = 0$$

If $T: X \to X$ is an operator (endomorphism) (don't worry about boundedness) the index of T is $\operatorname{ind} T = \dim \operatorname{coker} T - \dim \ker T$ when both of these terms are finite (i.e. T is Fredholm).

Theorem: Let $S, T: X \to X$ be operators (lin. transf). Of the three operators S, T, ST, then whenever two are Fredholm then so is the third and in this case $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.

In general (i.e. for any lin. transf. $S, T: X \to X$ (or abel. gps) we have an exact sequence

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0$$

So its Euler characteristic is zero. i.e. $\operatorname{ind} S + \operatorname{ind} T - \operatorname{ind} ST = 0$.

Snake Lemma In an abel. category we have a commutative diagram with exact rows

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

then we have a six-term exact seq.

$$\ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c.$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[ddll, bend left] & \\
 & A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r] \arrow[d, "c"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} a \arrow[r] \arrow[d] & \operatorname{coker} b \arrow[r] \arrow[d] & \operatorname{coker} c \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Group Cohomology : used in the study of group extensions

If G and H are groups then an extension of H by G is a group X giving an exact sequence $1 \longrightarrow H \longrightarrow X \longrightarrow G \longrightarrow 1$

Note: Groups are not necessarily abelian. We are asking for a new group X having a normal subgp $\cong H$ s.t. $X/H \cong G$. G on top, H on the bottom.

Trivial: $X = G \times H$. (split extension)

G is now an arbitrary group and A is an abelian group (G multiplicative ; A additive notation) on which G acts (each $g \in G$ gives $g \in GL(A)$ (automorphisms of A as an abel. gp or $\mathbb{Z}$-module).

$(g_1 g_2)(a) = g_1(g_2(a))$; $g(a+b) = ga + gb$; $\underset{\hat{G}}{1}a = a$. $G \xrightarrow[\text{(fixed)}]{\text{homo.}} \operatorname{Aut} A = GL(A)$

We construct an extension of A by G i.e. an exact sequence of gps

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & \hat{G} \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \arrow[d] \\
1 \arrow[r] & A \arrow[r] & \hat{\hat{G}} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

i.e. $\hat{G}$ is a gp with normal subgp. iso. to A with $\hat{G}/A \cong G$

Two extensions $\hat{G}$ $\hat{\hat{G}}$ are equivalent if we have a commutative diagram as shown with α, β, γ isomorphisms of groups (with exact rows). Note that the action of G on A is fixed throughout.

Cohomology of groups is the tool for this.

$\cdots \xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \xleftarrow{\delta} 0$ is an exact sequence of additive abel. gps where $C^k = C^k(G;A)$ is the set of all maps $\underbrace{G^k}_{G\times G\times\cdots\times G} \to A$ as an additive abel. gp. i.e. $\mathbb{Z}$-module

i.e. k-tuples of G

$C^0 = A$ (maps $\{1\} \to A$)

$C^1 = A^G$ = maps $G \to A$ i.e. $f: G \to A$

$C^2 = A^{G\times G}$ = maps $G\times G \to A$ etc.

Given $a \in C^0$ i.e. $a \in A$, $\delta a \in C^1$ is $\delta a : G \to A$

$g \mapsto ga - a$

Given $f \in C^1$ i.e. $f: G \to A$

construct $\delta f \in C^2$ i.e. $(\delta f): G \times G \to A$

$$(\delta f)(g,h) = g\underbrace{f(h)}_{\in A} - \underbrace{f(gh)}_{\in A} + \underbrace{f(g)}_{\in A} \in A.$$

Given $f \in C^2$ i.e. $f: G\times G \to A$

construct $(\delta f): G\times G\times G \to A$

$$(\delta f)(g,h,k) = gf(h,k) - f(gh,k) + f(g,hk) - f(g,h)$$

See p. 2 bottom of handout for $\delta: C^k \to C^{k+1}$ in general

Check: $C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0$ $\delta^2 = 0$?

Take $a \in C^0 = A$.

$(\delta a): G \to A$

$(\delta a)(g) = ga - a$.

$\delta^2 a : G\times G \to A$

$$(\delta^2 a)(f,g) = f(\delta a)(g) - (\delta a)(fg) + \delta a(f)$$

$$= f(ga-a) - ((fg)(a) - a) + (fa - a)$$

$$= \cancel{fga} - \cancel{fa} - \cancel{fga} + \cancel{a} + \cancel{fa} - \cancel{a}$$

$$= 0$$

Classify extensions $1 \to A \to \hat{G} \to G \to 1$ where G is a group acting on an abelian gp A i.e. $\hat{G}$ is a group with normal subgp A with $\hat{G}/A \cong G$, using cohomology. Start with a split extension i.e. A has a complementary subgp in $\hat{G}$. So $\hat{G}$ acts on the subgps complementary to A by conjugation.

$H^1(G;A)$ classifies the complementary subgps up to conjugacy.

Fix an action of $\underset{\text{mult.}}{G}$ on $\underset{\text{additively}}{A}$.

$$a(g_1g_2) = (ag_1)g_2$$
$$a\underset{\uparrow \text{ id. of } G}{1} = a$$
$$(a+a')g = ag + a'g \qquad \text{for } a,a'\in A;\quad 1,g,g_1,g_2\in G.$$

here A is a <u>right</u> G-module.

$\hat{G}$ is isomorphic to the semidirect product $A\rtimes G = \{(a,g) : a\in A,\ g\in A\}$

$G\ltimes A$ for left action

$$(a_1,g_1)(a_2,g_2) = (a_1g_2+a_2,\ g_1g_2) \qquad \text{identity } (\underset{\text{in } A}{0}, \underset{\text{in } G}{1})$$

Alternative notation: $A\rtimes G = \left\{ \begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} : a\in A,\ g\in G \right\}$

$$\begin{bmatrix} g_1 & 0 \\ a_1 & 1 \end{bmatrix}\begin{bmatrix} g_2 & 0 \\ a_2 & 1 \end{bmatrix} = \begin{bmatrix} g_1g_2 & 0 \\ a_1g_2+a_2 & 1 \end{bmatrix}$$

Complements of A in $\hat{G}$ are given by 1-cocycles.

$$C^2 \xleftarrow{\ \delta^1\ } C^1 \xleftarrow{\ \delta^0\ } \underset{A}{\overset{\shortparallel}{C^0}} \longleftarrow 0$$

For $a\in A = C^0$, $(\delta a)(g) = ag - a$

How do we construct a subgp of $A\rtimes G$ complementary to A?

Any such subgp $H\le A\rtimes G$ has the form $\{(t_g,g) : g\in G\} = \left\{ \begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix} : g\in G \right\}$

Here $g \overset{f}{\longmapsto} t_g$, $G\longrightarrow A$. This will automatically be a complement to A as long as it is a subgp. eg. $t_1=0$ but most importantly, closure.

$$\begin{bmatrix} g & 0 \\ t_g & 1 \end{bmatrix}\begin{bmatrix} g' & 0 \\ t_{g'} & 1 \end{bmatrix} = \begin{bmatrix} gg' & 0 \\ t_{gg'} & 1 \end{bmatrix} \quad \text{i.e. } \underbrace{t_{gg'}}_{A} = \underbrace{t_g}_{A}\underbrace{g'}_{G} + \underbrace{t_{g'}}_{A}$$

so $(\delta f)(g,g') = -f(gg') + f(g)g' + f(g') = 0$.

When are two complements of A conjugate in $\hat{G}$? $\quad \hat{G} = A \rtimes G$

If A has complementary subgps $H_1, H_2 \le \hat{G}$ given by $H_i = \{ (f_i(g), g) : g \in G \}$, $\quad f_i \in Z^1(G;A)$

$$(\delta f_i)(g,g') = f_i(g') - f_i(gg') + f_i(g)g' = 0$$

when are H_1, H_2 conjugate in $\hat{G}$?

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}^{-1} = \begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix}$$

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use $\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix} \in \hat{G}$ (a, g fixed) to conjugate H_1:

$$\begin{bmatrix} g & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ f_1(x) & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gx & 0 \\ ax+f_1(x) & 1 \end{bmatrix}\begin{bmatrix} g^{-1} & 0 \\ -ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ axg^{-1}+f_1(x)g^{-1}-ag^{-1} & 1 \end{bmatrix} = \begin{bmatrix} gxg^{-1} & 0 \\ f_2(gxg^{-1}) & 1 \end{bmatrix}$$

The 1-cycle defining this conjugate subgp would have to be f_2: so

$$f_2(gxg^{-1}) = axg^{-1} + f_1(x)g^{-1} - ag^{-1} = f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1}$$

$$\begin{aligned} f_2(gxg^{-1}) &= f_2(g(xg^{-1})) \\ &= f_2(g)xg^{-1} + f_2(xg^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} + f_2(g^{-1}) \\ &= f_2(g)xg^{-1} + f_2(x)g^{-1} - f_2(g)g^{-1} \end{aligned}$$

$$\begin{aligned} ax + f_1(x) - a &= f_2(g)x + f_2(x) - f_2(g) \\ f_2(x) - f_1(x) &= (a - f_2(g))x - (a - f_2(g)) \\ &= \delta(a - f_2(g))(x) \end{aligned}$$

$f \in C^1$ i.e. $f: G \to A$

f is a 1-cocycle: $f \in Z^1$

$(\delta f)(x,y) = f(x)y - f(xy) + f(y) = 0$

f is a crossed homomorphism or derivations

$$\boxed{f(xy) = f(x)y + f(y)}$$

$$\boxed{f(1) = f(1 \cdot 1) = f(1)1 + f(1) \Rightarrow f(1) = 0}$$

(If G acts trivially on A

i.e. $ag = a$ for all $a \in A, g \in G$)

then f is a homo. $G \to A$.

$$\boxed{0 = f(1) = f(gg^{-1}) = f(g)g^{-1} + f(g^{-1}) \Rightarrow f(g^{-1}) = -f(g)g^{-1}}$$

Extensions of A by G correspond to elements of $H^1 = Z^1/B^1$.

$f \in B^1$ (1-coboundary) iff

$f(x) = ax - a = (\delta a)(x)$, $\quad a \in A$.

(principal crossed homomorphisms)

(inner derivations)

Eg. Classify extensions of $C_4 = \langle x : x^4 = 1 \rangle$ by $C_2 = \langle y : y^2 = 1 \rangle$

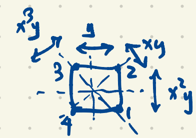
$x = (1234)$
$y = (14)(23)$
$x^2y = (12)(34)$
$xy = (13)$
$x^3y = (24)$

$$1 \longrightarrow C_4 \longrightarrow \hat{G} \longrightarrow C_2 \longrightarrow 1$$

Two cases depending on the action of C_2 on C_4

Case I: y inverts x i.e. $yxy^{-1} = x^{-1} = x^3$

$|H^1| = 2$. = how many complementary subgps of C_4 up to conjugacy.



C_4 has four complementary subgps in $\hat{G} \cong$ dihedral gp of order 8.

$\langle y \rangle$, $\langle x^2y \rangle$ are conjugate to each other in $\hat{G}$

$\langle xy \rangle, \langle x^3y \rangle$ are conjugate to each other in $\hat{G}$

Not conjugate.

Case II: y commutes with x. $xy = yx$ $\hat{G} = C_4 \times C_2$

$\langle x \rangle$ has two complements in $\hat{G}$, namely $\langle y \rangle$, $\langle x^2y \rangle$. They are not conjugate. $|H^1| = 2$.

How many extensions $1 \to C_4 \to \hat{G} \to C_2 \to 1$ are there up to equivalence, if we don't require the extension to be split? (Split $\Longleftrightarrow$ there is a complementary subgp for C_4)

eg. C_8 is a nonsplit extension of C_4 by C_2.

Case I: C_2 acts trivially on C_4. Here there are two extensions: C_8 (nonsplit) and $C_4 \times C_2$ (split).

Case II: C_2 acts nontrivially on C_4. Here there are two extensions: dihedral of order 8 (split), quaternion gp of order 8 (nonsplit).

The Schur-Zassenhaus Theorem

Given groups G, N with G acting N (the action of G on N is fixed) we consider exact sequences of groups

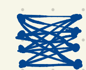
$$1 \longrightarrow N \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

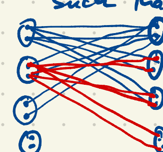
i.e. extensions of N by G i.e. groups $\hat{G}$ having a normal subgp isomorphic to N.
If $|N|$, $|G|$ are relatively prime then $H^1 = 1$ and $H^2 = 1$.
This says that the extension splits i.e. $\hat{G}$ has a subgp complementary to N and any two complements of N are conjugate in $\hat{G}$.
This generalizes Sylow theory; N and its complements are Hall subgps.

Note: We do not require N to be abelian. If N is abelian then the formulas are simpler. Even simpler if $N \subset Z(\hat{G})$.
(central extension of N by G)

A loop is a set L with a binary operation $(x,y) \longmapsto xy$, $L \times L \longrightarrow L$ such that any two of x, y, xy uniquely determine the other.
We also assume $\exists\, 1 \in L$ such that $1x = x1 = x$ for all $x \in L$.
A Bol loop satisfies $((xy)z)y = x((yz)y)$ for all $x, y, z \in L$.

A Hadamard matrix is an $n \times n$ matrix H with entries ± 1 such that $HH^T = nI = H^T H$

eg. $H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$ gives a (distance regular) double cover of $K_{4,4}$: 



A complex Hadamard matrix is an $n \times n$ matrix H with entries in $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that $HH^* = nI = H^* H$

I classified complex Had. matrices with an automorphism group which is doubly transitive on rows.

A proj. plane has points and lines satisfying  

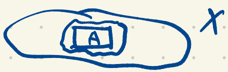
Any $(n^2+n+1)\times(n^2+n+1)$ matrix with 0/1 entries, $n+1$ ones in any row/col, row · row = 1 = col · col

n = The order of the plane

In all known cases, $n = p^k$, p prime, $k \geq 1$.

Some long exact sequences in homology

Take a top. space X having a closed subspace $A \subset X$ which is the deformation retract of an open nbhd in X.



X/A : collapse A to a single point, leaving points outside A untouched.

quotient space

Then we have a long exact sequence

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A)$$
$$\to \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(X/A) \to \cdots$$
$$\cdots \to \tilde{H}_0(X/A) \to 0$$

$$\tilde{H}_n(X) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

$$\to C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

has $H_n(X)$

$$\to C_{n+1} \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

has reduced homology $\tilde{H}_n(X)$.

Theorem ($k \geq 1$) $H_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n \in \{0, k\} \\ 0 & \text{otherwise} \end{cases}$

i.e. $\tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{else} \end{cases}$

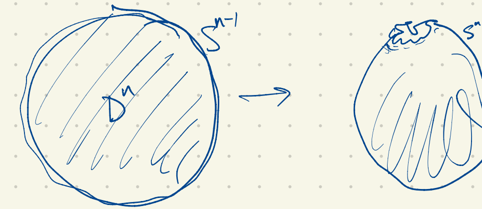
$$A \longrightarrow X \longrightarrow X/A$$

Proof: Take $X = D^n$ = closed n-ball $(n \geq 1)$

$A = \partial X = S^{n-1}$

$X/A \cong S^n$ = one-point compactification of $\mathbb{R}^n$

$\tilde{H}_k(D_n) = 0$ for all k



$$\cdots \to \underbrace{\tilde{H}_k(D^n)}_{X} \to \tilde{H}_k(\underbrace{S^n}_{X/A}) \to \tilde{H}_{k-1}(\underbrace{S^{n-1}}_{A}) \to \tilde{H}_{k-1}(\underbrace{D^n}_{X}) \to \cdots$$

$\tilde{H}_k(D^n) = 0$, $\tilde{H}_{k-1}(D^n) = 0$

$\Rightarrow \tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ depends only on $k-n$.

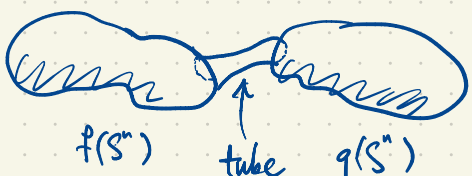
$\tilde{H}_n(S^n) \cong \cdots \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$

$$\tilde{H}_k(S^1) = \begin{cases} \mathbb{Z} & \text{if } k=1 \\ 0 & \text{else.} \end{cases}$$

Homotopy groups $\pi_n(X)$ "counts" ways to map $S^n \xrightarrow{f,g} X$

X is assumed to be path-connected.

In X:



Cut holes $\cong D^n$ in $f(S^n)$, $g(S^n)$ and join using $\underbrace{S^{n-1} \times [0,1]}_{\text{tube}}$

Identity element: constant.

Elements of $\pi_n(X)$ are up to homotopy.

$f_0, f_1 : S^n \to X$ are homotopic iff there exists $f: S^n \times [0,1] \to X$, $(x,t) \mapsto f_t(x)$

The group operation is associative.
It's commutative for $n \geq 2$.

If X, Y have the same homotopy type then $\pi_n(X) \cong \pi_n(Y)$ for all n. (Not conversely)

If $\pi_k(X) = 1$ for $k < n$ and $\pi_n(X) \neq 1$, then $H_n(X) =$ abelianization of $\pi_n(X)$.
(theorem of Hurewicz)

Homotopy groups are easier to define than homology groups
much harder to compute

$\pi_n(S^k)$ is not known in general

But some is known:

$\pi_k(S^n) = 0$ for $1 \leq k \leq n-1$

$\pi_n(S^n) \cong \mathbb{Z}$

$\pi_3(S^2) \cong \mathbb{Z}$ is due to the Hopf fibration $1 \to S^1 \to S^3 \to S^2 \to 1$

This means we have a map $f: S^3 \to S^2$ which is surjective and all fibres are circles. In other words we can partition S^3 into circles.

$S^3 =$ unit sphere in $\mathbb{R}^4$ = unit sphere in $\mathbb{H}$.
$= \{ (z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1 \}$

$$f(z,w) = \begin{cases} (1, \frac{w}{z}), & \text{if } z \neq 0 \\ (0, 1), & \text{if } z = 0 \end{cases} \in S^2 = P^1\mathbb{C} = \text{Riemann sphere}$$

eg. $(1, 2) \in P^1\mathbb{C}$
has preimage (fibre)
$\{ (z, 2z) : |z|^2 + |2z|^2 = 1 \}$
$|z|^2 = \frac{1}{5}$
$\left\{ e^{i\theta}\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) : \theta \in [0, 2\pi) \right\}$

$1 \longrightarrow S^0 \longrightarrow S^1 \longrightarrow S^1 \longrightarrow 1$ $\mathbb{R} \subset \mathbb{C}$

$1 \longrightarrow S^1 \longrightarrow S^3 \longrightarrow S^2 \longrightarrow 1$ $\mathbb{C} \subset \mathbb{H}$

$1 \longrightarrow S^3 \longrightarrow S^7 \longrightarrow S^4 \longrightarrow 1$ $\mathbb{H} \subset \mathbb{O}$

$1 \longrightarrow S^7 \longrightarrow S^{15} \longrightarrow S^8 \longrightarrow 1$ $\mathbb{O} \subset \mathbb{S}$

$1 \longrightarrow S^1 \longrightarrow S^1 \times S^2 \longrightarrow S^2 \longrightarrow 1$ trivial bundle has (global) nonvanishing section

The Hopf fibration does not have such a section; it is a nontrivial fibre bundle.

A fibre bundle over B (B = base space) with fibres F is a way of continuously attaching a copy of F at each point of B.

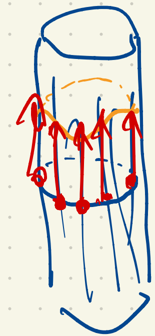
Trivial bundle: $E = F \times B$

Another way: eg. S^1 has more than one kind of line bundle (vector bundle over $\mathbb{R}$ of dimension 1) having fibre space $F = \mathbb{R}$

$$\mathbb{R} \longrightarrow E \longrightarrow \boxed{S^1}$$

E: "Total space"

Trivial bundle $\mathbb{R} \longrightarrow \underbrace{S^1 \times \mathbb{R}}_{\text{"infinite cylinder"}} \longrightarrow S^1$

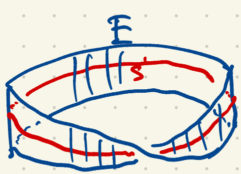


S^1

A section of a fibre bundle $F \longrightarrow E \xrightarrow{p} B$ is a right inverse for p

i.e. $g: B \to E$ such that $p \circ g = id_B$.

Nontrivial:



$\mathbb{R} \longrightarrow E \longrightarrow S^1$

Möbius strip (infinite)

This fibre bundle is nontrivial because it has no nonvanishing global sections.

For every fibre bundle $F \to E \to B$ we have a long exact sequence of homotopy groups

$$\cdots \to \pi_k(F) \to \pi_k(E) \to \pi_k(B) \to \pi_{k-1}(F) \to \cdots$$

A line bundle is a bundle with fibre space given by some field (usually $\mathbb{R}$ or $\mathbb{C}$)

A vector bundle has fibre space F^n (e.g. $\mathbb{R}^n$, $\mathbb{C}^n$, ...)

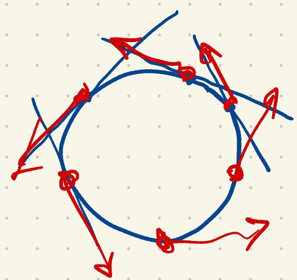
Sections of a vector bundle are the same thing as vector fields.

A circle bundle has fibre space $\cong S^1$.

A tangent bundle for a surface (hypersurface) $M \subset \mathbb{R}^n$ (M an $n-1$-manifold) has fibres $\cong \mathbb{R}^{n-1}$.

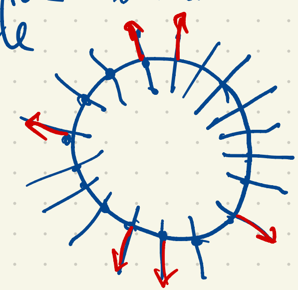
Eg. the tangent line bundle on S^1

Which one is it (trivial or Möbius strip)?

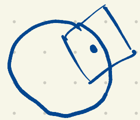


also trivial.

Eg. the normal line bundle



Eg. $S^2 \subset \mathbb{R}^3$ Tangent bundle E



$\mathbb{R}^2 \to E \to S^2$

2 dim'l 4-dim'l 2-dim'l

$(s,v) \mapsto s$

$(s,v) \leftarrow s$ v constant

Is this bundle trivial? No.

The trivial bundle $S^2 \times \mathbb{R}^2$ has a nonvanishing section.

$\cup\!\!\!\vert$

$(s,v) \longrightarrow S^2$

$(s,v) \longmapsto s \in S^2$

Hairy Ball Theorem: The tangent bundle on S^2 (or S^{2k}) has no nonvanishing (global) section ie. on S^2 there is no nonvanishing vector field. On the Earth at any instant in time, any tangent vector field (eg. wind) must be zero at some point.

Proof: Suppose $g: S^2 \to E$ is a nonvanishing section, WLOG $g(s) = (s,v)$ where $\|v\| = 1$. Such a map g defines a homotopy from id_{S^2} to the antipodal map on S^2

$f_t : S^2 \to S^2$, $0 \le t \le 1$

$f_t(s) = f(s,t)$, $(s,t) \in S^2 \times [0,1]$

$f_0(s) = s$ identity on S^2

$f_1(s) = -s$ (antipodal point on S^2)

$f_t(s)$: start at $s \in S^2$ and go $t\pi$ radians on S^2 in the direction of $g(s)$.

Remark: If $g: S^2 \to S^2$ then g induces a map on $H_2(S^2) \cong \mathbb{Z}$ (take singular homology). or $\pi_2(S^2) = \mathbb{Z}$

$g_* \in H_2(S^2)$ induced by g has a well-defined degree of g in $\mathbb{Z}$.

The identity $S^2 \to S^2$ has degree 1.

A constant map $S^2 \to S^2$ 0

The antipodal map $S^2 \to S^2$ has degree -1.