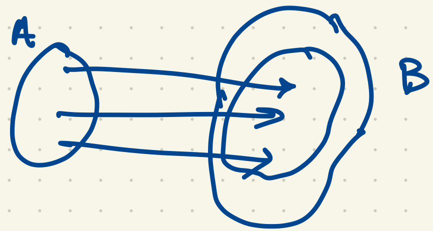
Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \leq |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.

Bernstein-Cantor-Schröder Theorem uses ZF



Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?
There is an injection $(0,1) \to [0,1]$, $x \longmapsto x$. So $|(0,1)| \leq |[0,1]|$.
There is an injection $[0,1] \to (0,1)$, $x \longmapsto \frac{1}{3}(x+1)$. So $|[0,1]| \leq |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \to [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \to [0,1]$, $(x,y,z) \mapsto 0.x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 x_4 y_4 z_4 \cdots$

$$x = 0.x_1 x_2 x_3 x_4 \cdots$$
$$y = 0.y_1 y_2 y_3 y_4 \cdots$$
$$z = 0.z_1 z_2 z_3 z_4 \cdots$$

<u>Theorem</u> $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$|X| = |\mathbb{R}| = 2^{\aleph_0}$

And how many lines do we need to (partition) cover X?

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \to \mathbb{R}^3$ so $|\Sigma| \leq |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \to \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} l \xrightarrow{1:1} \Sigma$

Let l be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \quad ?$$

hoping that "in the limit" we cover all of X.

$\Sigma_0 = \emptyset$.

$\Sigma_1 = \{ l_0 \}$

$\Sigma_2 = \{ l_0, l_1 \}$

$\Sigma_3 = \{ l_0, l_1, l_2 \}$

$X = \mathbb{R}^3 - \{0\}$

Well-order the points of X as P_α, $\alpha \in A$ where A is well-ordered.

Actually we can take $A = \kappa$ the smallest ordinal such that $|\kappa| = 2^{\aleph_0}$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers P_α whenever $\alpha < \beta$.
- $|\Sigma_\beta| \leq |\beta| < |\kappa| = 2^{\aleph_0}$.
- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$

Take $\Sigma = \bigcup_{\beta \in A} \Sigma_\beta$

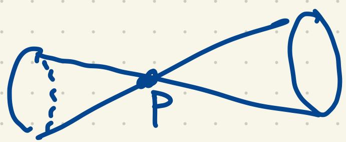
Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\mathbb{R}| = 2^{\aleph_0}$

with $P \in X$ not covered by Σ $\quad (P \notin \underbrace{\bigcup \Sigma}_{\text{union of lines in } \Sigma})$,

there exists line ℓ in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.



By the Pigeonhole Principle, ℓ exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizatizability ?
- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y, we have the disjoint union $X \sqcup Y$ which can be viewed as $\underbrace{(X \times \{0\})}_{\{(x,0)\,:\, x \in X\}} \cup \underbrace{(Y \times \{1\})}_{\{(y,1)\,:\, y \in Y\}}$

e.g. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$

$\mathbb{R} \times \{1\}$ = the line $y = 1$

$\mathbb{R} \times \{0\}$ = x-axis $(y = 0)$

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y, a <u>coproduct</u> of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \to X \sqcup Y$, $\iota_1 : Y \to X \sqcup Y$ such that whenever Z is a top. space and $f : X \to Z$, $g : Y \to Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \to Z$ such that this 'diagram commutes' i.e. $(f \sqcup g) =$ see over

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dotted, "f \sqcup g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$\iota_0(x) = (x, 0)$, $\iota_1(y) = (y, 1)$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, "f \sqcup g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$
$$(f \sqcup g)(x,0) = f(x) \in Z$$
$$(f \sqcup g)(y,1) = g(y) \in Z$$

Any $X \sqcup Y$ together with ι_0, ι_1 satisfying this universal property is a (the) coproduct of X and Y. It exists by our construction; and it is unique. If $\begin{tikzcd}[column sep=small] X \arrow[r, "j_0"] & W & Y \arrow[l, "j_1"'] \end{tikzcd}$ also satisfies the same universal property then

$$\begin{tikzcd} & W \arrow[d, dashed, shift left] & \\ X \arrow[ur, "j_0"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, shift left] & Y \arrow[ul, "j_1"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$$\begin{tikzcd} & W \arrow[dd] & \\ X \arrow[ur, "j_0"] \arrow[dr, "j_0"'] & & Y \arrow[ul, "j_1"'] \arrow[dl, "j_1"] \\ & W & \end{tikzcd}$$

Given top. spaces X, Y, a **product** is a top. space $X \times Y$ together with morphisms (cont. maps) $\pi_0 : X \times Y \longrightarrow X$, $\pi_1 : X \times Y \longrightarrow Y$ such that for every top. space Z and morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique $h : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
X & X\times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y
\end{tikzcd}$$

Existence of direct product: $X\times Y = \{(x,y) : x\in X,\ y\in Y\}$.

Topology: $U\times V \subseteq X\times Y$ ($U\subseteq X$, $V\subseteq Y$ open) are a basis for top. on $X\times Y$.

$$\pi_0 : (X,Y) \to X \qquad \pi_1 : X\times Y \longrightarrow Y$$
$$(x,y) \mapsto x \qquad (x,y) \longmapsto y$$

Given $\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$ we have $h(z) = (f(z), g(z))$.

The product topology $X\times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U\times Y$ to be open in $X\times Y$ whenever $U\subseteq X$ is open. Also

" " $\pi_1^{-1}(V) = X\times V$ . . . . $V\subseteq Y$ . . . . .

Then $U\times V = (U\times Y)\cap(X\times V)$ must be open in $X\times Y$.

Eg. $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ has topology generated by  $U\times V$ $(U, V \in \mathbb{R}$ open$)$. which is the standard topology.

A <u>topological group</u> is a group G endowed with a topology such that the maps
$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$
is continuous
and
$$\begin{aligned} G\times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$
is also continuous.

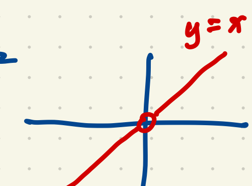
Eg. Consider $f : \underbrace{\mathbb{R}^2}_{\mathbb{R}\times\mathbb{R}} \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0); \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

The map $\mathbb{R}\to\mathbb{R}$, $x \mapsto f(x,b)$ is continuous for every $b\in\mathbb{R}$.
" " " " " " " $y \mapsto f(a,y)$ " " " " " " $a\in\mathbb{R}$.

But f is not continuous.

$$f^{-1}(1) = \left\{ (x,y)\in\mathbb{R}^2 : f(x,y) = \frac{2xy}{x^2+y^2} = 1 \right\}$$
$$= \{ (x,x)\in\mathbb{R}^2 : x\neq 0 \}. \text{ is not closed in } \mathbb{R}^2.$$

$2xy = x^2+y^2$
$(x-y)^2 = 0$



$(\mathbb{R}, +)$ is a topological group.

$(\mathbb{R}^*, \times)$ " " " "

$+, \times$ are continuous maps $\mathbb{R}^2 \to \mathbb{R}$.

If $f, g : \mathbb{R} \to \mathbb{R}$ is continuous then so are $f+g$, fg.

One way to see this is

$$(f \times g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x, y) \longmapsto (f(x), g(y))$$ is continuous.

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (x, x) & \longmapsto & (f(x), g(x)) & \longmapsto & f(x) + g(x). \end{array}$$

diagonal embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

Similarly for multiplication.

Given a top. space X, is the diagonal embedding $X \to X \times X$, $x \mapsto (x, x)$ always continuous?

Given a metric space (X, d), $d: X \times X \to [0, \infty]$,
metric

d is continuous.

This description of product spaces generalizes easily to $X_1 \times X_2 \times \cdots \times X_n$ including $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ as a special case.

Infinite products are a little bit more subtle.

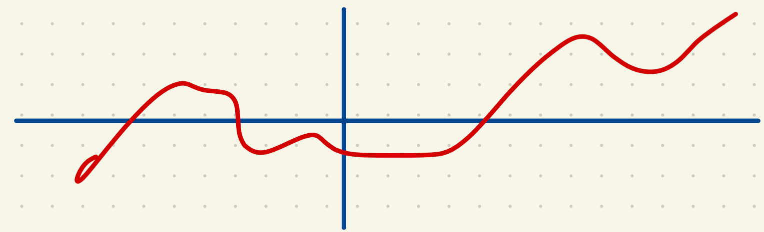
Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega = \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}\}$

Every function $\omega \to \mathbb{R}$

$n \mapsto a_n$

$\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$



$f \in \mathbb{R}^{\mathbb{R}}$

The product topology for $\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R}\to\mathbb{R}\}$ is the coarsest topology for which the projections $f \mapsto f(a)$ $(a\in\mathbb{R})$ are continuous.

This means we require: for every $\underbrace{\varepsilon>0,\ b\in\mathbb{R}}$, $\{f\in\mathbb{R}^{\mathbb{R}} : f(a) \in B_\varepsilon(b)\}$ is open in $\mathbb{R}^{\mathbb{R}}$.

$B_\varepsilon(b)$ or any open set in $\mathbb{R}$.

$$(\underbrace{\cdot\ -\ -\ -}_{\text{no restriction}},\ f(a),\ \underbrace{\cdot\ -\ -\ -\ -}_{\text{no restriction}}) \in \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times U\times\mathbb{R}\times\cdots$$

General product: Let X_α ($\alpha\in A$, some index set A) be top. spaces. The <u>product space</u> $\prod_{\alpha\in A} X_\alpha$ has the Cartesian product as its underlying set.

As a set, an element $x = (x_\alpha)_{\alpha\in A} \in \prod_{\alpha\in A} X_\alpha$ is really a function $A \longrightarrow \bigcup_{\alpha\in A} X_\alpha$ subject to $x_\alpha \in X_\alpha$ for all $\alpha\in A$.

(Special case: all X_α isomorphic to X; $\alpha \mapsto x_\alpha$ is a map $A \to X_\alpha = X$).

If $X_\alpha \neq \emptyset$ for all $\alpha\in A$, then $\prod_{\alpha\in A} X_\alpha \neq \emptyset$. This uses AC = Axiom of Choice

If all $X_\alpha = X$ for all $\alpha \in A$ then $\prod_{\alpha\in A} X_\alpha = X^A = \{$ functions $A \to X\} \neq \emptyset$ assuming $X \neq \emptyset$. This holds in ZF without requiring AC. Let $x \in X$ and consider the constant function $f(\alpha) = x$ for all $\alpha \in A$. This gives the diagonal embedding $X \to X^A$.

Topology on $\prod_{\alpha\in A} X_\alpha$: A <u>subbasis</u> consists of the open cylinders

$$\{x = (x_\alpha)_\alpha : x_\alpha \in X_\alpha \text{ arbitrary for } \alpha \neq \beta;\ x_\beta \in U\} \quad \text{where } \beta \in A,\ U \subseteq X_\beta \text{ open}$$

$$= \underbrace{U}_{\text{in coordinate } \beta} \times \prod_{\substack{\alpha\in A\\ \alpha\neq\beta}} X_\alpha = \pi_\beta^{-1}(U) \quad \text{where } \pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$$

$$x = (x_\alpha)_{\alpha\in A} \mapsto x_\beta.$$

Under finite intersections, these generate a basis for the topology on the product space. Basic open sets have the form

$$\{x \in (x_\alpha)_{\alpha\in A} : x_{\alpha_i} \in U_{\alpha_i} \text{ for } i = 1, \dots, k\}$$

where $k \geq 1$ is a positive integer;
$\alpha_1, \dots, \alpha_k \in A$;
$U_{\alpha_i} \subseteq X_{\alpha_i}$ for each $i = 1, \dots, k$ are open sets

Arbitrary open sets are arbitrary unions of basic open sets.
This is the <u>product topology</u> (or the Tychonoff topology).

If instead one takes as basic open sets $\prod_{\alpha\in A} U_\alpha$ ($U_\alpha \subseteq X_\alpha$ open), then one gets the <u>box</u> <u>topology</u>. ← Cartesian product

This is a refinement of the product topology.

Unless otherwise specified, the topology on $\prod_{\alpha\in A} X_\alpha$ is understood to be the product topology.

Eg. $\mathbb{R}^{\mathbb{R}} = \prod_{\alpha\in\mathbb{R}} \mathbb{R} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$

Each function $f: \mathbb{R} \to \mathbb{R}$ determines a point $(f(\alpha))_{\alpha\in\mathbb{R}}$ (a generalized sequence).

A basic open nbhd of $f \in \mathbb{R}^{\mathbb{R}}$ has the form

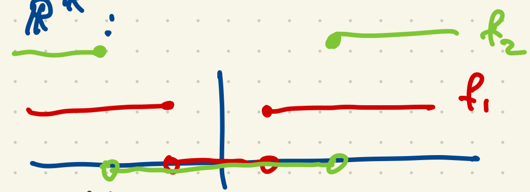
$$\{g \in \mathbb{R}^{\mathbb{R}} : g(x_i) \in U_i,\ i = 1, 2, \cdots, k\}, \quad U_i \text{ is an open nbhd of } f(x_i) \text{ in } \mathbb{R}\}.$$

or specifically

$$\{g \in \mathbb{R}^{\mathbb{R}} : |g(x_i) - f(x_i)| < \varepsilon_i,\ i = 1, \cdots, k\}$$

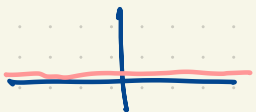
Varying $x_1, \cdots, x_k$, k, $\varepsilon_1, \cdots, \varepsilon_k$ we get a basis for the topology of $\mathbb{R}^{\mathbb{R}}$ in this way.

A convergent sequence of functions in $\mathbb{R}^{\mathbb{R}}$:



$$f_n(x) = \begin{cases} 0, & \text{if } |x| < n; \\ n, & \text{if } |x| \geq n. \end{cases}$$

$f_n \to \underbrace{0}_{\text{zero function}}$ i.e. for any basic open nbhd $\overset{U}{\ }$ of 0, $f_n \in U$ for all $n \gg 0$.



In usual language, $f_n \to 0$ pointwise meaning for all $x \in \mathbb{R}$,
$f_n(x) \to 0$.

In the box topology, $f_n \not\to 0$.

$\omega = \{0, 1, 2, 3, \dots\}$

Take $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{\overbrace{(a_0, a_1, a_2, a_3, \dots)}^{x} : a_i \in \mathbb{R}\} = \mathbb{R}^{\omega}$
as a set (Cartesian product). Compare product topology, box topology, and topologies from a few norms including

$\|x\|_1 = \sum_{i \in \omega} |a_i| = |a_0| + |a_1| + |a_2| + \cdots$ $\qquad \|x\|_\infty = \sup |a_i|$.

$\|x\|_2 = \left(\sum |a_i|^2\right)^{1/2}$

$$\ell^1 = \{x \in \mathbb{R}^\omega : \|x\|_1 < \infty\}$$
$$\ell^2 = \{x \in \mathbb{R}^\omega : \|x\|_2 < \infty\}$$
$$\ell^\infty = \{x \in \mathbb{R}^\omega : \|x\|_\infty < \infty\}$$

$$x_1 = (1, 1, 1, 1, 1, 1, \cdots)$$
$$x_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \cdots)$$
$$x_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \cdots)$$
$$\vdots$$
$$x_n = (\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \cdots) \in \mathbb{R}^\omega$$

$x_n \to 0 = (0, 0, 0, \cdots)$ in the product topology but not in the box topology

In the uniform norm topology, $x_n \to 0$ ($x_n \to 0$ in ℓ^∞).

In the box topology, $\prod_{n=0}^{\infty} \left(-\frac{1}{n+1}, \frac{1}{n+1}\right)$ is a basic open nbhd of 0 and it contains no terms of the sequence $(x_n)_{n \in \omega}$.

Now consider

$$y_1 = (1, 0, 0, 0, 0, \cdots)$$

$$y_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \cdots)$$

$$y_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \cdots)$$

etc.

$\|y_n\|_1 = 1 < \infty$

$\|y_n\|_2 = \frac{1}{\sqrt{n}} < \infty$

$\|y_n\|_\infty = \frac{1}{n} < \infty$

$$y_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \cdots, \tfrac{1}{n}}_{n}, 0, 0, \cdots) \longrightarrow 0$$

in l^2, l^∞, product topology but not in the box topology.

The box topology has $\prod_{n=0}^{\infty} \left(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right)$ as a basic open nbhd of 0 and it contains no term of the sequence of points $(y_n)_n$.

The product topology is sometimes called the topology of pointwise convergence.

The box topology is not usually as useful the other topologies.

A sequence f_n in $\mathbb{R}^A$ converges uniformly to f if for all $\varepsilon > 0$ there exists N such that $|f_n(a) - f(a)| < \varepsilon$ for all $a \in A$ whenever $n > N$

Basic open sets in the uniform topology look like $U^A = \prod_{a \in A} U$, $U \subseteq \mathbb{R}$ is open.
(finer than the product topology but coarser than the box topology).

If $|A| < \infty$ then the product topology on $\prod_{\alpha \in A} X_\alpha$ agrees with the box topology.

If $|A| = |B|$ then products $\prod_{\alpha \in A} X_\alpha$ and $\prod_{\beta \in B} Y_\beta$ are essentially the same.
(The order of the ~~factors~~ does not alter the definition of the product or box topology.)

$\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R} \times \mathbb{R}}$ in the product topology

The Cantor Space (as a topological space)

$K_1 = [0,1]$

$K_2 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$K_3 = [0,\frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

etc.

$C = \bigcap_{n=1}^{\infty} K_n$ is a compact top. space. $C \subset \mathbb{R}$ and we take the standard topology. It is a metric space.

It is totally disconnected: given $x \neq y$ in C, there exists a partition $C = U \sqcup V$, $U, V \subset C$ open, $x \in U$, $y \in V$.

Equivalently, $C = \{0,1\}^{\omega} = 2^{\omega}$ with the product topology. ($\{0,1\}$ is discrete)

Points of C have the form $(a_0, a_1, a_2, a_3, \dots)$ where $a_i \in \{0,1\}$.

$|C| = |\mathbb{R}| = 2^{\aleph_0}$

A set of basic open nbhds of $a=(a_0, a_1, a_2, a_3, \dots) \in C$ is the set of $\{ b \in C : \ b_i = a_i \text{ for } i \leq n \}$.

A metric defining this topology is

$$d(a,b) = \begin{cases} 0, & \text{if } a = b \\ \frac{1}{2^n}, & \text{if } a_n \neq b_n \text{ for some } n \text{ and we take the smallest such } n. \end{cases}$$

This is really $\mathbb{Z}_2$ = 2-adic integers

$= \{ a \in \mathbb{Q} : \ \|a\|_2 \leq 1 \}$.

A homeomorphism

$\{0,1\}^{\mathbb{N}} \longrightarrow$ Usual Cantor set $\bigcap_{n=1}^{\infty} K_n$ is

$$(a_1, a_2, a_3, \dots) \longmapsto \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

The Cantor Space is the unique compact Hausdorff space without isolated points which is second countable having a countable base of clopen sets.

Second countable: having a countable base.

Separable: having a countable dense subset.

$A \subseteq X$ is dense if $A \cap U \neq \emptyset$ for every open $U \neq \emptyset$.

Tychonoff's Theorem A product of compact spaces is compact.

That is, if X_α ($\alpha \in A$) is an indexed family of compact spaces, then

$\prod_{\alpha \in A} X_\alpha$ is compact. (NB: We are using the product topology here.)

NB means "take note"

eg. $[0,1]^\omega$ is compact in the product topology.

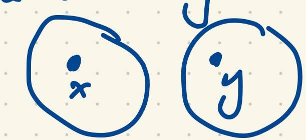
Not in the box topology eg. for every $a \in \{0,1\}^\omega$ i.e. $a = (a_0, a_1, a_2, \ldots)$ $a_i \in \{0,1\}$

the sets $U_a = \prod_{i \in \omega} U_{a(i)}$ $U_0 = [0, \frac{2}{3})$, $U_1 = (\frac{1}{3}, 1]$ open in $[0,1]$

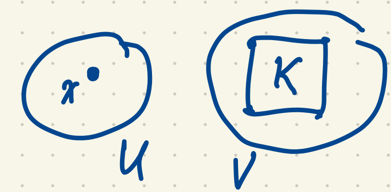
including $U_{(0,1,0,1,1,1,0,1,0,0,\ldots)} = U_0 \times U_1 \times U_0 \times U_1 \times U_1 \times \cdots$

covers $[0,1]^\omega$. No finite number of these U_a's cover $[0,1]^\omega$.

X is **Hausdorff** if for all $x \neq y$ in X, there are disjoint open nbhds of x and y.



X is **regular** if for every closed set K and every point $x \notin K$, there are open sets U, V with $U \cap V = \emptyset$, $x \in U$, $K \subseteq V$.



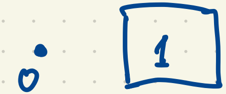
X is **normal** if

<u>Warning</u> normal spaces are not necessarily regular (unless points are closed)

Eg. $X = \{0,1\}$ This space is normal. It's not regular.

Open sets: $\emptyset, \{0\}, X$

Closed sets: $\emptyset, \{1\}, X$.



<u>Urysohn's</u> <u>Lemma</u> X is a normal top. space iff for every pair of disjoint closed sets, K, L, there exists a continuous function $X \to [0,1]$ such that $f(x)=0$ for all $x \in K$, $f(x)=1$ for all $x \in L$.

Metric spaces are Hausdorff, normal and regular.

In any metric space (X, d), $d: X \times X \to [0,\infty)$ is continuous.

If $A \subseteq X$, we can define distance from $x \in X$ to A: $d(x,A) = \inf_{a \in A} d(x,a)$.

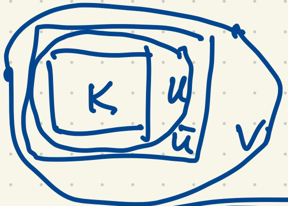
This is a continuous map $X \to [0,\infty)$. $d(x,A)=0$ iff $x \in \overline{A}$ = closure of A.

$d(A,B) = \inf_{a\in A} d(a,B)$. If A, B are disjoint closed sets then $d(A,B) > 0$.

$$f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$$

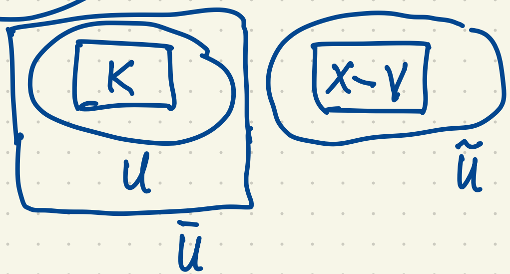
Wed Oct 19 } prerecorded lectures on Baire Category - see website (Lecture videos + pdfs)
Fri Oct 21 }

Lemma X is normal iff whenever $K \subseteq V$ with K closed and V open, there exists an open set U such that $K \subseteq U \subseteq \bar{U} \subseteq V$.



($\bar{U}$ = closure of U = smallest closed set containing U).

Proof



$X \setminus V = \{x \in X : x \notin V\}$
closed

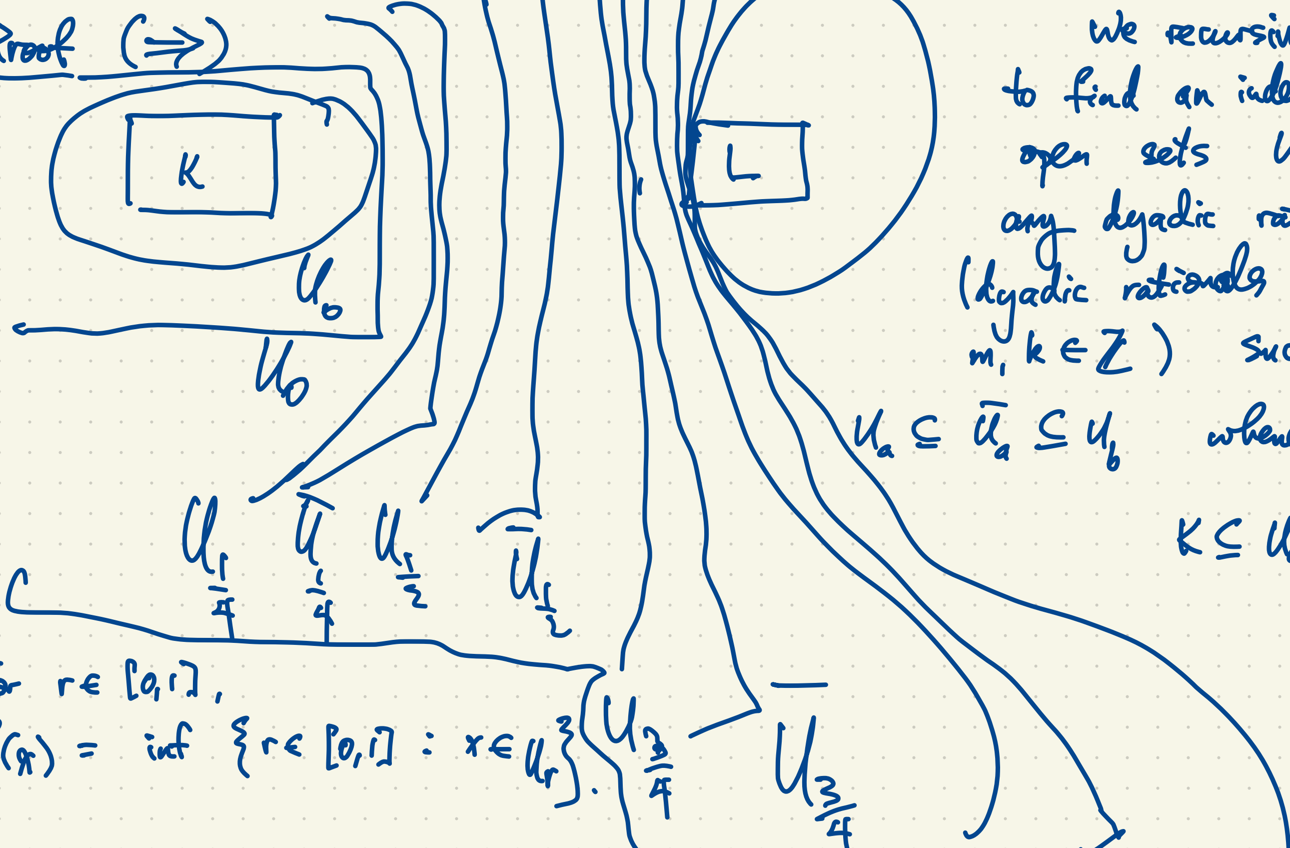
Proof of Urysohn's Lemma ($\Leftarrow$) Suppose K, L disjoint closed sets in a space X and $f: X \to [0,1]$ is continuous with $f|_K = 0$, $f|_L = 1$.

Let $U = f^{-1}([0,\frac{1}{3})) \subseteq X$ is open, $V = f^{-1}((\frac{2}{3},1]) \subseteq X$ is open.

$U \cap V = \emptyset$, $K \subseteq U$, $L \subseteq V$. So X is normal.

Urysohn's Lemma X is a normal top. space iff for every pair of disjoint closed sets, K, L, there exists a continuous function $X \to [0,1]$ such that $f(x) = 0$ for all $x \in K$, $f(x) = 1$ for all $x \in L$.

Proof $(\Rightarrow)$



We recursively use the Lemma to find an indexed collection of open sets U_a where a is any dyadic rational in $[0,1]$ (dyadic rationals have the form $\frac{m}{2^k}$, $m, k \in \mathbb{Z}$) such that

$$U_a \subseteq \overline{U}_a \subseteq U_b \quad \text{whenever} \quad 0 \le a < b \le 1$$

$$K \subseteq U_a, \qquad \overline{U}_a \cap L = \emptyset.$$

For $r \in [0,1]$,

$$f(x) = \inf\{r \in [0,1] : x \in U_r\}.$$

$\square$

Question: If X is regular, i.e. $\overset{\bullet}{x}$ $\boxed{K}$ $x \notin K$

must there exist a continuous function $f: X \to [0,1]$ such that $f(x)=0$, $f|_K = 1$?

No! There is no analogue of Urysohn's Lemma for regularity.

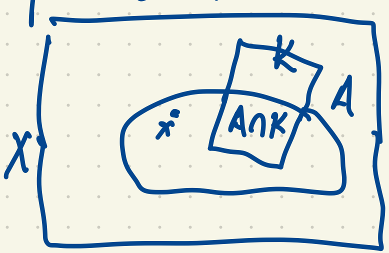
X is <u>completely regular</u> if whenever $\overset{\bullet}{x}$ $\boxed{K}$ K closed, $x \notin K$, there exist continuous $f: X \to [0,1]$, $f(x)=0$, $f|_K = 1$.

There exist top. spaces which are regular but not completely regular (e.g. Tychonoff corkscrew) but we will omit this.

X is <u>completely normal</u> if every subspace of X is normal.

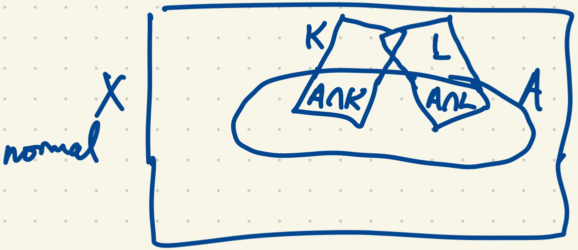
<u>Remarks</u>: If X is completely regular then X is regular (easy) and every subspace of X is also completely regular.

In X:



There exist continuous $f: X \to [0,1]$ such that $f(x)=0$, $f|_K = 1$. Restricting f to $f|_A$, we see that A is also completely regular.

Is every subspace of a normal space normal? No: see Tychonoff's Plank.



$\omega =$ discrete

$= \{0,1,2,3,\cdots\}$

$\omega+1 = \{0,1,2,\cdots\} \cup \{\omega\}$

$=$ in which $\{\omega\}$ is not open

$(m-1,\ n-1)$

$\omega 2 = \quad = \omega+\omega$

$\omega^2 = \underbrace{\omega+\omega+\omega+\cdots}_{\omega \text{ times}} = \quad \cdots \simeq \left\{ m-\frac{1}{n} : m,n \text{ positive integers} \right\} \subset \mathbb{R}$

$$\begin{array}{ccccccccccccccc}
(0,0) & < & (0,1) & < & (0,2) & < & (0,3) & < \cdots & < & (1,0) & < & (1,1) & < & (1,2) & < (1,3) \cdots \\
0 & & \frac{1}{2} & & \frac{2}{3} & & \frac{3}{4} & & & 1 & & 1\frac{1}{2} & & 1\frac{2}{3} & 1\frac{3}{4} \cdots
\end{array}$$

$$\begin{array}{ccccccc}
< (2,0) & < & (2,1) & < & (2,2) & < & (2,3) \cdots \\
2 & & 2\frac{1}{2} & & 2\frac{2}{3} & & 2\frac{3}{4}
\end{array}$$

For $\{a_\alpha : \alpha \in A\}$ any indexed set of ~~positive~~ real numbers,

$$\sum \{a_\alpha : \alpha \in A\} = \sum_{\alpha \in A} a_\alpha = \sup \{ a_{\alpha_1} + a_{\alpha_2} + \cdots + a_{\alpha_k} : \begin{array}{c} \text{distinct} \\ \alpha_1, \cdots, \alpha_k \in A \\ k \geq 1 \end{array} \}$$

$$\sum \{1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \dots\} = 2$$

<u>But</u> if A is uncountable and $a_\alpha > 0$ (positive reals) then $\sum_{\alpha \in A} a_\alpha = \infty$ (always diverges)!

Why? In other words, if $\underbrace{\sum_{\alpha \in A} a_\alpha < \infty}$, why must A be countable?

i.e. there exists M real such that $a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_k} < M$ for all $k \geq 1$; $\alpha_1, \dots, \alpha_k \in A$ distinct.

$$\sum_{m,n \in \mathbb{Z}} \frac{1}{(m^4 + n^4 + 1)}$$

$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \dots$ where $|A_n| < \infty$ for all n

$A_1 = \{\alpha \in A : a_\alpha \in [1, \infty)\}$
$A_2 = \{\alpha \in A : a_\alpha \in [\tfrac{1}{2}, 1)\}$
$A_3 = \{\alpha \in A : a_\alpha \in [\tfrac{1}{4}, \tfrac{1}{2})\}$ etc.

So A is a countable union of finite sets so it's countable.

Recall order topology on a totally ordered set $(X, <)$

Sub-basic open sets $\{x \in X : x < \beta\}$ for $\beta \in X$

$\{x \in X : x > \alpha\}$ for $\alpha \in X$

Basic open sets $(\alpha, \beta) = \{x \in X : \alpha < x < \beta\}$ including the sub-basic sets above

Ordinals: well-ordered sets ; ordinals are the canonical examples of well-ordered sets.

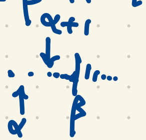
Given an ordinal γ, $[0, \gamma] = \{\text{ordinals } \alpha \leq \gamma\}$.

In γ, every nonempty subset $A \subseteq [0, \gamma]$ has a least element $\inf A$.

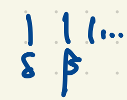
Also A has a supremum $\sup A$ which may or may not be an element of A.

($\sup A$ = least upper bound for A ; since γ is an upper bound for A, so $\{$upper bounds for $A\}$ is a nonempty subset of $[0, \gamma]$ so $\sup A$ exists by well-ordering).

Moreover: $[0, \gamma]$ is compact Hausdorff. If $\alpha \neq \beta$ in $[0, \gamma]$, WLOG $\alpha < \beta$ so



(i)



(ii)

Case (i). β is a limit ordinal.

$\alpha \in [0, \alpha+1)$, $\beta \in (\alpha+1, \gamma]$.

Case (ii) β is a successor ordinal, $\beta = \delta + 1$, $\alpha \leq \delta$, $\alpha \in [0, \alpha+1)$, $\beta \in (\delta, \gamma]$.

Why is $[0,\gamma]$ compact? (If γ is a limit ordinal then $\gamma = [0,\gamma)$ is not compact.)

$[0,\gamma] = \gamma \cup \{\gamma\}$

Let $\mathcal{O}$ be an open cover of $[0,\gamma]$.

A set $A \subseteq [0,\gamma]$ is <u>finitely</u> <u>covered</u> by $\mathcal{O}$ if there exist $U_1,\dots,U_n \in \mathcal{O}$ such that $A \subseteq U_1 \cup U_2 \cup \dots \cup U_n$.

We must that $X = [0,\gamma]$ is finitely covered by $\mathcal{O}$.

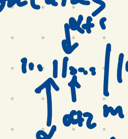
Let $S = \{\alpha \in [0,\gamma] : [\alpha,\gamma]$ is finitely covered by $\mathcal{O}\}$.

$S \subseteq [0,\gamma]$ is nonempty, since $\gamma \in S$. So S has a least element m.

If $m = 0$ we're done. Otherwise $[m,\gamma] \subseteq U_1 \cup \dots \cup U_n$ for some $U_1,\dots,U_n \in \mathcal{O}$.

m must be a limit ordinal (if $m = \alpha+1$ then some $U_0 \in \mathcal{O}$ covers α and $[\alpha,\gamma] \subseteq U_0 \cup U_1 \cup \dots \cup U_n$, contradicting the minimality of $m \in S$.)

Now $m \in [0,\gamma]$ is a limit ordinal. $m \in U_0$ for some $U_0 \in \mathcal{O}$ and U_0 contains some basic open nbhd of m i.e. $m \in (\alpha,\beta) \subseteq U_0$. Here $\alpha < m$



Now $U_0 \cup U_1 \cup U_2 \cup \dots \cup U_n$ covers $[\alpha+1,\gamma]$. Again $\alpha+1 \in S$, $\alpha+1 < m$ contradicting the minimality of $m \in S$. $\square$

If points are closed then

completely normal $\xRightarrow{\text{easy}}$ normal $\xRightarrow{\text{Urysohn's lemma}}$ completely regular $\xRightarrow{\text{easy}}$ regular

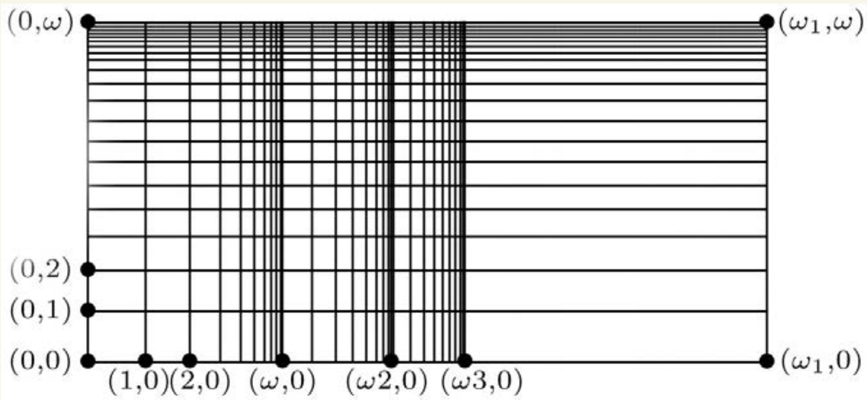
$T_{4\frac{1}{2}}$ $\underset{\text{Tychonoff plank}}{\not\Leftarrow}$ T_4 $\underset{\text{Tychonoff plank}}{\not\Leftarrow}$ $T_{3\frac{1}{2}}$ $\underset{\text{Tychonoff corkscrew}}{\not\Leftarrow}$ T_3

Prop 1.1 on Separation Properties

Tychonoff plank $X = [0,\omega_1] \times [0,\omega]$. is compact Hausdorff. X is normal.

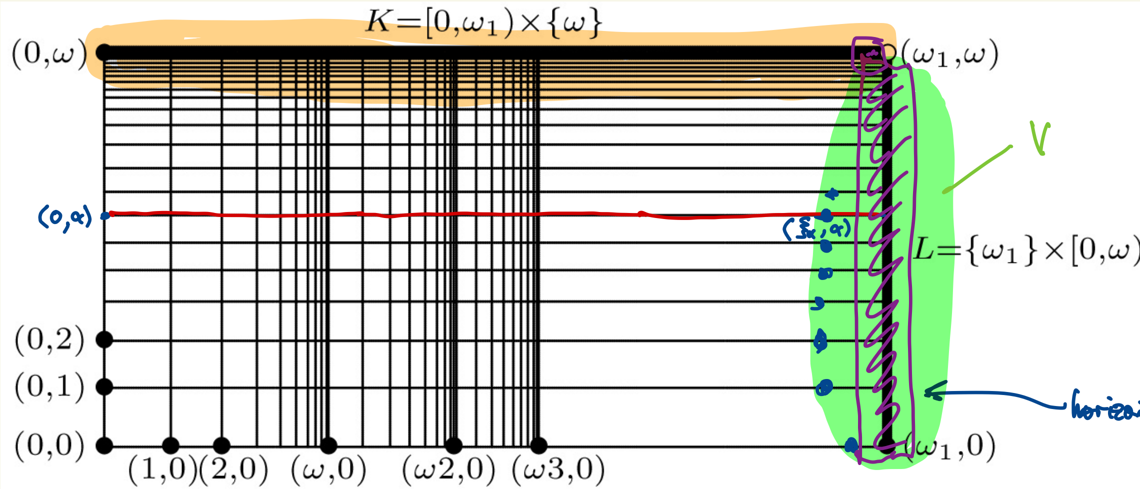
But X is not completely normal

i.e. it has a subspace that is not normal. Delete (ω_1, ω)



Warning: (α, β) can be either an interval or a point.

Deleted Tychonoff Plank $X = ([0,\omega_1]\times[0,\omega]) - \{(\omega_1,\omega)\} \subset$ Tychonoff Plank : X is not normal.



horizontal lines are copies of $[0,\omega_1]$

K = top edge

L = right edge

$K, L \subset X$ closed subsets

$K \cap L = \emptyset$

Claim: We can't find open sets $U, V \subset X$, $K \subseteq U$, $L \subseteq V$, $U \cap V \neq \emptyset$

Suppose $V \subseteq X$ open, $L \subseteq V$. For each $\alpha \in \omega$, there is a minimal $\xi_\alpha \in [0,\omega_1)$ such that $[\xi_\alpha, \omega_1] \times \{\alpha\} \subseteq V$. ξ_α is countable (since $\xi_\alpha < \omega_1$). So $\xi = \sup_{\alpha<\omega} \xi_\alpha = \bigcup_{\alpha<\omega} \xi_\alpha$ is countable. So $[\xi,\omega_1] \times [0,\omega) \subseteq V$.

Now $K \subseteq \underset{\text{open}}{U} \subseteq X$. U contains points $[\xi,\alpha]$ for α sufficiently close to ω. (α large positive integers).

Then $[\xi,\alpha] \in U \cap V$, contradiction.

What's Next?

- Ultrafilters
- Nonstandard Reals
- Proof of Tychonoff's Theorem
- Construction of the Stone-Čech Compactification

But first let's review the real numbers.

One way to construct $\mathbb{R}$ from $\mathbb{Q}$ is via Dedekind cuts

Better: completion via Cauchy sequences.

Start with the set of all Cauchy sequences in $\mathbb{Q}$.

$X = \{ x \in \mathbb{Q}^{\omega} : x \text{ is Cauchy} \}$

$x = (x_0, x_1, x_2, x_3, \dots)$, $x_i \in \mathbb{Q}$

For all $\varepsilon > 0$ there exists N such that $|x_i - x_j| < \varepsilon$ whenever $i, j > N$.

Here $|x_i - x_j|$ can be replaced by $d(x_i, x_j)$ in any metric space so the same notion leads to a completion of a general metric space.

Introduce an equivalence relation $\sim$ on X

Given a set X, we want to distinguish every subset $A \subseteq X$ as large ($A \in \mathcal{U}$) or small ($A \notin \mathcal{U}$). Here $\mathcal{U} \subseteq \mathcal{P}(X)$ (collection of subsets of X) called an ultrafilter, satisfying:

- $\emptyset \notin \mathcal{U}$, $X \in \mathcal{U}$.
- For every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \smallsetminus A \in \mathcal{U}$ (but not both).
- If $A \in \mathcal{U}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{U}$.
- If $A, A' \in \mathcal{U}$ then $A \cap A' \in \mathcal{U}$.

Note: If $X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$ (note: disjoint union) then exactly one of $A_1, \dots, A_n$ is $\in \mathcal{U}$.

Every nonempty set X has an ultrafilter. But we are not so interested in principal ultrafilters: If $a \in X$, $\mathcal{U}_a = \{A \subseteq X : a \in A\}$. If $|X| < \infty$ then every ultrafilter on X is principal. If $|X| = \infty$ then X has many ultrafilters including lots of nonprincipal ultrafilters. But AC is required to find them. Let's "find" a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N} = \{1,2,3,4,5,\dots\}$.

Since $\{1\} \notin \mathcal{U}$, $\mathbb{N} \smallsetminus \{1\} \in \mathcal{U}$. If $A \subseteq \mathbb{N}$, $|A| < \infty$ then $\mathbb{N} \smallsetminus A \in \mathcal{U}$. Complements of finite sets are large. What about $2\mathbb{N} = \{2,4,6,8,\dots\}$ and $2\mathbb{N}-1 = \{1,3,5,7,9,\dots\}$? We must chose one of these to be large and the other to be small. My choice: $2\mathbb{N}$ is small, $2\mathbb{N}-1$ is large.

$$(\mathbb{N} = \underbrace{2\mathbb{N}}_{\notin \mathcal{U}} \sqcup \underbrace{(2\mathbb{N}-1)}_{\in \mathcal{U}}). \qquad \mathbb{N} = \underbrace{(3\mathbb{N})}_{\substack{\{3,6,9,12,\dots\} \\ \notin \mathcal{U}}} \sqcup \underbrace{(3\mathbb{N}-2)}_{\substack{\{1,4,7,10,\dots\} \\ \in \mathcal{U}}} \sqcup \underbrace{(3\mathbb{N}-1)}_{\substack{\{2,5,8,11,\dots\} \\ \notin \mathcal{U}}}$$

My choice: $3\mathbb{N}-2 = \{1,4,7,10,\dots\} \in \mathcal{U}$.

$$\mathbb{N} = \underbrace{\{1,4,6,8,9,10,12,14,\dots\}}_{\text{non-primes}} \sqcup \underbrace{\{2,3,5,7,11,13,\dots\}}_{\text{primes}}.$$

Once again either of these can be chosen to be "large".

I choose $\{2,3,5,7,11,13,\dots\} \in \mathcal{U}$. Note: By Dirichlet's theorem, there are infinitely many primes $\equiv 1 \bmod 6$ so the intersection of our "large" sets is infinite as required.

A <u>filter</u> on X is a collection $\mathcal{F}$ consisting of subsets of X such that

- $\varnothing \notin \mathcal{F}$, $X \in \mathcal{F}$
- If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
- If $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.

Every ultrafilter is a filter, but not conversely.

A collection $\mathcal{S}$ of subsets of X has the <u>finite intersection property</u> (f.i.p.) if for all $A_1, \cdots, A_n \in \mathcal{S}$, $A_1 \cap A_2 \cap \cdots \cap A_n \neq \varnothing$.

A filter has the f.i.p. If $\mathcal{S}$ is any collection of subsets of X having f.i.p. then $\mathcal{S}$ generates a filter:

$$\mathcal{F}_{\mathcal{S}} = \{\text{supersets of finite intersections of sets in } \mathcal{S}\}$$
$$= \{B \subseteq X : A_1 \cap A_2 \cap \cdots \cap A_n \subseteq B \text{ for some } A_1, A_2, \cdots, A_n \in \mathcal{S}\}.$$

This is the (unique) smallest collection of subsets of X which contains $\mathcal{S}$ and is a filter.

If $\mathcal{F}, \mathcal{F}'$ are filters on X, we say $\mathcal{F}'$ <u>refines</u> $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$.

The collection of all filters on X is partially ordered by refinement.

Given a filter $\mathcal{F}_0$ on X, the collection of filters refining $\mathcal{F}_0$ has a maximal member by Zorn's Lemma. This is guaranteed to be an ultrafilter.