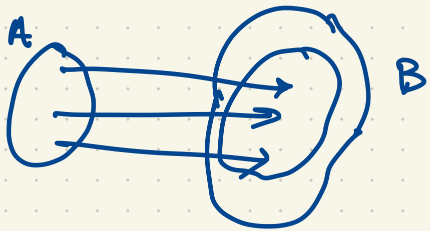
Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \le |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.



Bernstein-Cantor-Schröder Theorem uses ZF

Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?
There is an injection $(0,1) \to [0,1]$, $x \mapsto x$. So $|(0,1)| \le |[0,1]|$.
There is an injection $[0,1] \to (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \le |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \to [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \to [0,1]$, $(x,y,z) \mapsto 0.x_1y_1z_1x_2y_2z_2x_3y_3z_3x_4y_4z_4\cdots$

$$x = 0.x_1x_2x_3x_4\cdots$$
$$y = 0.y_1y_2y_3y_4\cdots$$
$$z = 0.z_1z_2z_3z_4\cdots$$

<u>Theorem</u> $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$|X| = |\mathbb{R}| = 2^{\aleph_0}$

And how many lines do we need to (partition) cover X?

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $\ell \in \Sigma$. This gives an injection $\Sigma \to \mathbb{R}^3$ so

$|\Sigma| \leq |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \to \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} \ell \xrightarrow{1:1} \Sigma$

Let ℓ be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \quad ?$$

hoping that "in the limit" we cover all of X.

$$\Sigma_0 = \emptyset.$$

$$\Sigma_1 = \{ l_0 \}$$

$$\Sigma_2 = \{ l_0, l_1 \}$$

$$\Sigma_3 = \{ l_0, l_1, l_2 \}$$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers P_α whenever $\alpha < \beta$.
- $|\Sigma_\beta| \leq |\beta| < |\kappa| = 2^{\aleph_0}$.
- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$

$X = \mathbb{R}^3 - \{0\}$

Well order the points of X as P_α, $\alpha \in A$ where A is well-ordered.

Actually we can take $A = \kappa$ the smallest ordinal such that $|\kappa| = 2^{\aleph_0}$

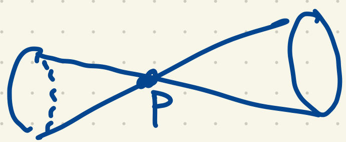
Take $\Sigma = \bigcup_{\beta \in A} \Sigma_\beta$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\mathbb{R}| = 2^{\aleph_0}$ with $P \in X$ not covered by Σ $(P \notin \underbrace{\bigcup \Sigma}_{\text{union of lines in } \Sigma})$,

there exists line ℓ in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.



By the Pigeonhole Principle, ℓ exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizatizability ?
- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y, we have the disjoint union $X \sqcup Y$ which can be viewed as $\underbrace{(X \times \{0\})}_{\{(x,0)\,:\, x \in X\}} \cup \underbrace{(Y \times \{1\})}_{\{(y,1)\,:\, y \in Y\}}$

eg. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$

$\longleftrightarrow$ $\mathbb{R} \times \{1\} =$ the line $y = 1$

$\longleftrightarrow$ $\mathbb{R} \times \{0\} =$ x-axis $(y=0)$

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y, a <u>coproduct</u> of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \to X \sqcup Y$, $\iota_1 : Y \to X \sqcup Y$ such that whenever Z is a top. space and $f : X \to Z$, $g : Y \to Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \to Z$ such that this diagram commutes i.e. $(f \sqcup g) =$ see over

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, "f \sqcup g"'] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$\iota_0(x) = (x, 0)$, $\iota_1(y) = (y, 1)$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, "f \sqcup g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \sqcup g)(x,0) = f(x) \in Z$$

$$(f \sqcup g)(y,1) = g(y) \in Z$$

Any $X \sqcup Y$ together with ι_0, ι_1 satisfying this universal property is a (the) coproduct of X and Y. It exists by our construction; and it is unique. If W $\left(X \xrightarrow{j_0} W \xleftarrow{j_1} Y \right)$ also satisfies the same universal property then

$$\begin{tikzcd} & W \arrow[d, dashed, shift left] & \\ X \arrow[ur, "j_0"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, shift left] & Y \arrow[ul, "j_1"'] \arrow[l, "\iota_1"] \end{tikzcd}
\qquad
\begin{tikzcd} & W \arrow[dd] & \\ X \arrow[ur, "j_0"] \arrow[dr, "j_0"'] & & Y \arrow[ul, "j_1"'] \arrow[dl, "j_1"] \\ & W & \end{tikzcd}$$

Given top. spaces X, Y, a **product** is a top. space $X \times Y$ together with morphisms (cont. maps) $\pi_0 : X \times Y \to X$, $\pi_1 : X \times Y \to Y$ such that for every top. space Z and morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique $h : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

Existence of direct product: $X \times Y = \{(x,y) : x \in X, y \in Y\}$.

Topology: $U \times V \subseteq X \times Y$ ($U \subseteq X$, $V \subseteq Y$ open) are a basis for top. on $X \times Y$.