Point Set Topology

Book 1

Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X (called the open sets) such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.

if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

(So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

Example

The standard topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon\}$.

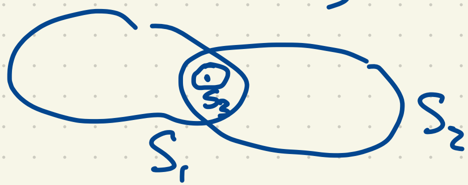
$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$

(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let $\mathcal{S}$ be a collection of subsets of X which cover X, i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \dots, S_k \in \mathcal{S}$ is a topology on X. The members of $\mathcal{S}$ are called a sub-basis for this topology and the topology is said to be *generated* by $\mathcal{S}$.

$\mathcal{S}$ is called a *base* (or a *basis*) for the topology if the topology is the collection of arbitrary unions of elements of $\mathcal{S}$. This holds iff



for all $S_1, S_2 \in \mathcal{S}$,
and all $u \in S_1 \cap S_2$,
there exists $S_3 \in \mathcal{S}$ such that

$$u \in S_3 \subseteq S_1 \cap S_2 .$$

Eg. Let X be any set. The *discrete topology* on X is the collection of *all* subsets of X. (2^X)
The *indiscrete* topology on X is $\{\emptyset, X\}$.

If $X = \{0,1\}$ then there are four possible topologies on X: $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let $\mathcal{J}$ be the collection of complements of finite sets, and $\emptyset$

i.e. $\mathcal{J} = \{\emptyset\} \cup \{X \setminus A : A \subseteq X,\ |A| < \infty\}$, $\quad X \setminus A = \{x \in X : x \notin A\}$.

set difference

This is a topology on X, called the <u>finite complement topology</u>.

$X - A$, $X \setminus A$, $X \backslash A$

$\emptyset$, ϕ, $\varnothing$, 0

\varnothing

A <u>topological space</u> is a pair $(X, \mathcal{J})$ where $\mathcal{J}$ is a topology on a set X.

Note: $\bigcup \mathcal{J} = X$. By abuse of language, we often say that X is a topological space.

Let X be a set. A <u>distance function</u> (or <u>metric</u>) on X is a function $d : X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

$$d(x,y) = d(y,x)$$

$d(x,y) \geq 0$ and equality holds iff $x = y$.

$$d(x,z) \leq d(x,y) + d(y,z)$$

The standard topology on $\mathbb{R}^n$ is a metric topology.

The metric $d_2(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n - y_n)^2}$ (the Euclidean metric)

$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ all give the standard topology on $\mathbb{R}^n$.

In $\mathbb{R}^2$, open balls with respect to d_2, d_1, d_∞ look like

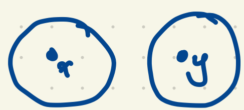
respectively. These three metrics define the same topology.

The metric $d(x,y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ defines the discrete topology.

A topological space is <u>metrizable</u> if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.

A topology is <u>Hausdorff</u> if for any two points $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

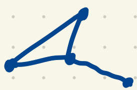


Every metric space is Hausdorff since if $x \neq y$, $\delta = d(x,y) > 0$. Take $U = B_{\delta/3}(x)$, $V = B_{\delta/3}(y)$

An <u>open</u> <u>neighbourhood</u> of a point $x \in X$ is an open set containing x.

nbhd

A <u>basic</u> <u>open</u> <u>nbhd</u> of a point $x \in X$ is an open nbhd of x which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.

Consider $X = \mathbb{Q}$. A <u>norm</u> on $\mathbb{Q}$ is a function $\mathbb{Q} \to [0,\infty)$,

$x \mapsto \|x\|$ <u>satisfying</u>

(i) $\|x\| \geq 0$; equality holds iff $x = 0$.

(ii) $\|xy\| = \|x\| \cdot \|y\|$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

From any norm on $\mathbb{Q}$, we obtain a metric $d(x,y) = \|x-y\|$.

One way to do this is with the usual absolute value $\|x\| = |x| = |x|_\infty = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0 \end{cases}$

This gives the standard topology on $\mathbb{Q}$.

An alternative is: given $x \in \mathbb{Q}$, if $x = 0$ define $\|0\|_2 = 0$.

If $x \neq 0$, write $x = 2^k \cdot \frac{a}{b}$, $a,b,k \in \mathbb{Z}$, $b \neq 0$; a,b odd. Then define $\|x\|_2 = 2^{-k}$.

This is the 2-adic norm on $\mathbb{Q}$. In fact it satisfies a stronger form of (iii), the ultrametric inequality $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$.

E.g. $\left\|\frac{20}{21}+\frac{5}{14}\right\|_2 = \left\|\frac{40+15}{42}\right\|_2 = \left\|\frac{55}{42}\right\|_2 = 2. = \max\left\{\underbrace{\left\|\frac{20}{21}\right\|_2}_{\frac{1}{4}}, \underbrace{\left\|\frac{5}{14}\right\|_2}_{2}\right\} = 2$

$\left\|\frac{20}{21}\right\|_2 = \frac{1}{4}, \quad \left\|\frac{5}{14}\right\|_2 = 2$

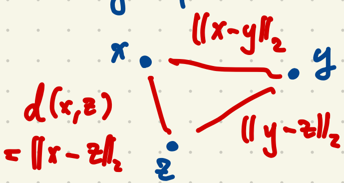
Compare: $\left\|\frac{20}{21}\right\|_2 + \left\|\frac{5}{14}\right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

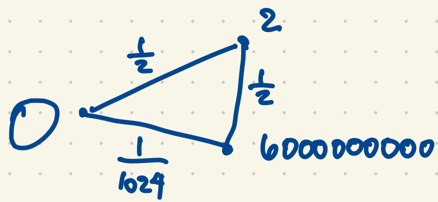
$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \{\frac{a}{b} : a, b \in \mathbb{Z}, \ a \text{ even}, \ b \text{ odd}\}.$

Every point in the ball is a centre of the ball i.e. if $c \in B_1(0)$ then $B_1(c) = B_1(0)$.



Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1+2+4+8+16+32+64+\cdots = -1$$

The partial sums $1, 3, 7, 15, 31, 63, \ldots$ converge to -1 in the 2-adic norm.