Cohomology

1. Graded Rings

Denote by $A = F[X_1, X_2, \ldots, X_n]$ the ring of polynomials in n indeterminates $X_1, X_2, \ldots, X_n$ with coefficients in F. This is in fact an algebra over F (a ring which is also a vector space over F). As a vector space we have a direct sum decomposition

$$A = \bigoplus_{k \geq 0} A_k$$

where A_k is the subspace consisting of all homogeneous polynomials of degree k. A basis for A_k is given by the monomials of degree k:

$$X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n} \quad \text{where} \quad i_1, i_2, \ldots, i_n \geq 0, \quad i_1+i_2+\cdots+i_n = k.$$

In particular the dimension of A_k is the number of such monomials, namely $\binom{n-1+k}{k}$. We have

$$A_k A_\ell \subseteq A_{k+\ell}\,,$$

i.e. the product of homogeneous polynomials of degree k and ℓ is a homogeneous polynomial of degree $k+\ell$. Thus A is an example of a *graded ring* (in this case a *graded algebra*). More generally a ring A is said to be *graded* if we have a direct sum decomposition $A = \bigoplus_{k \geq 0} A_k$ where the A_k's are additive subgroups satisfying $A_k A_\ell \subseteq A_{k+\ell}$.

Another example of a graded ring is the quotient ring $F[X]/(X^{n+1})$; thus for example $F[X]/(X^3)$ has elements $a_0 + a_1 X + a_2 X^2$ [actually $a_0 + a_1 X + a_2 X^2 + (X^3)$ but we simply denote this coset by its unique representative of smallest degree] and multiplication defined by

$$(a_0 + a_1 X + a_2 X^2)(b_0 + b_1 X + b_2 X^2) = a_0 b_0 + (a_0 b_1 + a_1 b_0)X + (a_0 b_2 + a_1 b_1 + a_2 b_0)X^2.$$

In this case the ring $A = A_0 \oplus A_1 \oplus A_2$ is three-dimensional, with each homogeneous part A_0, A_1, A_2 of dimension 1, and $A_k = 0$ for $k \notin \{0, 1, 2\}$. We will see that the cohomology ring of the real projective plane with coefficients in the field $\mathbb{F}_2$ of order two, is of this form:

$$H^*(P^2\mathbb{R}; \mathbb{F}_2) \cong \mathbb{F}_2[X]/(X^3).$$

2. Exterior Algebra

Let V be an n-dimensional vector space over a field F. The *k-th exterior power of V* is defined as

$$\bigwedge^k V = \left(\bigotimes^k V\right)/S$$

where $\bigotimes^k V = V \otimes V \otimes \cdots \otimes V$ (the k-th tensor power of V) and S is the subspace spanned by all pure tensors of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ such that $v_i = v_j$ for some $i \neq j$. We denote the image of a typical pure tensor $v_1 \otimes \cdots \otimes v_k \in \bigotimes^k V$ by

$$v_1 \wedge \cdots \wedge v_k \;=\; (v_1 \otimes \cdots \otimes v_k) + S \;\in\; \bigwedge^k V.$$

Let $1 \leq i < j \leq k$ and denote

$$f(v_i, v_j) = v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n$$

where the vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k \in V$ are fixed. Using the bilinearity of the wedge product we obtain

$$\begin{aligned} 0 &= f(v_i + v_j, v_i + v_j) - f(v_i, v_i) - f(v_j, v_j) \\ &= f(v_i, v_j) + f(v_j, v_i). \end{aligned}$$

Thus the expression $v_1 \wedge \cdots \wedge v_k$ reverses in sign whenever two of the vectors $v_1, \ldots, v_k$ are interchanged. It follows that more generally for any permutation $\sigma \in S_k$ we have

$$v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} = sgn(\sigma)(v_1 \wedge v_2 \wedge \cdots \wedge v_k)\,.$$

If $\{e_1, e_2, \ldots, e_n\}$ is a basis for V then a basis for $\bigwedge^k V$ is formed by the $\binom{n}{k}$ expressions $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. In particular the dimension of $\bigwedge^k V$ is $\binom{n}{k}$. In particular $\bigwedge^k V = 0$ whenever $k > n$, and $\bigwedge^n V$ is 1-dimensional with single basis vector $e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Note that the wedge product of n arbitrary vectors $v_1, \ldots, v_n \in V$ is given by

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = (\det M)\, e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

where M is the $n \times n$ matrix over F with columns formed by the coordinates of $v_1, \ldots, v_n$ with respect to the basis $e_1, \ldots, e_n$.

The *exterior algebra of V* is the graded algebra

$$\bigwedge^* V = \bigoplus_{k \geq 0} \bigwedge^k V$$

with respect to the wedge product. (Note that $\bigotimes^0 V = \bigwedge^0 V = F$.) For example when $n = 2$, elements of $\bigwedge^* V$ have the form $a_0 + a_1e_1 + a_2e_2 + a_3e_1{\wedge}e_2$ where $a_i \in F$, and the product becomes

$$\begin{aligned}(a_0 + a_1e_1 + a_2e_2 + a_3e_1{\wedge}e_2) \wedge (b_0 + b_1e_1 + b_2e_2 + b_3e_1{\wedge}e_2)\\ = a_0b_0 + (a_0b_1{+}a_1b_0)e_1 + (a_0b_2{+}a_2b_0)e_2 + (a_0b_3{+}a_3b_0{+}a_1b_2{-}a_2b_1)e_1{\wedge}e_2.\end{aligned}$$

Note that in general the exterior algebra has dimension

$$\dim(\textstyle\bigwedge^* V) = \displaystyle\sum_{k\geq 0} \binom{n}{k} = 2^n.$$

3. De Rham Cohomology

As our first example of cohomology rings we consider the de Rham cohomology ring of an open region $X \subseteq \mathbb{R}^n$. This is a graded ring of the form

$$H^*_{\text{de Rham}}(X) = \bigoplus_{k\geq 0} H^k_{\text{de Rham}}(X)$$

where $H^k_{\text{de Rham}}(X)$ denotes the k-th de Rham cohomology group of X. The de Rham cohomology groups for nice spaces $X \subset \mathbb{R}^n$ turn out to agree with the simplicial and singular cohomology groups $H^k(X;\mathbb{R})$ with real coefficients. We will certainly not prove this but this fact may be observed in our examples.

In order to define these groups we first consider the ring R consisting of smooth real-valued functions defined on X. (Without worrying too much about what 'smooth' means, let us say that $f \in R$ means that $f : X \to \mathbb{R}$ is infinitely differentiable, and in particular f has continuous partial derivatives of all orders.) Now for each $k \geq 0$, consider the real vector space $C^k = C^k(X)$ spanned by expressions of the form

$$f(x_1, \ldots, x_n)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and f is a smooth function $X \to \mathbb{R}$. Thus C^1 consists of expressions of the form

$$f_1(x)dx_1 + f_2(x)dx_2 + \cdots + f_n(x)dx_n$$

where $f_1(x), \ldots, f_n(x)$ are smooth functions of $x = (x_1, \ldots, x_n) \in X$. Also by convention $C^0 = R$ consists of smooth functions. We refer to elements of C^k as *(alternating) differential k-forms,* or simply *k-forms.* Although C^k is infinite-dimensional as a real vector space (at least for $k = 0, 1, 2, \ldots, n$), it has finite rank $\binom{n}{k}$ as a free module over the ring R.

We define the *differential* of a smooth function $f : X \to \mathbb{R}$ by

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n \in C^1(X).$$

More generally the differential operator $d : C^k \to C^{k+1}$ is the real linear map defined by

$$d(f(x)\, dx_{i_1} \wedge \cdots dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots dx_{i_k}$$

with df defined as above. With this notation one can state Stoke's Theorem: Given a k-dimensional subset $A \subseteq \mathbb{R}^n$ and a $(k-1)$-form ω, we have

$$\int_{\partial A} \omega = \int_A d\omega.$$

Here ∂A denotes the boundary of A, which is $(k-1)$-dimensional; and ∂A has the appropriate orientation induced by A. Note that one integrates the k-form $d\omega$ over the k-dimensional subset A; and the $(k-1)$-form ω over the $(k-1)$-dimensional subset ∂A.

3.1 Lemma. $d^2 = 0$.

Proof. We consider a differential k-form $\omega = f(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. (The most general k-form is a linear combination of expressions of this type.) Then

$$\begin{aligned} d\omega &= df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum_{1 \le j \le n} \frac{\partial f}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\,; \\ d^2\omega &= \sum_{1 \le i \le n} \sum_{1 \le j \le n} \frac{\partial^2 f}{\partial x_i\, \partial x_j}\, dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \;= 0 \end{aligned}$$

since the terms with i and j interchanged cancel each other. □

The *k-th de Rham cohomology group of* X is the quotient group

$$H^k_{\text{de Rham}}(X) = Z^k / B^k$$

where Z^k denotes the kernel of $d : C^k \to C^{k+1}$ (the group of *closed k-forms*) and $B^k = B^k$ denotes the image of $d : C^{k-1} \to C^k$ (the group of *exact k-forms*). Note that every exact form is closed; and if the converse fails, then the degree to which closed forms may fail to be exact, is measured by the cohomology groups of X. These groups depend only on topological properties of X which may be thought of as higher dimensional versions of connectedness. For example $H^0_{\text{de Rham}}(X) = 0$ iff X is connected, and in general $H^0_{\text{de Rham}}(X) \cong \mathbb{R}^{m-1}$ where m is the number of path-connected components of X. Also if X is path-connected, then $H^1_{\text{de Rham}}(X) = 0$ iff X is simply connected, and in general we view $H^1_{\text{de Rham}}(X)$ as counting the number of 'holes' in X.

Compare the following with Lemma 3.6 (page 206) and Theorem 3.14 (page 215) in the textbook.

3.2 Lemma. *If* $\omega \in C^k$ *and* $\rho \in C^\ell$ *then*

(a) $d(\omega \wedge \rho) = d\omega \wedge \rho + (-1)^k \omega \wedge d\rho$;

(b) $\rho \wedge \omega = (-1)^{k\ell} \omega \wedge \rho$.

Proof. Suppose $\omega = f(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\rho = g(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$. (Recall that general choices of k-form and ℓ-form will be linear combinations of such expressions.) Then

$$\begin{aligned} d(\omega \wedge \rho) &= d(f(x)g(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}) \\ &= \sum_{1 \le i \le n} \big(\frac{\partial f}{\partial x_i} g(x) + f(x) \frac{\partial g}{\partial x_i}\big)\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\ &= \Big[\sum_{1 \le i \le n} \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \Big] \wedge g(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\ &\qquad + (-1)^k f(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \Big[\sum_{1 \le i \le n} \frac{\partial g}{\partial x_i}\, dx_i \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \Big] \\ &= d\omega \wedge \rho + (-1)^k \omega \wedge d\rho. \end{aligned}$$

Note that $(-1)^k$ appears as the sign of the cyclic permutation of the first $k+1$ differentials. This proves (a). Now

$$\begin{aligned} \rho \wedge \omega &= f(x)g(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^{k\ell} f(x)g(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell} \\ &= (-1)^{k\ell} \omega \wedge \rho. \end{aligned}$$

To understand the appearance of the factor $(-1)^{k\ell}$, denote by σ a cyclic shift of all $k+\ell$ differentials appearing above, so that $sgn(\sigma) = (-1)^{k+\ell+1}$. The permutation of differentials appearing in the identity above is actually σ^k (or σ^ℓ, depending on whether you cycle to the left or to the right) and we have $sgn(\sigma^k) = (-1)^{(k+\ell+1)k} = (-1)^{k\ell + k(k+1)} = (-1)^{k\ell}$ since $k(k+1)$ is even. □

A consequence of Lemma 3.2(a) is that the wedge product for differential forms gives a well-defined product on cohomology classes

$$H^k_{\text{de Rham}}(X) \times H^\ell_{\text{de Rham}}(X) \xrightarrow{\wedge} H^{k+\ell}_{\text{de Rham}}(X).$$

In order to show that this operation is well-defined, we must show that if either of the forms ω or ρ is exact, then the wedge product $\omega \wedge \rho$ is exact (and so represents the zero element of $H^{k+\ell}_{\text{de Rham}}(X)$). To see this, note that if $\omega \in B^k$, say $\omega = d\psi$ where $\psi \in C^{k-1}$, and $\rho \in Z^\ell$, then

$$\omega \wedge \rho = d\psi \wedge \rho = d(\psi \wedge \rho) - (-1)^k \psi \wedge d\rho = d(\psi \wedge \rho) \in B^{k+\ell}$$

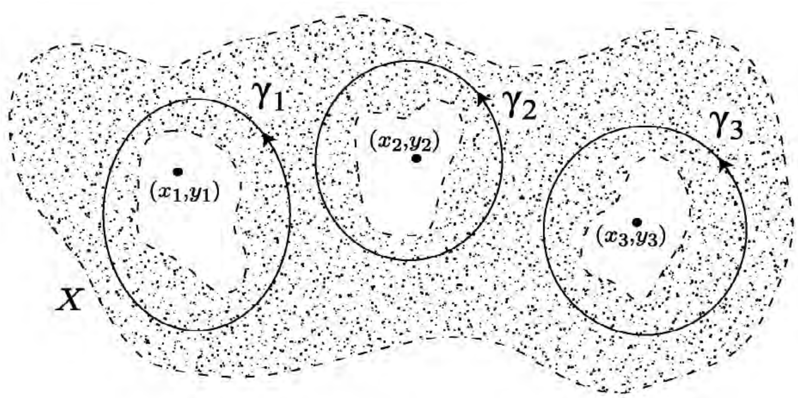
since $d\rho = 0$. A similar argument (or an application of Lemma 3.2(b)) gives the same result for $\omega \in Z^k$ and $\rho \in B^\ell$. Now the wedge product makes

$$H^*_{\text{de Rham}}(X) = \bigoplus_{k \geq 0} H^k_{\text{de Rham}}(X)$$

into a ring, and hence an algebra over $\mathbb{R}$.

3.3 Example: Plane Regions

Consider a connected open plane region $X \subseteq \mathbb{R}^2$ with k 'holes'. Let $(x_i, y_i) \in \mathbb{R}^2$ ($i = 1, 2, \ldots, k$) be points in the corresponding holes, and let $\gamma_i : [0,1] \to \mathbb{R}^2$ be closed paths such that γ_i winds once around (x_i, y_i) in the counterclockwise direction, but not around the other points (x_j, y_j) for $j \neq i$. Shown is the case $k = 3$:



In this case we have $H^0_{\text{de Rham}}(X) \cong \mathbb{R}$ because X is connected. A basis for $H^0_{\text{de Rham}}(X)$ is given by the constant function 1. This is because every closed 0-form is a smooth function $f : X \to \mathbb{R}$ satisfying $df = 0$ and so f is constant. Also $H^1_{\text{de Rham}}(X)$ is k-dimensional with basis $\{\omega_1 + B^0, \ldots, \omega_k + B^0\}$ where

$$\omega_i = \frac{1}{2\pi} \, \frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2}, \quad i = 1, 2, \ldots, k$$

and B^0 is the collection of exact forms $df = (\partial f / \partial x)dx + (\partial f / \partial y)dy$ where $f : X \to \mathbb{R}$ is smooth. Thus every closed 1-form on X is expressible as

$$\omega = a_1\omega_1 + a_2\omega_2 + \cdots + a_k\omega_k + df$$

for some $a_1, \ldots, a_k \in \mathbb{R}$ and $f : X \to \mathbb{R}$ smooth. Moreover this expression is unique up to an additive constant term in f.

Recall that the first homology group $H_1(X; \mathbb{R})$ of X is a k-dimensional real vector space with basis $\gamma_1, \ldots, \gamma_k$. (The set X deformation retracts to a wedge product of k

circles, so the fundamental group $\pi_1(X)$ is a free group on k generators.) We observe that $H^k_{\text{de Rham}}(X)$ is naturally dual to $H_k(X;\mathbb{R})$ in this case. Every closed 1-form ω gives rise to a linear functional

$$H_k(X;\mathbb{R}) \to \mathbb{R}, \qquad \sum_{1\le i\le k} a_i\gamma_i \mapsto \sum_{1\le i\le k} a_i \int_{\gamma_i} \omega.$$

The latter integrals are well-defined on cosets of B^0 since by the Fundamental Theorem of Calculus, every exact form df integrates to 0 over closed paths. Thus we have a natural isomorphism

$$H^1_{\text{de Rham}}(X) \;\cong\; H_1(X;\mathbb{R})^* \;\cong\; H^1(X;\mathbb{R}).$$

Note that the 1-forms $\omega_1,\dots,\omega_k$ give a basis of $H^1_{\text{de Rham}}(X)$ which is dual to the basis $\gamma_1,\dots,\gamma_k$ of $H_1(X;\mathbb{R})$.

4. Cohomology of Simplicial Complexes

Let X be a topological space. As in the definition of singular homology, we consider a k-simplex $[v_0,v_1,\dots,v_k] \subset \mathbb{R}^k$ with vertices $v_0,v_1,\dots,v_k$ and let

$$\sigma : [v_0,v_1,\dots,v_k] \to X$$

be any continuous map. (Note that σ is not required to be injective.) A singular k-chain in X, with coefficients in R, is defined to be a formal R-linear combination of such maps $[v_0,v_1,\dots,v_k] \to X$. We denote by $C_k(X;R)$ the set of such k-chains. The boundary of σ is the $(k-1)$-chain $\partial\sigma \in C_{k-1}(X;R)$ defined by

$$\begin{aligned}\partial\sigma = \sigma|[v_1,v_2,v_3,\dots,v_k] - \sigma|[v_0,v_2,v_3,\dots,v_k] + \sigma|[v_0,v_1,v_3,\dots,v_k] - \cdots\\ + (-1)^k\sigma|[v_0,v_1,v_2,\dots,v_{k-1}] \in C_{k-1}(X;R)\end{aligned}$$

where each summand denotes the restriction of the map σ to the indicated $(k-1)$-dimensional face of the simplex $[v_0,v_1,\dots,v_k]$. (We may then rewrite each term in this sum as a continuous map from the standard $(k-1)$-simplex $[v_0,v_1,\dots,v_{k-1}] \subset \mathbb{R}^{k-1}$ to X.) Using linearity this extends to an R-linear map $C_k(X;R) \to C_{k-1}(X;R)$. The chain complex associated to X with coefficients in R is the sequence of R-modules

$$\cdots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

where we abbreviate $C_k = C_k(X;R)$. The *k-th homology group of X with coefficients in R* is the quotient group

$$H_k(X;R) = Z_k(X;R)/B_k(X;R)$$

where $Z_k(X;R)$ is the kernel of $\partial : C_k(X;R) \to C_{k-1}(X;R)$ (the group of k-cycles) and $B_k(X;R)$ is the image of $\partial : C_{k+1}(X;R) \to C_k(X;R)$ (the group of k-boundaries). Dualizing the above chain complex gives the cochain complex

$$\cdots \xleftarrow{\delta} C_2^* \xleftarrow{\delta} C_1^* \xleftarrow{\delta} C_0^* \longleftarrow 0$$

where $C_k^* = Hom_R(C_k, R)$ is the group of all R-module homomorphisms $C_k \to R$ and the map $\delta = \partial^* : C_k^* \to C_{k+1}^*$ is the dual of $\partial : C_{k+1} \to C_k$. Elements of C_k^* are called *k-cochains.* (Recall: a matrix for $\delta = \partial^*$ is obtained simply as the transpose of a matrix for ∂.) Also $\delta^2 = \partial^* \circ \partial^* = (\partial \circ \partial)^* = 0^* = 0$. We define the *$k$-th cohomology group with coefficients in R* as the quotient group

$$H^k(X;R) = Z^k(X;R)/B^k(X;R)$$

where $Z^k = Z^k(X;R)$ is the kernel of $\delta : C_k^* \to C_{k+1}^*$ (the group of *k-cocycles*) and $B^k = B^k(X;R)$ is the image of $\delta : C_{k-1}^* \to C_k^*$ (the group of *k-coboundaries*).

The *cup product* of $\phi \in C_k^*$ and $\psi \in C_\ell^*$ is the $(k+\ell)$-cochain $\phi \cup \psi \in C_{k+\ell}^*$ defined as follows for a typical continuous map $\sigma : [v_0, v_1, \ldots, v_{k+\ell}] \to X$ where $[v_0, v_1, \ldots, v_{k+\ell}] \subset \mathbb{R}^{k+\ell}$ is a $(k+\ell)$-simplex:

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \ldots, v_k])\psi(\sigma|[v_k, \ldots, v_{k+\ell}]).$$

As in Section 3 we obtain

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi \;+\; (-1)^k \phi \cup \delta\psi;$$
$$\psi \cup \phi = (-1)^{k\ell} \phi \cup \psi.$$

It follows (as in Section 3) that the cup product gives a well-defined bilinear map (a 'product' operation)

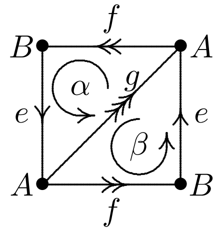
$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R).$$

Thus we may define the cohomology ring of X with coefficients in R as the graded ring

$$H^*(X;R) = \bigoplus_{k \geq 0} H^k(X;R).$$

5. Example: The Real Projective Plane

As before we use the following triangulation of the real projective plane $X = P^2\mathbb{R}$:



We compute homology and cohomology groups with coefficients in $\mathbb{F}_2$, so that $-1 = 1$. This will simplify some of our matrices, so that the orientation of edges and triangles becomes irrelevant; however we must still be careful with labeling of vertices when defining the cup product. The corresponding chain complex is given by the following sequence, in which explicit matrices for the boundary operator ∂ are given relative to the indicated bases for the chain groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2 & \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\\1&1\end{smallmatrix}\right]} & C_1 & \xrightarrow{\left[\begin{smallmatrix}1&1&0\\1&1&0\end{smallmatrix}\right]} & C_0 & \longrightarrow & 0 \\ & & \langle \alpha, \beta \rangle & & \langle e, f, g \rangle & & \langle A, B \rangle & & \end{array}$$

$$\begin{array}{cc} \text{kernel:} & \text{kernel:} \\ Z_2 = \left\langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\rangle & Z_1 = \left\langle \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \right\rangle \\ \text{image:} & \text{image:} \\ B_1 = \left\langle \left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right] \right\rangle & B_0 = \left\langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\rangle \end{array}$$

Now select dual bases for the cochain groups. For example the basis $\{\phi_A, \phi_B\}$ of C_0^* dual to the basis $\{A, B\}$ of C_0 is defined by

$$\phi_A(xA + yB) = x, \qquad \phi_B(xA + yB) = y$$

for all $x, y \in \mathbb{F}_2$. The resulting cochain complex, in which explicit matrices are given for the coboundary operator δ relative to the chosen bases of the cochain groups, is given by

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C_2^* & \xleftarrow{\left[\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right]} & C_1^* & \xleftarrow{\left[\begin{smallmatrix}1&1\\1&1\\0&0\end{smallmatrix}\right]} & C_0^* & \longleftarrow & 0 \\ & & \langle \phi_\alpha, \phi_\beta \rangle & & \langle \phi_e, \phi_f, \phi_g \rangle & & \langle \phi_A, \phi_B \rangle & & \end{array}$$

$$\begin{array}{cc} \text{kernel:} & \text{kernel:} \\ Z^1 = \left\langle \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right] \right\rangle & Z^0 = \left\langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\rangle \\ \text{image:} & \text{image:} \\ B^2 = \left\langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\rangle & B^1 = \left\langle \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right] \right\rangle \end{array}$$

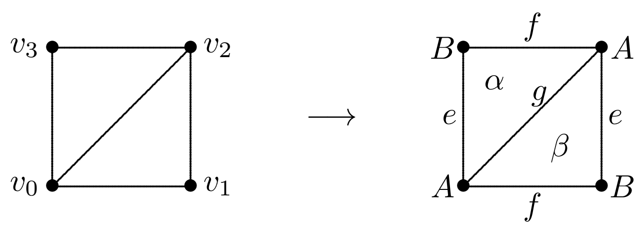
The resulting homology and cohomology groups are all one-dimensional over $\mathbb{F}_2$:

$$\begin{aligned} H_0 &= Z_0/B_0 = \langle A{+}B_0 \rangle, & H^0 &= Z^0/B^0 = \langle \phi_A{+}\phi_B \rangle, \\ H_1 &= Z_1/B_1 = \langle g{+}B_1 \rangle, & H^1 &= Z^1/B^1 = \langle \phi_e{+}\phi_g{+}B^1 \rangle, \\ H_2 &= Z_2/B_2 = \langle \alpha{+}\beta \rangle, & H^2 &= Z^2/B^2 = \langle \phi_\alpha{+}B^2 \rangle. \end{aligned}$$

It is now straightforward to verify that the cohomology ring $H^*(X;\mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[X]/(X^3)$ as claimed in Section 1. The least trivial part of this calculation is to check that the cup product of a generator of $H^1(X;\mathbb{F}_2)$ with itself yields a generator of $H^2(X;\mathbb{F}_2)$, i.e. that

$$(\phi_e+\phi_g)\cup(\phi_e+\phi_g) = \phi_\alpha\,.$$

This means that the element $(\phi_e+\phi_g)\cup(\phi_e+\phi_g) : H_2(X;\mathbb{F}_2)\to\mathbb{F}_2$ maps the nontrivial element $\alpha+\beta\in H_2(X;\mathbb{F}_2)$ to 1 (rather than to 0). We verify this by the following calculation. First consider a 3-simplex $[v_0,v_1,v_2,v_3]\subset\mathbb{R}^3$ and maps implied by the diagram



Thus for example $\alpha : [v_0,v_2,v_3]\to X$ is a continuous map sending vertices $v_0,v_2\mapsto A$, $v_3\mapsto B$; and sending edges $[v_0,v_2]\mapsto g$, $[v_0,v_3]\mapsto e$, $[v_2,v_3]\mapsto f$. Similarly $\beta : [v_0,v_1,v_2]\to X$. Then

$$\begin{aligned}
&[(\phi_e+\phi_g)\cup(\phi_e+\phi_g)](\alpha+\beta)\\
&\qquad = (\phi_e+\phi_g)(\alpha|[v_0,v_2]+\beta|[v_0,v_1])\cdot(\phi_e+\phi_g)(\alpha|[v_2,v_3]+\beta|[v_1,v_2])\\
&\qquad = (\phi_e+\phi_g)(g+f)\cdot(\phi_e+\phi_g)(f+e)\\
&\qquad = 1\cdot 1 = 1
\end{aligned}$$

as required. More generally we have

$$H^*(P^n\mathbb{R};\mathbb{F}_2)\;\cong\;\mathbb{F}_2[X]/(X^{n+1})$$

and this fact leads very directly to a proof of the Borsuk-Ulam Theorem.

6. Universal Coefficient Theorem

We have seen a universal coefficient theorem for homology which indicates how to obtain the homology groups $H_k(X;R)$ of a space X with coefficients in a commutative ring R, directly from the homology groups $H_k(X)=H_k(X;\mathbb{Z})$ with integer coefficients. Here we indicate how the cohomology groups $H^k(X;R)$ may be similarly obtained from $H_k(X)$. The theorem makes use of the group $Ext(H,G)$ defined for additive abelian groups G and H. This group is fully defined in Chapter 3 of the textbook, but for our present purposes one does not require the full definition of $Ext(H,G)$; in the case H is a finitely generated abelian group one can determine $Ext(H,G)$ rather quickly using the rules

$$Ext(H_1 \oplus H_2, G) \cong Ext(H_1, G) \oplus Ext(H_2, G);$$

$$Ext(H, G) = 0 \quad \text{whenever } H \text{ is a free } \mathbb{Z}\text{-module};$$

$$Ext(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG.$$

The definition of $Ext(H,G)$ is rather similar to (actually dual to) the definition of $Tor(A, B)$ encountered earlier, but beware: $Ext(G,H)$ is not generally isomorphic to $Ext(H,G)$ (in contrast with the identity $Tor(B,A) \cong Tor(A,B)$).

6.1 Universal Coefficient Theorem for Cohomology. *There is a split exact sequence*

$$0 \longrightarrow Ext(H_{k-1}(X), R) \longrightarrow H^k(X;R) \longrightarrow Hom(H_k(X), R) \longrightarrow 0.$$

In particular

$$H^k(X;R) \cong Ext(H_{k-1}(X), R) \oplus Hom(H_k(X), R)$$

although in general there is no canonical choice of subgroup (isomorphic to $Hom(H_k(X), R)$) complementary to $Ext(H_{k-1}(X), R)$.

6.2 Example: The Real Projective Plane

Let $X = P^2\mathbb{R}$. We make use of the previously computed homology groups

$$H_0(X) = H_0(X;\mathbb{Z}) \cong \mathbb{Z};$$

$$H_1(X) = H_1(X;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2;$$

$$H_2(X) = H_2(X;\mathbb{Z}) = 0.$$

From this we obtain the cohomology groups of X with integer coefficients:

$$H^0(X) \cong 0 \oplus Hom(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z};$$

$$H^1(X) \cong Ext(\mathbb{Z},\mathbb{Z}) \oplus Hom(\mathbb{F}_2,\mathbb{Z}) = 0;$$

$$H^2(X) \cong Ext(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \oplus 0 \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2 .$$

And we obtain the cohomology groups of X with coefficients in $\mathbb{F}_2$:

$$H^0(X;\mathbb{F}_2) \cong 0 \oplus Hom(\mathbb{Z},\mathbb{F}_2) \cong \mathbb{F}_2 ;$$

$$H^1(X;\mathbb{F}_2) \cong Ext(\mathbb{Z},\mathbb{F}_2) \oplus Hom(\mathbb{F}_2,\mathbb{F}_2) \cong \mathbb{F}_2 \oplus 0 = \mathbb{F}_2;$$

$$H^2(X;\mathbb{F}_2) \cong Ext(\mathbb{Z}/2\mathbb{Z},\mathbb{F}_2) \oplus 0 \cong \mathbb{F}_2/2\mathbb{F}_2 \cong \mathbb{F}_2 .$$

In this example we observe

$$H^k(X) \cong \begin{pmatrix}\text{free part}\\ \text{of } H_k(X)\end{pmatrix} \oplus \begin{pmatrix}\text{torsion part}\\ \text{of } H_{k-1}(X)\end{pmatrix}$$

which holds generally as a consequence of Theorem 6.1. Another general fact reflected in this example is that when F is a field, we have the isomorphism

$$H^k(X;F) \cong H_k(X;F),$$

although not canonically; there is however a natural isomorphism

$$H^k(X;F) \cong H_k(X;F)^* = Hom_F(H_k(X,F), F).$$