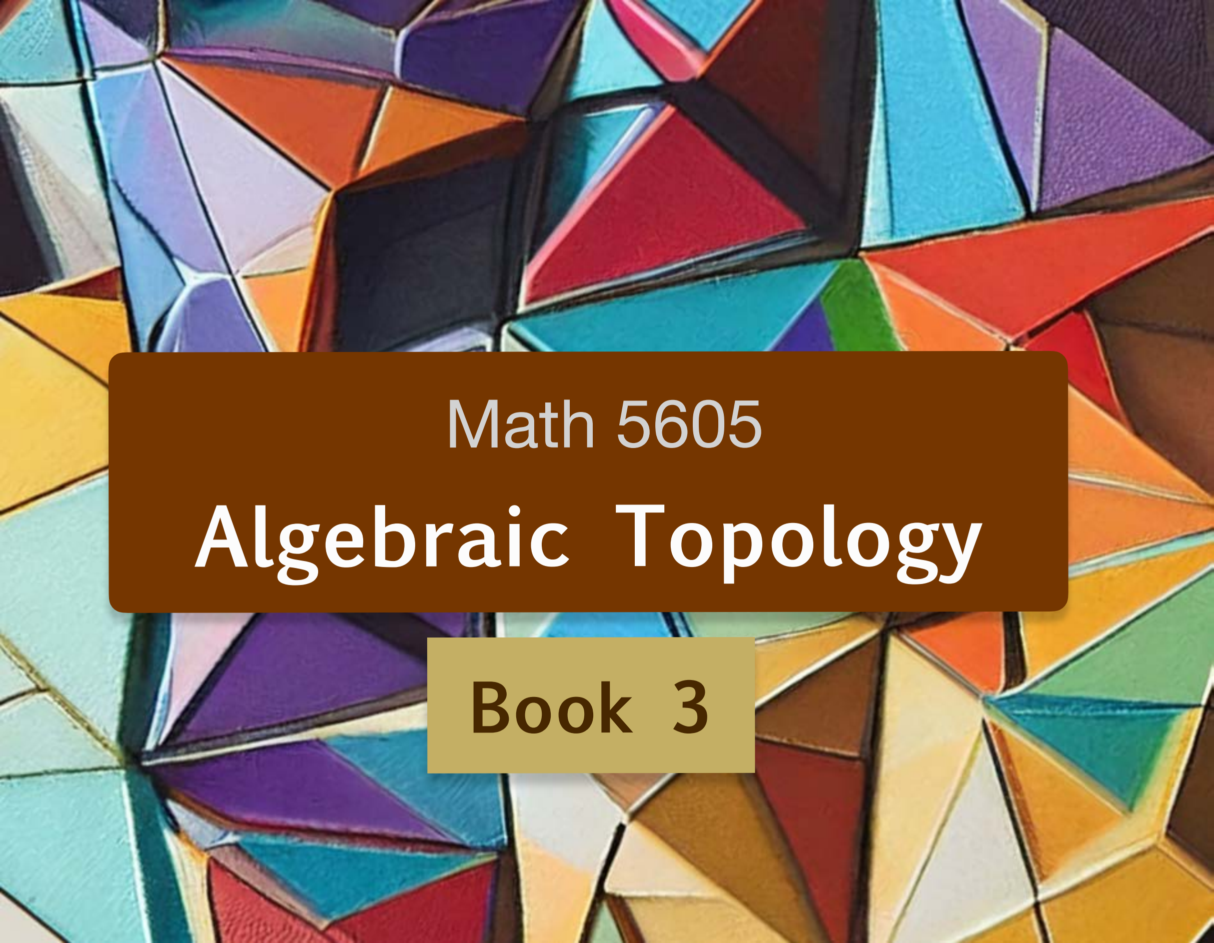


Math 5605

Algebraic Topology

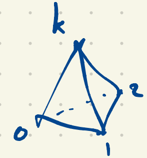
Book 3

Cup product for simplicial cohomology $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ makes $H^*(X;\mathbb{Z})$ or $H^*(X;R)$ into a graded ring.

To explain, let's talk about singular homology and cohomology.

Singular k-chains : $(k=0,1,2,3,\dots)$ ways of mapping k-simplices into X, not necessarily embeddings.

Take an abstract k-simplex $\{$all subsets of $\{0,1,2,\dots,k\}\}$.

This has a geometric realization  $\xrightarrow{\sigma} X$

$\Delta = \Delta^n = \{\underbrace{(v_0,v_1,\dots,v_n)}_{\text{barycentric coordinates}} : v_i \geq 0,\ \Sigma v_i = 1\} \subset \mathbb{R}^{n+1}$ (convex combinations of $e_0 = (1,0,\dots,0)$, $e_1, \dots, e_n = (0,\dots,0,1)$)

An n-chain is a formal linear combination of maps $\sigma : \Delta^n \longrightarrow X$.

$C_n = \{n\text{-chains in } X\} = C_n(X;R)$, R any commutative ring with 1 eg. $\mathbb{R}, \mathbb{Z}, \mathbb{F}_2$

$C^n = C_n^* = \{n\text{-cochains in } X\} = \mathrm{Hom}(C_n, R) = \{R\text{-homomorphisms } C_n \longrightarrow R\}$

$\partial : C_n \longrightarrow C_{n-1}$, $\partial\sigma = \sum_{i=0}^{n} \sigma | [v_0,\dots,\hat{v}_i,\dots,v_n]$ $\qquad \partial^2 = 0,\ (\partial^*)^2 = 0$

$\partial^* : C^{n-1} \longrightarrow C^n$

If $\phi \in C^k$ k-cochain, then $\phi \cup \psi \in C^{k+l}$ cochain: for any $(k+l)$-chain $\sigma : \Delta^{k+l} \longrightarrow X$

$\psi \in C^l$ l-cochain

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_k])\,\psi(\sigma|[v_k, \dots, v_{k+l}])$$

$[v_0, \dots, v_{k+l}] \longmapsto \sigma(v_0, \dots, v_{k+l})$

This gives a bilinear product $C^k \times C^l \xrightarrow{\cup} C^{k+l}$
inducing a bilinear product $H^k \times H^l \xrightarrow{\cup} H^{k+l}$ (cup product)
making $H^*(X;R)$ into a graded ring

$$H^*(X;R) = \bigoplus_{i \geq 0} H^i(X;R).$$

Eg. $X = P^n\mathbb{R}$, $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$; $H^i(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & 0 \leq i \leq n \\ 0, & \text{else} \end{cases}$

$P^n\mathbb{R} = \{\text{1-dim'l subspaces of } \mathbb{R}^{n+1}\} = S^n/\text{antipodality}$

$P^1\mathbb{R} \cong S^1/\text{antipodality} \cong S^1$

$P^n\mathbb{R}$ is orientable iff n is odd.

$P^2\mathbb{R} = S^2/\text{antipodality} =$ 

$H^*(X;\mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^{n+1})$ Additively: $\{a_0 + a_1x + \dots + a_nx^n : a_i \in \mathbb{F}_2\}$

Borsuk-Ulam Theorem: There is no antipodal map $S^n \xrightarrow{f} S^{n-1}$ for $n \geq 2$.
i.e. $f(-x) = -f(x)$

Proof is by contradiction

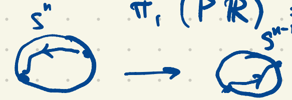
Suppose $f: S^n \to S^{n-1}$ is antipodal. $(f(-x) = -f(x))$

Then f induces a well-defined map

$$\begin{array}{ccc} P^n\mathbb{R} & \xrightarrow{f} & P^{n-1}\mathbb{R} \\ \Downarrow & & \Downarrow \\ \pm x & & \pm f(x) \end{array}$$

$(x \in S^n)$

$\pi_1(P^n\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$



f^* maps a generator of $\pi_1(P^{n-1}\mathbb{R})$ to a generator of $\pi_1(P^n\mathbb{R})$

f induces $f^*: H^*(P^{n-1}\mathbb{R};\mathbb{F}_2) \to H^*(P^n\mathbb{R};\mathbb{F}_2)$ mapping $x \mapsto x$

$$\begin{array}{ccc} H^*(P^{n-1}\mathbb{R};\mathbb{F}_2) & \longrightarrow & H^*(P^n\mathbb{R};\mathbb{F}_2) \\ \wr\Vert & & \wr\Vert \\ \mathbb{F}_2[x]/(x^n) & & \mathbb{F}_2[x]/(x^{n+1}) \end{array}$$

$x^n \longmapsto x^{n+1}$; contradiction.