

# Point Set Topology

Book 1

Let  $X$  be a set. A topology on  $X$  is a collection  $\mathcal{J}$  of subsets of  $X$  (called the open sets) such that

(i)  $\emptyset, X \in \mathcal{J}$

(ii)  $\mathcal{J}$  is closed under finite intersection and arbitrary union, i.e.

if  $U, V \in \mathcal{J}$  then  $U \cap V \in \mathcal{J}$ ;

if  $\mathcal{U} \subseteq \mathcal{J}$  then  $\bigcup \mathcal{U} \in \mathcal{J}$ .

(So for  $U, V \in \mathcal{J}$ ,  $U \cup V \in \mathcal{J}$ . If  $\{U_\alpha : \alpha \in I\}$  is an indexed collection of open sets, then  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$ .)

Example

The standard topology on  $\mathbb{R}^n$ :  $X = \mathbb{R}^n$ . A set  $U \subseteq \mathbb{R}^n$  is open if (standard open set)  
for all  $u \in U$ , there exists  $\varepsilon > 0$  such that



$$B_\varepsilon(u) \subseteq U.$$

Here  $B_\varepsilon(u) = \{x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon\}$ .

Euclidean distance

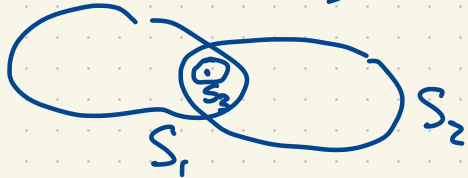
(the open  $\varepsilon$ -ball centered at  $u$ ).

$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$$

In other words, a standard open set in  $\mathbb{R}^n$  is a union of open balls.

Eg. (More generally) Let  $X$  be any set and let  $\mathcal{S}$  be a collection of subsets of  $X$  which cover  $X$ , i.e.  $\bigcup \mathcal{S} = X$ . Then the collection of all unions of finite intersections  $S_1 \cap S_2 \cap \dots \cap S_k$ ,  $S_1, \dots, S_k \in \mathcal{S}$  is a topology on  $X$ . The members of  $\mathcal{S}$  are called a sub-basis for this topology and the topology is said to be generated by  $\mathcal{S}$ .

$\mathcal{S}$  is called a base (or a basis) for the topology if the topology is the collection of arbitrary unions of elements of  $\mathcal{S}$ . This holds iff



for all  $S_1, S_2 \in \mathcal{S}$ ,  
and all  $u \in S_1 \cap S_2$ ,  
there exists  $S_3 \in \mathcal{S}$  such that  
 $u \in S_3 \subseteq S_1 \cap S_2$ .

Eg. Let  $X$  be any set. The discrete topology on  $X$  is the collection of all subsets of  $X$ . ( $2^X$ )

The indiscrete topology on  $X$  is  $\{\emptyset, X\}$ .

If  $X = \{0, 1\}$  then there are four possible topologies on  $X$ :  $\{\emptyset, X\}$ ,  $\{\emptyset, \{0\}, \{1\}, X\}$ ,  $\{\emptyset, \{0\}, X\}$ ,  $\{\emptyset, \{1\}, X\}$ .

Let  $X$  be an infinite set. Let  $\mathcal{J}$  be the collection of complements of finite sets, and  $\emptyset$   
 i.e.  $\mathcal{J} = \{\emptyset\} \cup \{X - A : A \subseteq X, |A| < \infty\}$ ,  $X - A = \{x \in X : x \notin A\}$ .  
 set difference

This is a topology on  $X$ , called the finite complement topology.

$X - A, X - A, X \setminus A$   
 $\emptyset, \emptyset, \emptyset, \emptyset$   
 \varnothing nothing

A topological space is a pair  $(X, \mathcal{J})$  where  $\mathcal{J}$  is a topology on a set  $X$ .

Note:  $\bigcup \mathcal{J} = X$ . By abuse of language, we often say that  $X$  is a topological space.

Let  $X$  be a set. A distance function (or metric) on  $X$  is a function

$d : X \times X \rightarrow [0, \infty]$  such that for all  $x, y, z \in X$ ,

$$d(x, y) = d(y, x)$$

$d(x, y) \geq 0$  and equality holds iff  $x = y$ .

$$d(x, z) \leq d(x, y) + d(y, z)$$

The standard topology on  $\mathbb{R}^n$  is a metric topology.

The metric  $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$  (the Euclidean metric)

$$d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

$d_\infty(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$  all give the standard topology on  $\mathbb{R}^n$ .

In  $\mathbb{R}^2$ , open balls with respect to  $d_2$ ,  $d_1$ ,  $d_\infty$  look like



respectively.

These three metrics define the same topology.

The metric  $d(x,y) = \begin{cases} 0, & \text{if } x=y \\ 1, & \text{if } x \neq y \end{cases}$  defines the discrete topology.

A topological space is metrizable if its topology can be given by some metric. (not uniquely however)

If  $X$  is an infinite set, then its finite complement topology is not metrizable.

A topology is Hausdorff if for any two points  $x \neq y$ , there exist open sets  $U, V$  such that  $x \in U$ ,  $y \in V$ ,  $U \cap V = \emptyset$ .



Every metric space is Hausdorff since if  $x \neq y$ ,  $d = d(x,y) > 0$ . Take  $U = B_{d/3}(x)$ ,  $V = B_{d/3}(y)$

An open neighbourhood of a point  $x \in X$  is an open set containing  $x$ .

A basic open nbhd of a point  $x \in X$  is an open nbhd of  $x$  which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.

Consider  $X = \mathbb{Q}$ . A norm on  $\mathbb{Q}$  is a function  $\mathbb{Q} \rightarrow [0, \infty)$ ,  $x \mapsto \|x\|$  satisfying

- (i)  $\|x\| \geq 0$ ; equality holds iff  $x = 0$ .
- (ii)  $\|xy\| = \|x\| \cdot \|y\|$ .
- (iii)  $\|x+y\| \leq \|x\| + \|y\|$ .

From any norm on  $\mathbb{Q}$ , we obtain a metric  $d(x, y) = \|x - y\|$ .

One way to do this is with the usual absolute value  $\|x\| = |x| = |x|_{\infty} = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

This gives the standard topology on  $\mathbb{Q}$ .

An alternative is: given  $x \in \mathbb{Q}$ , if  $x = 0$  define  $\|0\|_2 = 0$ .

If  $x \neq 0$ , write  $x = 2^k \cdot \frac{a}{b}$ ,  $a, b, k \in \mathbb{Z}$ ,  $b \neq 0$ ;  $a, b$  odd. Then define  $\|x\|_2 = 2^{-k}$ .

This is the 2-adic norm on  $\mathbb{Q}$ . In fact it satisfies a stronger form of (iii), the ultrametric inequality  $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$ .

E.g.  $\left\| \frac{20}{21} + \frac{5}{14} \right\|_2 = \left\| \frac{10+15}{42} \right\|_2 = \left\| \frac{55}{42} \right\|_2 = 2. = \max \left\{ \underbrace{\left\| \frac{20}{21} \right\|_2}_{\frac{1}{4}}, \underbrace{\left\| \frac{5}{14} \right\|_2}_2 \right\} = 2$

$\left\| \frac{20}{21} \right\|_2 = \frac{1}{4}, \left\| \frac{5}{14} \right\|_2 = 2$

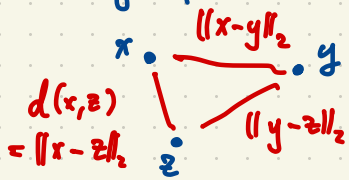
Compare:  $\left\| \frac{20}{21} \right\|_2 + \left\| \frac{5}{14} \right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

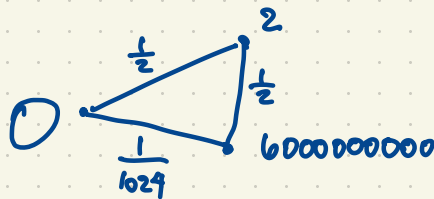
$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, a \text{ even}, b \text{ odd} \right\}.$

Every point in the ball is a centre of the ball i.e. if  $c \in B_1(0)$  then  $B_1(c) = B_1(0).$



Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots = -1$$

The partial sums  $1, 3, 7, 15, 31, 63, \dots$  converge to  $-1$  in the 2-adic norm.

Note: If  $(x_n)_n$  is a sequence of points in a top. space  $X$ , we say  $(x_n)_n$  converges to  $x \in X$  if for every open nbhd  $U$  of  $x$ ,  $x_n \in U$  for all  $n$  sufficiently large. (This means: for all  $U$  open nbhd of  $x$ , there exists  $N$  such that  $x_n \in U$  whenever  $n > N$ .)



In place of arbitrary open nbhds of  $x$ , it suffices to check basic open nbhds. For metric topology, it suffices to check open balls. In this case,  $x_n \rightarrow x$  provided that for all  $\varepsilon > 0$ , there exists  $N$  such that

$$\left. \begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array} \right\} \text{ whenever } n > N.$$

In our example above,  $d(x_n, x) = 2^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ .

$$\|2^{-n}\| = \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

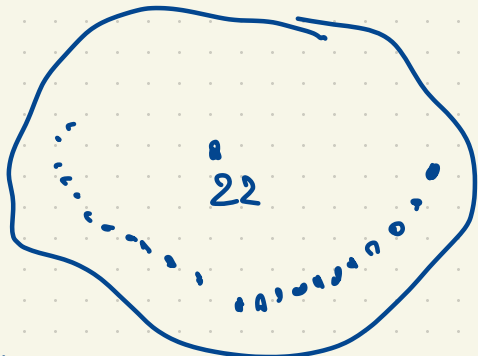
Find the inverse of 5 mod 64.



In  $\mathbb{Z}/64\mathbb{Z}$ ,  $\frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - 64 + 256 - 1024 + \dots$   
 $= 1 - 4 + 16$  (zero)  
 $= 13.$

Eg. in  $\mathbb{Z}$  with the finite complement topology, the sequence  $(n)_n = (1, 2, 3, \dots)$  converges. It converges to 22.

$(n)_n \rightarrow 22.$



1, 13, 25, 84

In fact for every  $a \in \mathbb{Z}$ ,  
 $(n)_n \rightarrow a.$

- 1
- 13
- 25
- 84

Theorem If  $X$  is Hausdorff, then every sequence in  $X$  has at most one limit. (it converges to at most one point.)

Proof Suppose  $a \neq b$  in a Hausdorff space  $X$  where a sequence  $(x_n)_n \rightarrow a$  and  $(x_n)_n \rightarrow b$ . Choose disjoint open nbhds  $U, V$  of  $a, b$ .



There exists  $N_1$  such that  $x_n \in U$  for all  $n > N_1$ ; also  $N_2$  such that  $x_n \in V$  for all  $n > N_2$ . respectively.

then pick  $n > \max\{N_1, N_2\}$  to obtain a contradiction.

We prefer to write  $(x_n)_n \rightarrow a$  rather than  $\lim_{n \rightarrow \infty} x_n = a$  in general.

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In any top. space, closed sets are the complements of open sets.

$\emptyset, X$  are closed

If  $K, K'$  are closed then  $K \cup K'$  is closed. (So finite unions of closed sets are closed.)

Arbitrary intersections of closed sets are closed.

De Morgan laws:  $X - \left( \bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X - A_\alpha)$

$$X - \left( \bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X - A_\alpha)$$

Given an infinite set  $X$ , the finite complement topology has as its closed sets the finite sets and  $X$  itself.

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Let  $X$  be a top. space. Given  $A \subseteq X$ , the closure of  $A$  is the (unique) smallest closed set containing  $A$  i.e.  $\bar{A} = \bigcap \{K \subseteq X : K \text{ closed, } K \supseteq A\}$ .

The interior of  $A$  is the largest open set contained in  $A$ , i.e.  $A^\circ = \bigcup \{U \subseteq A : U \text{ open in } X\}$ .  $(X - A)^\circ = X - \bar{A}$ ;  $\overline{X - A} = X - A^\circ$ .

Theorem There are infinitely many primes.

Known proofs: Euclid's proof (elementary)

Euler's proof (analytic proof:  $\sum \frac{1}{p}$  diverges)

This proof is topological.

Proof Form a topology on  $X = \mathbb{Z}$  whose basic open sets are the <sup>(infinite)</sup> arithmetic progressions

$\dots, -6, -1, 4, 9, 14, 19, \dots$

Every nonempty open set is infinite.

Suppose there are only finitely many primes:  $|P| < \infty$  is the set of all primes.

$\{-1, 1\} = \{a \in \mathbb{Z} : a \text{ is not divisible by any prime}\}.$