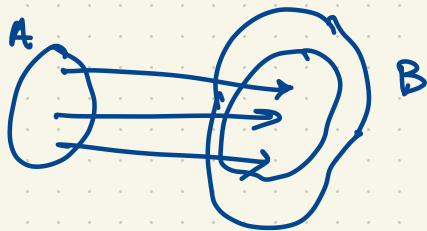


Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. I.e. if there is an injection $A \rightarrow B$ and an injection $B \rightarrow A$ then there is a bijection $A \rightarrow B$.

Here $|A| \leq |B|$ means there is an injection $A \rightarrow B$ i.e. A is in one-to-one correspondence with a subset of B . This is equivalent to the existence of a surjection $B \rightarrow A$ under the Axiom of Choice.



Bernstein-Cantor-Schröder Theorem uses ZF

Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?

There is an injection $(0,1) \rightarrow [0,1]$, $x \mapsto x$. So $|(0,1)| \leq |[0,1]|$.

There is an injection $[0,1] \rightarrow (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \leq |(0,1)|$.

$$\underline{|R| = |R^3| = |[0,1]| = |[0,1]^3|}$$

$[0,1] \rightarrow [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \rightarrow [0,1]$, $(x,y,z) \mapsto 0.x_1y_1z_1x_2y_2z_2x_3y_3z_3x_4y_4z_4 \dots$

$$x = 0.x_1x_2x_3x_4 \dots$$

$$y = 0.y_1y_2y_3y_4 \dots$$

$$z = 0.z_1z_2z_3z_4 \dots$$

Theorem $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$$|X| = |\mathbb{R}| = 2^{\aleph_0}$$

And how many lines do we need to cover X ? (partition)

Let Σ be a set of lines partitioning X . Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \rightarrow \mathbb{R}^3$ so

$|\Sigma| \leq |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \rightarrow \Sigma$? $\mathbb{R}^3 \xrightarrow{!} \mathbb{R} \xrightarrow{!} l \xrightarrow{!} \Sigma$

Let l be any line in X which is not in Σ .

To construct Σ , we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \dots ?$$

hoping that "in the limit" we cover all of X .

$$\Sigma_0 = \emptyset.$$

$$\Sigma_1 = \{l_0\}$$

$$\Sigma_2 = \{l_0, l_1\}$$

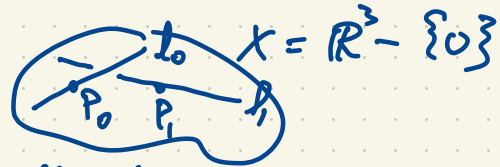
$$\Sigma_3 = \{l_0, l_1, l_2\}$$

Inductively construct Σ_β , $\beta \in A$, a set of disjoint lines in X , such that

• Σ_β covers P_α whenever $\alpha < \beta$.

• $|\Sigma_\beta| \leq |\beta| < |K| = 2^{\aleph_0}$.

• $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$



Well-orders the points of X

as P_α , $\alpha \in A$

where A is well-ordered.

Actually we can take $A = \kappa$
the smallest ordinal such that

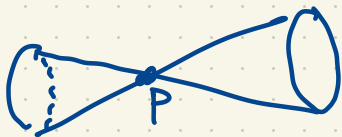
$$|K| = 2^{\aleph_0}$$

$$\text{Take } \Sigma = \bigcup_{\beta \in A} \Sigma_\beta$$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\kappa| = 2^{\aleph_0}$
with $P \in X$ not covered by Σ ($P \notin \bigcup_{\text{in } \Sigma} \text{lines}$),

there exists line l in X disjoint from all lines in Σ passing through P .
Consider a cone with vertex P . Every line of Σ hits this cone in at most 2 points. There are 2^{\aleph_0} lines in this cone passing through P , at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ .



By the Pigeonhole Principle, l exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizability?

- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y , we have the disjoint union $X \sqcup Y$ which can be viewed as $(X \times \{0\}) \cup (Y \times \{1\})$

$$\{(x, 0) : x \in X\}$$

$$\{(y, 1) : y \in Y\}$$

eg. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0, 1\} \subset \mathbb{R}^2$

$\mathbb{R} \times \{1\} =$ the line $y=1$

$\mathbb{R} \times \{0\} =$ x-axis ($y=0$)

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

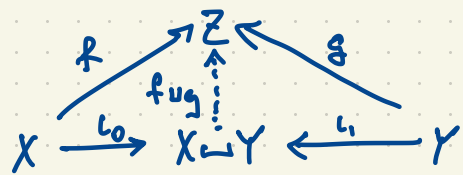
Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y , a coproduct of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \rightarrow X \sqcup Y$, $\iota_1 : Y \rightarrow X \sqcup Y$

such that whenever Z is a top. space and $f : X \rightarrow Z$, $g : Y \rightarrow Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \rightarrow Z$ such that this diagram commutes i.e. $(f \sqcup g) \circ \iota_0 = f$ and $(f \sqcup g) \circ \iota_1 = g$ see over

$$\begin{array}{ccc} & f & \\ & \nearrow & \\ X & \xrightarrow{\iota_0} & X \sqcup Y & \xleftarrow{\iota_1} & Y \\ & \searrow & & \nearrow & \\ & f \sqcup g & & g & \\ & \downarrow & & \downarrow & \\ & Z & & Z & \end{array}$$

$$\iota_0(x) = (x, 0), \quad \iota_1(y) = (y, 1)$$

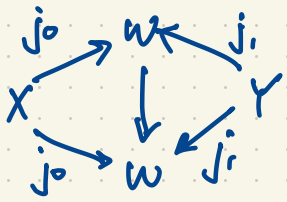
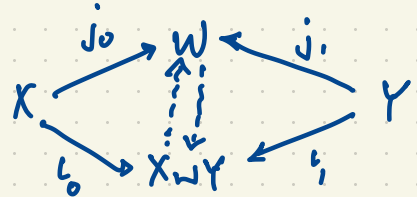
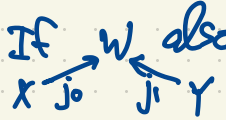


$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \sqcup g)(x, 0) = f(x) \in Z$$

$$(f \sqcup g)(y, 1) = g(y) \in Z$$

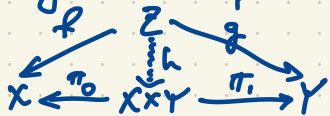
Any $X \sqcup Y$ together with l_0, l_1 satisfying this universal property is a (the) coproduct of X and Y . It exists by our construction; and it is unique. If W also satisfies the same universal property then

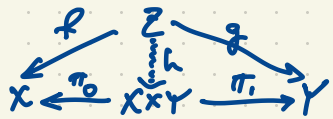


(cont maps)

Given top. spaces X, Y , a product is a top. space $X \times Y$ together with morphisms $\pi_0: X \times Y \rightarrow X, \pi_1: X \times Y \rightarrow Y$ such that for every top. space Z and morphisms $f: Z \rightarrow X, g: Z \rightarrow Y$, there exists a unique $h: Z \rightarrow X \times Y$ such that the following diagram

Commutates:





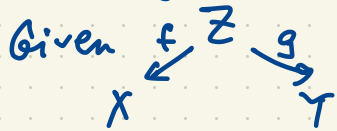
Existence of direct product: $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

Topology: $U \times V \subseteq X \times Y$ ($U \subseteq X, V \subseteq Y$ open)

are a basis for top. on $X \times Y$.

$$\begin{aligned}
 \pi_0: (X, Y) &\rightarrow X \\
 (x, y) &\mapsto x
 \end{aligned}$$

$$\begin{aligned}
 \pi_1: X \times Y &\rightarrow Y \\
 (x, y) &\mapsto y
 \end{aligned}$$




we have $h(z) = (f(z), g(z))$.

The product topology $X \times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U \times Y$ to be open in $X \times Y$ whenever $U \subseteq X$ is open. Also

" " $\pi_1^{-1}(V) = X \times V$ $V \subseteq Y$

Then $U \times V = (U \times Y) \cap (X \times V)$ must be open in $X \times Y$.

Ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has topology generated by  $u \times v$ $(u, v \in \mathbb{R})$
open
which is the standard topology.

A topological group is a group G endowed with a topology

such that the maps $G \rightarrow G$ is continuous
 $g \mapsto g^{-1}$

and $G \times G \rightarrow G$ is also continuous.
 $(g, h) \mapsto gh$

Ex. Consider $f: \underbrace{\mathbb{R}^2}_{\mathbb{R} \times \mathbb{R}} \rightarrow \mathbb{R}$, $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

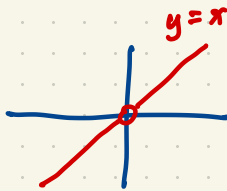
The map $\mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto f(x, b)$ is continuous for every $b \in \mathbb{R}$.

... .. $y \mapsto f(a, y)$ $a \in \mathbb{R}$.

But f is not continuous.

$$\begin{aligned} f^{-1}(1) &= \left\{ (x, y) \in \mathbb{R}^2 : f(x, y) = \frac{2xy}{x^2+y^2} = 1 \right\} \\ &= \left\{ (x, x) \in \mathbb{R}^2 : x \neq 0 \right\}. \text{ is not closed in } \mathbb{R}^2. \end{aligned}$$

$$\begin{aligned} 2xy &= x^2 + y^2 \\ (x-y)^2 &= 0 \end{aligned}$$



$(\mathbb{R}, +)$ is a topological group.

(\mathbb{R}^2, \times)

$+, \times$ are continuous maps $\mathbb{R}^2 \rightarrow \mathbb{R}$.

If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous then so are $f+g, fg$.

One way to see this is

$(f \times g): \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $(x, y) \mapsto (f(x), g(y))$ is continuous.

$\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$
 $x \mapsto (x, x) \mapsto (f(x), g(x)) \mapsto f(x) + g(x)$.

Similarly for multiplication.

diagonal
embedding of
 \mathbb{R} in \mathbb{R}^2 .

Given a top. space X , is the diagonal embedding $X \rightarrow X \times X$, $x \mapsto (x, x)$ always continuous?

Given a metric space (X, d) , $d: X \times X \rightarrow [0, \infty]$,
 d is continuous.

This description of product spaces generalizes easily to $X_1 \times X_2 \times \dots \times X_n$
including $X^n = \underbrace{X \times X \times \dots \times X}_{n \text{ times}}$ as a special case.

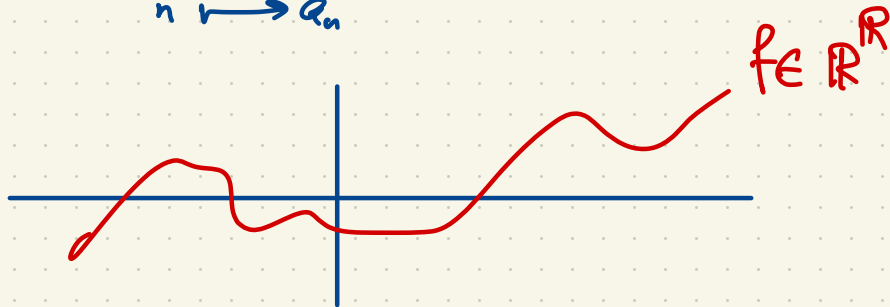
Infinite products are a little bit more subtle.

Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega = \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}\}$

Every function $\omega \mapsto \mathbb{R}$
 $n \mapsto a_n$

$\mathbb{R}^\mathbb{R} = \{\text{functions } \mathbb{R} \rightarrow \mathbb{R}\}$



The product topology for $\mathbb{R}^{\mathbb{R}} = \{ \text{functions } \mathbb{R} \rightarrow \mathbb{R} \}$ is the coarsest topology for which the projections $f \mapsto f(a)$ ($a \in \mathbb{R}$) are continuous.

This means we require: for every $\varepsilon > 0$, $b \in \mathbb{R}$, $\{ f \in \mathbb{R}^{\mathbb{R}} : f(a) \in \underbrace{B_{\varepsilon}(b)} \}$ is open in $\mathbb{R}^{\mathbb{R}}$.
 $B_{\varepsilon}(b)$ or any open set in \mathbb{R} .

$(\underbrace{\dots}_{\text{no restriction}}, \underbrace{f(a)}_m, \underbrace{\dots}_{\text{no restriction}}) \in \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times U \times \mathbb{R} \times \dots$