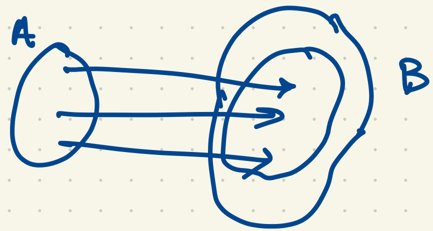
Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \leq |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.

Bernstein-Cantor-Schröder Theorem uses ZF



Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?
There is an injection $(0,1) \to [0,1]$, $x \mapsto x$. So $|(0,1)| \leq |[0,1]|$.
There is an injection $[0,1] \to (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \leq |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \to [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \to [0,1]$, $(x,y,z) \mapsto 0.x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 x_4 y_4 z_4 \cdots$

$$x = 0.x_1 x_2 x_3 x_4 \cdots$$
$$y = 0.y_1 y_2 y_3 y_4 \cdots$$
$$z = 0.z_1 z_2 z_3 z_4 \cdots$$

Theorem $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$$|X| = |\mathbb{R}| = 2^{\aleph_0}$$

And how many lines do we need to (partition) cover X?

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \to \mathbb{R}^3$ so $|\Sigma| \le |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \to \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} l \xrightarrow{1:1} \Sigma$

Let l be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \quad ?$$

hoping that "in the limit" we cover all of X.

$X = \mathbb{R}^3 - \{0\}$

$$\Sigma_0 = \emptyset.$$

$$\Sigma_1 = \{l_0\}$$

$$\Sigma_2 = \{l_0, l_1\}$$

$$\Sigma_3 = \{l_0, l_1, l_2\}$$

Well order the points of X as P_α, $\alpha \in A$ where A is well-ordered.

Actually we can take $A = \kappa$ the smallest ordinal such that $|\kappa| = 2^{\aleph_0}$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers P_α whenever $\alpha < \beta$.
- $|\Sigma_\beta| \leq |\beta| < |\kappa| = 2^{\aleph_0}$.
- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$

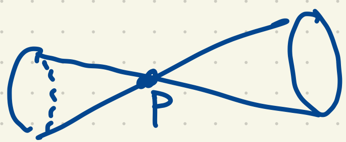
Take $\Sigma = \bigcup_{\beta \in A} \Sigma_\beta$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\mathbb{R}| = 2^{\aleph_0}$ with $P \in X$ not covered by Σ $\left(P \notin \underbrace{\bigcup \Sigma}_{\text{union of lines in } \Sigma}\right)$,

there exists line l in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.



By the Pigeonhole Principle, l exists.