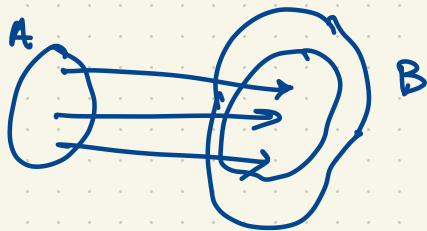


Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. I.e. if there is an injection $A \rightarrow B$ and an injection $B \rightarrow A$ then there is a bijection $A \rightarrow B$.

Here $|A| \leq |B|$ means there is an injection $A \rightarrow B$ i.e. A is in one-to-one correspondence with a subset of B . This is equivalent to the existence of a surjection $B \rightarrow A$ under the Axiom of Choice.



Bernstein-Cantor-Schröder Theorem uses ZF

Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?

There is an injection $(0,1) \rightarrow [0,1]$, $x \mapsto x$. So $|(0,1)| \leq |[0,1]|$.

There is an injection $[0,1] \rightarrow (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \leq |(0,1)|$.

$$\underline{|R| = |R^3| = |[0,1]| = |[0,1]^3|}$$

$[0,1] \rightarrow [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \rightarrow [0,1]$, $(x,y,z) \mapsto 0.x_1y_1z_1x_2y_2z_2x_3y_3z_3x_4y_4z_4 \dots$

$$x = 0.x_1x_2x_3x_4 \dots$$

$$y = 0.y_1y_2y_3y_4 \dots$$

$$z = 0.z_1z_2z_3z_4 \dots$$

Theorem $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$$|X| = |\mathbb{R}| = 2^{\aleph_0}$$

And how many lines do we need to cover X ? (partition)

Let Σ be a set of lines partitioning X . Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \rightarrow \mathbb{R}^3$ so

$|\Sigma| \leq |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \rightarrow \Sigma$? $\mathbb{R}^3 \xrightarrow{!} \mathbb{R} \xrightarrow{!} l \xrightarrow{!} \Sigma$

Let l be any line in X which is not in Σ .

To construct Σ , we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \dots ?$$

hoping that "in the limit" we cover all of X .

$$\Sigma_0 = \emptyset.$$

$$\Sigma_1 = \{l_0\}$$

$$\Sigma_2 = \{l_0, l_1\}$$

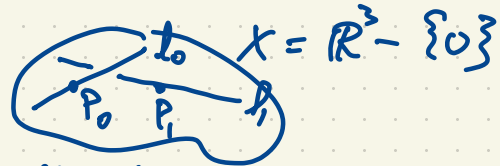
$$\Sigma_3 = \{l_0, l_1, l_2\}$$

Inductively construct Σ_β , $\beta \in A$, a set of disjoint lines in X , such that

- Σ_β covers P_α whenever $\alpha < \beta$.

- $|\Sigma_\beta| \leq |\beta| < |K| = 2^{\aleph_0}$.

- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$



Well-orders the points of X as P_α , $\alpha \in A$

where A is well-ordered.

Actually we can take $A = \kappa$ the smallest ordinal such that

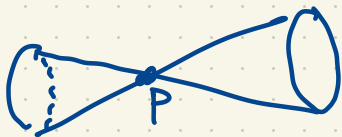
$$|K| = 2^{\aleph_0}$$

$$\text{Take } \Sigma = \bigcup_{\beta \in A} \Sigma_\beta$$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |K| = 2^{\aleph_0}$
with $P \in X$ not covered by Σ ($P \notin \bigcup_{\text{in } \Sigma} \text{lines}$),

there exists line l in X disjoint from all lines in Σ passing through P .
Consider a cone with vertex P . Every line of Σ hits this cone in at most 2 points. There are 2^{\aleph_0} lines in this cone passing through P , at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ .



By the Pigeonhole Principle, l exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizability?

- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y , we have the disjoint union $X \sqcup Y$ which can be viewed as $(X \times \{0\}) \cup (Y \times \{1\})$

$$\{(x, 0) : x \in X\}$$

$$\{(y, 1) : y \in Y\}$$

eg. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0, 1\} \subset \mathbb{R}^2$

$\mathbb{R} \times \{1\} =$ the line $y=1$

$\mathbb{R} \times \{0\} =$ x-axis ($y=0$)

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

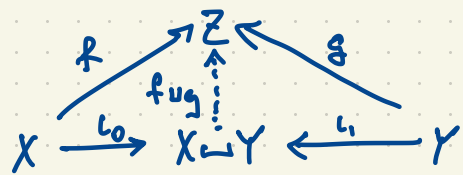
Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y , a coproduct of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \rightarrow X \sqcup Y$, $\iota_1 : Y \rightarrow X \sqcup Y$

such that whenever Z is a top. space and $f : X \rightarrow Z$, $g : Y \rightarrow Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \rightarrow Z$ such that this diagram commutes i.e. $(f \sqcup g) \circ \iota_0 = f$ and $(f \sqcup g) \circ \iota_1 = g$ see over

$$\begin{array}{ccc} & f & \\ & \nearrow & \\ X & \xrightarrow{\iota_0} & X \sqcup Y & \xleftarrow{\iota_1} & Y \\ & \searrow & \\ & f \sqcup g & \end{array}$$

$$\iota_0(x) = (x, 0), \quad \iota_1(y) = (y, 1)$$

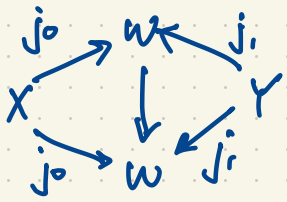
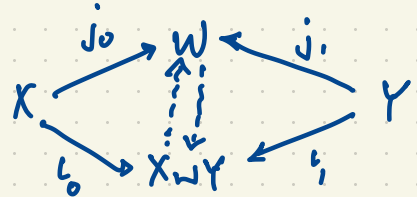


$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \cup g)(x, 0) = f(x) \in Z$$

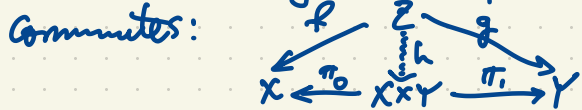
$$(f \cup g)(y, 1) = g(y) \in Z$$

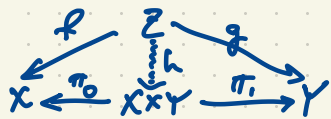
Any $X \sqcup Y$ together with l_0, l_1 satisfying this universal property is a (the) coproduct of X and Y . It exists by our construction; and it is unique. If W also satisfies the same universal property then



(cont maps)

Given top. spaces X, Y , a product is a top. space $X \times Y$ together with morphisms $\pi_0: X \times Y \rightarrow X, \pi_1: X \times Y \rightarrow Y$ such that for every top. space Z and morphisms $f: Z \rightarrow X, g: Z \rightarrow Y$, there exists a unique $h: Z \rightarrow X \times Y$ such that the following diagram





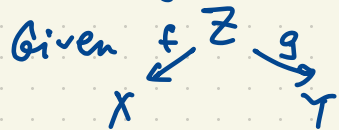
Existence of direct product: $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

Topology: $U \times V \subseteq X \times Y$ ($U \subseteq X, V \subseteq Y$ open)

are a basis for top. on $X \times Y$.

$$\begin{aligned}
 \pi_0: (X, Y) &\rightarrow X \\
 (x, y) &\mapsto x
 \end{aligned}$$

$$\begin{aligned}
 \pi_1: X \times Y &\rightarrow Y \\
 (x, y) &\mapsto y
 \end{aligned}$$




we have $h(z) = (f(z), g(z))$.

The product topology $X \times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U \times Y$ to be open in $X \times Y$ whenever $U \subseteq X$ is open. Also

" " $\pi_1^{-1}(V) = X \times V$ $V \subseteq Y$

Then $U \times V = (U \times Y) \cap (X \times V)$ must be open in $X \times Y$.

Ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has topology generated by  $U \times V$ $(u, v \in \mathbb{R})$
 which is the standard topology. open

A topological group is a group G endowed with a topology such that the maps $G \rightarrow G$ $g \mapsto g^{-1}$ is continuous and $G \times G \rightarrow G$ $(g, h) \mapsto gh$ is also continuous.

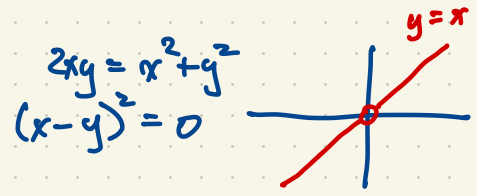
Ex. Consider $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

The map $\mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto f(x, b)$ is continuous for every $b \in \mathbb{R}$.

... .. $y \mapsto f(a, y)$ $a \in \mathbb{R}$.

But f is not continuous.

$$f^{-1}(1) = \left\{ (x, y) \in \mathbb{R}^2 : f(x, y) = \frac{2xy}{x^2+y^2} = 1 \right\} \\ = \left\{ (x, x) \in \mathbb{R}^2 : x \neq 0 \right\} \text{ is not closed in } \mathbb{R}^2.$$



$(\mathbb{R}, +)$ is a topological group.

(\mathbb{R}^*, \cdot)

$+, \cdot$ are continuous maps $\mathbb{R}^2 \rightarrow \mathbb{R}$.

If $f, g: \mathbb{R} \rightarrow \mathbb{R}$ is continuous then so are $f+g, fg$.

One way to see this is

$(f \times g): \mathbb{R}^2 \rightarrow \mathbb{R}^2$
 $(x, y) \mapsto (f(x), g(y))$ is continuous.

$\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}$
 $x \mapsto (x, x) \mapsto (f(x), g(x)) \mapsto f(x) + g(x)$.

Similarly for multiplication.

diagonal
embedding of
 \mathbb{R} in \mathbb{R}^2 .

Given a top. space X , is the diagonal embedding $X \rightarrow X \times X$, $x \mapsto (x, x)$ always continuous?

Given a metric space (X, d) , $d: X \times X \rightarrow [0, \infty]$,
 d is continuous.

This description of product spaces generalizes easily to $X_1 \times X_2 \times \dots \times X_n$
including $X^n = \underbrace{X \times X \times \dots \times X}_{n \text{ times}}$ as a special case.

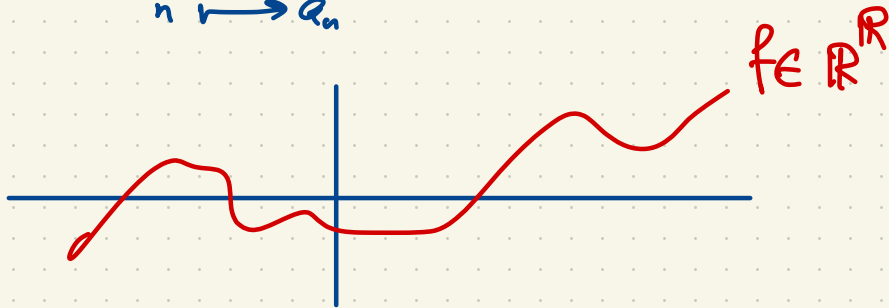
Infinite products are a little bit more subtle.

Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega \stackrel{\omega}{=} \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}\}$

Every function $\omega \mapsto \mathbb{R}$
 $n \mapsto a_n$

$\mathbb{R}^\mathbb{R} = \{\text{functions } \mathbb{R} \rightarrow \mathbb{R}\}$



The product topology for $\mathbb{R}^{\mathbb{R}} = \{ \text{functions } \mathbb{R} \rightarrow \mathbb{R} \}$ is the coarsest topology for which the projections $f \mapsto f(a)$ ($a \in \mathbb{R}$) are continuous.

This means we require: for every $\varepsilon > 0$, $b \in \mathbb{R}$, $\{ f \in \mathbb{R}^{\mathbb{R}} : f(a) \in \underbrace{B_{\varepsilon}(b)} \}$ is open in $\mathbb{R}^{\mathbb{R}}$.
 $B_{\varepsilon}(b)$ or any open set in \mathbb{R} .

$(\underbrace{\dots}_{\text{no restriction}}, \underbrace{f(a)}_m, \underbrace{\dots}_{\text{no restriction}}) \in \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \times U \times \mathbb{R} \times \dots$

General product: Let X_{α} ($\alpha \in A$, some index set A) be top. spaces. The product space $\prod_{\alpha \in A} X_{\alpha}$ has the Cartesian product as its underlying set.

As a set, an element $x = (x_{\alpha})_{\alpha \in A} \in \prod_{\alpha \in A} X_{\alpha}$ is really a function $A \rightarrow \bigcup_{\alpha \in A} X_{\alpha}$ subject to $x_{\alpha} \in X_{\alpha}$ for all $\alpha \in A$.

(Special case: all X_{α} isomorphic to X ; $x \mapsto x_{\alpha}$ is a map $A \rightarrow X = X$).
 If $X_{\alpha} \neq \emptyset$ for all $\alpha \in A$, then $\prod_{\alpha \in A} X_{\alpha} \neq \emptyset$. This uses AC = Axiom of Choice.

If all $X_\alpha = X$ for all $\alpha \in A$ then $\prod_{\alpha \in A} X_\alpha = X^A = \{ \text{functions } A \rightarrow X \} \neq \emptyset$ assuming $X \neq \emptyset$. This holds in ZF without requiring AC. Let $x \in X$ and consider the constant function $f(\alpha) = x$ for all $\alpha \in A$. This gives the diagonal embedding $X \rightarrow X^A$.

Topology on $\prod_{\alpha \in A} X_\alpha$: A subbasis consists of the open cylinders

$$\{ x = (x_\alpha)_\alpha : x_\alpha \in X_\alpha \text{ arbitrary for } \alpha \neq \beta; x_\beta \in U \} \quad \text{where } \beta \in A, U \subseteq X_\beta \text{ open}$$

$$= \underbrace{U}_{\text{in coordinate } \beta} \times \prod_{\substack{\alpha \in A \\ \alpha \neq \beta}} X_\alpha = \pi_\beta^{-1}(U) \quad \text{where } \pi_\beta : \prod_{\alpha \in A} X_\alpha \rightarrow X_\beta$$

$$x = (x_\alpha)_{\alpha \in A} \mapsto x_\beta.$$

Under finite intersections, these generate a basis for the topology on the product space. Basic open sets have the form

$$\{ x \in (X_\alpha)_{\alpha \in A} : x_{\alpha_i} \in U_{\alpha_i} \text{ for } i=1, \dots, k \} \quad \text{where } k \geq 1 \text{ is a positive integer;}$$

$$\alpha_1, \dots, \alpha_k \in A;$$

$$U_{\alpha_i} \subseteq X_{\alpha_i} \text{ for each } i=1, \dots, k \text{ are open sets}$$

Arbitrary open sets are unions of basic open sets. This is the product topology (or the Tychonoff topology).

If instead one takes as basic open sets $\prod_{\alpha \in A} U_\alpha$ ($U_\alpha \subseteq X_\alpha$ open), then one gets the box topology.

This is a refinement of the product topology. Unless otherwise specified, the topology on $\prod_{\alpha \in A} X_\alpha$ is understood to be the product topology.

Eg. $\mathbb{R}^{\mathbb{R}} = \prod_{x \in \mathbb{R}} \mathbb{R} = \{ \text{functions } \mathbb{R} \rightarrow \mathbb{R} \}$

Each function $f: \mathbb{R} \rightarrow \mathbb{R}$ determines a point $(f(x))_{x \in \mathbb{R}}$ (a generalized sequence).

A basic open nbhd of $f \in \mathbb{R}^{\mathbb{R}}$ has the form

$$\{ g \in \mathbb{R}^{\mathbb{R}} : g(x_i) \in U_i, i=1, 2, \dots, k \}, \quad U_i \text{ is an open nbhd of } f(x_i) \text{ in } \mathbb{R} \}.$$

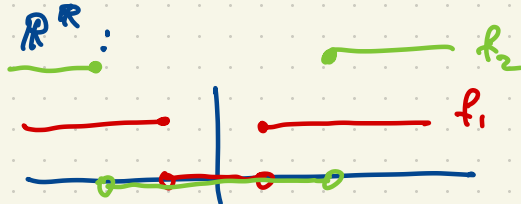
or specifically

$$\{ g \in \mathbb{R}^{\mathbb{R}} : |g(x_i) - f(x_i)| < \varepsilon_i, i=1, \dots, k \}$$

Varying $x_1, \dots, x_k, k, \varepsilon_1, \dots, \varepsilon_k$ we get a basis for the topology of $\mathbb{R}^{\mathbb{R}}$ in this way.

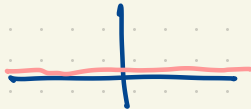
A convergent sequence of functions in $\mathbb{R}^{\mathbb{R}}$:

$$f_n(x) = \begin{cases} 0, & \text{if } |x| < n; \\ n, & \text{if } |x| \geq n. \end{cases}$$



$f_n \rightarrow 0$ i.e. for any basic open nbhd U of 0 , $f_n \in U$ for all $n \gg 0$.

$\underbrace{0}_{\text{zero function}}$



In usual language, $f_n \rightarrow 0$ pointwise meaning for all $x \in \mathbb{R}$, $f_n(x) \rightarrow 0$.

In the box topology, $f_n \not\rightarrow 0$.

$$\omega = \{0, 1, 2, 3, \dots\}$$

Take $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots = \{ \overbrace{(a_0, a_1, a_2, a_3, \dots)}^x : a_i \in \mathbb{R} \} = \mathbb{R}^\omega$ as a set (Cartesian product). Compare product topology, box topology, and topologies from a few norms including

$$\|x\|_1 = \sum_{i \in \omega} |a_i| = |a_0| + |a_1| + |a_2| + \dots$$

$$\|x\|_\infty = \sup |a_i|.$$

$$\|x\|_2 = \left(\sum |a_i|^2 \right)^{1/2}$$

$$l^1 = \{x \in \mathbb{R}^\omega : \|x\|_1 < \infty\}$$

$$l^2 = \{x \in \mathbb{R}^\omega : \|x\|_2 < \infty\}$$

$$l^\infty = \{x \in \mathbb{R}^\omega : \|x\|_\infty < \infty\}$$

$$x_1 = (1, 1, 1, 1, 1, 1, \dots)$$

$$x_2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots\right)$$

$$x_3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots\right)$$

$$\vdots$$

$$x_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots\right) \in \mathbb{R}^\omega$$

$x_n \rightarrow 0 = (0, 0, 0, \dots)$ in the product topology but not in the box topology

In the uniform norm topology, $x_n \rightarrow 0$ ($x_n \rightarrow 0$ in l^∞).

In the box topology, $\prod_{n=0}^{\infty} \left(-\frac{1}{n+1}, \frac{1}{n+1}\right)$ is a basic open nbhd of 0 and it contains no terms of the sequence $(x_n)_{n \in \omega}$.

Now consider

$$y_1 = (1, 0, 0, 0, 0, \dots)$$

$$y_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots)$$

$$y_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots)$$

etc.

$$y_n = (\underbrace{\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}}_n, 0, 0, \dots) \rightarrow 0$$

$$\|y_n\|_1 = 1 < \infty$$

$$\|y_n\|_2 = \frac{1}{\sqrt{n}} < \infty$$

$$\|y_n\|_\infty = \frac{1}{n} < \infty$$

in l^1, l^2, l^∞ product topology
but not in the box topology.

The box topology has $\prod_{n=0}^{\infty} (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$ as a basic open nbhd of 0 and it contains no term of the sequence of points $(y_n)_n$.

The product topology is sometimes called the topology of pointwise convergence.

The box topology is not usually as useful the other topologies.

A sequence f_n in \mathbb{R}^A converges uniformly to f if for all $\varepsilon > 0$ there exists N such that $|f_n(a) - f(a)| < \varepsilon$ whenever $n > N$ for all $a \in A$.

Basic open sets in the $\tilde{\text{uniform}}$ topology look like $U^A = \prod_{a \in A} U_a$, $U \subseteq \mathbb{R}$ is open.

(finer than the product topology but coarser than the box topology).

If $|A| < \infty$ then the product topology on $\prod_{a \in A} X_a$ agrees with the box topology.

If $|A| = |B|$ then products $\prod_{a \in A} X_a$ and

$\prod_{b \in B} Y_b$ are essentially the same.

(The order of the factors does not alter the definition of the product or box topology.)

$\mathbb{R}^{\mathbb{R}} \cong \mathbb{R}^{\mathbb{R} \times \mathbb{R}}$ in the product topology

The Cantor Space (as a topological space)

$$K_1 = [0, 1]$$

$$K_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$$

$$K_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

etc.

$C = \bigcap_{n=1}^{\infty} K_n$ is a compact top. space. $C \subset \mathbb{R}$ and we take the standard topology. It is a metric space.

It is totally disconnected: given $x \neq y$ in C , there exists a partition $C = U \sqcup V$, $U, V \subset C$ open, $x \in U, y \in V$.

Equivalently, $C = \{0, 1\}^{\omega} = 2^{\omega}$ with the product topology. ($\{0, 1\}$ is discrete)

Points of C have the form $(a_0, a_1, a_2, a_3, \dots)$ where $a_i \in \{0, 1\}$.

$$|C| = |\mathbb{R}| = 2^{\aleph_0}$$

A set of basic open nbhds of $a = (a_0, a_1, a_2, a_3, \dots) \in C$ is the set of $\{b \in C : b_i = a_i \text{ for } i \leq n\}$.

A metric defining this topology is

$$d(a, b) = \begin{cases} 0, & \text{if } a = b \\ \frac{1}{2^n}, & \text{if } a_n \neq b_n \text{ for some } n \text{ and we take the smallest such } n. \end{cases}$$

This is really $\mathbb{Z}_2 = 2\text{-adic integers}$
 $= \{a \in \mathbb{Q} : \|a\|_2 \leq 1\}$.

A homeomorphism

$\{0, 1\}^{\mathbb{N}} \rightarrow$ Usual Cantor set $\bigcap_{n=1}^{\infty} K_n$ is

$$(a_1, a_2, a_3, \dots) \mapsto \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

The Cantor Space is the unique compact Hausdorff space without isolated points which is second countable having a countable base of clopen sets.

Second countable: having a countable base.

Separable: having a countable dense subset.

$A \subseteq X$ is dense if $A \cap U \neq \emptyset$ for every open $U \neq \emptyset$.

Tychonoff's Theorem A product of compact spaces is compact.

That is, if X_α ($\alpha \in A$) is an indexed family of compact spaces, then

$\prod_{\alpha \in A} X_\alpha$ is compact.

(NB: We are using the product topology here.)

NB means "take note"

eg. $[0, 1]^{\omega}$ is compact in the product topology.

Not in the box topology eg. for every $a \in \{0, 1\}^{\omega}$ i.e. $a = (a_0, a_1, a_2, \dots)$ $a_i \in \{0, 1\}$

the sets $U_a = \prod_{i \in \omega} U_{a(i)}$

$U_0 = [0, \frac{2}{3})$, $U_1 = (\frac{1}{3}, 1]$ open in $[0, 1]$

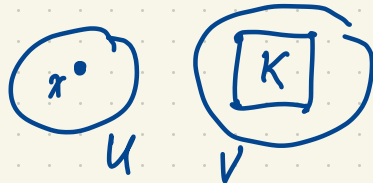
including $U_{(0, 0, 1, 1, 1, 0, 1, 0, 0, \dots)} = U_0 \times U_1 \times U_0 \times U_1 \times U_1 \times \dots$

covers $[0, 1]^{\omega}$. No finite number of these U_a 's cover $[0, 1]^{\omega}$.

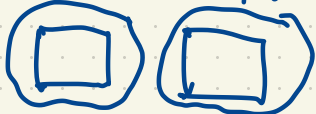
X is Hausdorff if for all $x \neq y$ in X , there are disjoint open nbhds of x and y .

X is regular if for every closed

set K and every point $x \in K$, there are open sets U, V with $U \cap V = \emptyset$, $x \in U$, $K \subseteq V$.



X is normal if



Warning normal spaces are not necessarily regular (unless points are closed)

Eg. $X = \{0, 1\}$

Open sets: $\emptyset, \{0\}, X$

Closed sets: $\emptyset, \{1\}, X.$

This space is normal. It's not regular.



Urysohn's Lemma X is a normal top. space iff for every pair of disjoint closed sets, K, L , there exists a continuous function $X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in K$; $f(x) = 1$ for all $x \in L$.

Metric spaces are Hausdorff, normal and regular.

In any metric space (X, d) , $d: X \times X \rightarrow [0, \infty)$ is continuous.

If $A \subseteq X$, we can define distance from $x \in X$ to A : $d(x, A) = \inf_{a \in A} d(x, a)$.

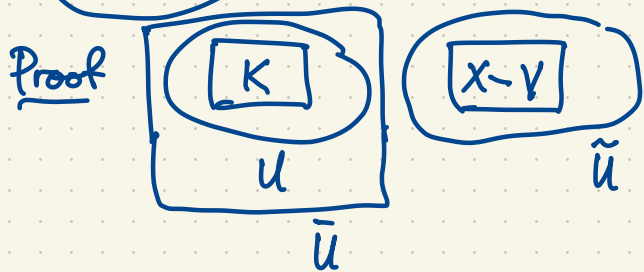
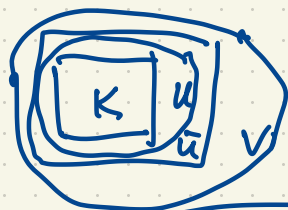
This is a continuous map $X \rightarrow [0, \infty)$. $d(x, A) = 0$ iff $x \in \bar{A}$ = closure of A .

$d(A, B) = \inf_{a \in A} d(a, B)$. If A, B are disjoint closed sets then $d(A, B) > 0$.

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

Wed Oct 19 } prerecorded lectures on Baire Category - see website (Lecture
Fri Oct 21 } videos + pdfs)

Lemma X is normal iff whenever $K \subseteq V$ with K closed and V open, there exists an open set U such that $K \subseteq U \subseteq \bar{U} \subseteq V$. (\bar{U} = closure of U = smallest closed set containing U).



$X - V = \{x \in X : x \notin V\}$
closed

Proof of Urysohn's Lemma (\Leftarrow) Suppose K, L disjoint closed sets in a space X and $f: X \rightarrow [0, 1]$ is continuous with $f|_K = 0$, $f|_L = 1$.

Let $U = f^{-1}([0, \frac{1}{3})) \subseteq X$ is open, $V = f^{-1}((\frac{2}{3}, 1]) \subseteq X$ is open.

$U \cap V = \emptyset$, $K \subseteq U$, $L \subseteq V$. So X is normal.

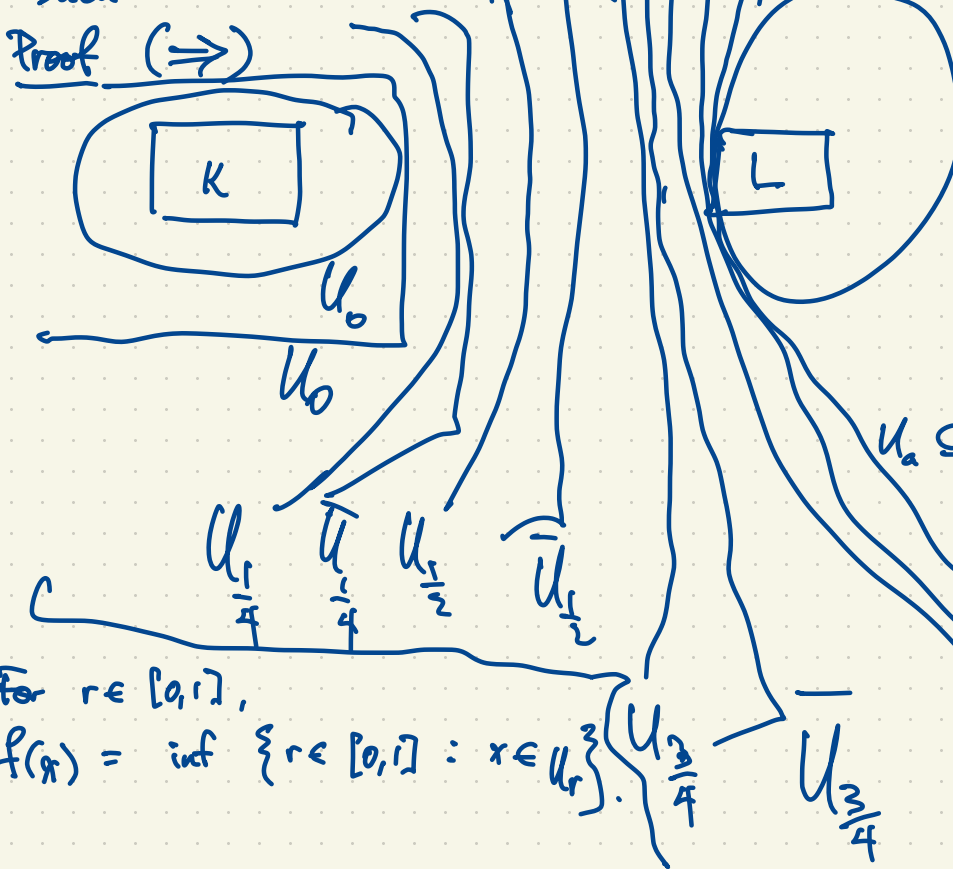
Urysohn's Lemma

disjoint closed sets,

such that $f(x) = 0$ for all $x \in K$,

$f(x) = 1$ for all $x \in L$.

Proof (\Rightarrow)



X is a normal top space iff for every pair of disjoint closed sets K, L there exists a continuous function $X \rightarrow [0, 1]$

We recursively use the Lemma to find an indexed collection of open sets U_a where a is any dyadic rational in $[0, 1]$ (dyadic rationals have the form $\frac{m}{2^k}$, $m, k \in \mathbb{Z}$) such that

$$U_a \subseteq \bar{U}_a \subseteq U_b \text{ whenever } 0 \leq a < b \leq 1$$

$$K \subseteq U_0, \quad \bar{U}_1 \cap L = \emptyset$$

For $r \in [0, 1]$,
 $f(x) = \inf \{ r \in [0, 1] : x \in U_r \}$



Question: If X is regular, i.e.

must there exist a continuous function $f: X \rightarrow [0,1]$ such that $f(x)=0$, $f|_K=1$?
No! There is no analogue of Urysohn's Lemma for regularity.

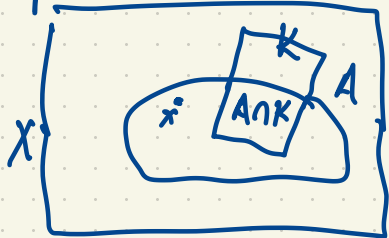
X is completely regular if whenever $\exists \boxed{K}$ K closed, $x \notin K$, there exist continuous $f: X \rightarrow [0,1]$, $f(x)=0$, $f|_K=1$.

There exist top. spaces which are regular but not completely regular (e.g. Tychonoff cork screw) but we will omit this.

X is completely normal if every subspace of X is normal.

Remarks: If X is completely regular then X is regular (easy) and every subspace of X is also completely regular.

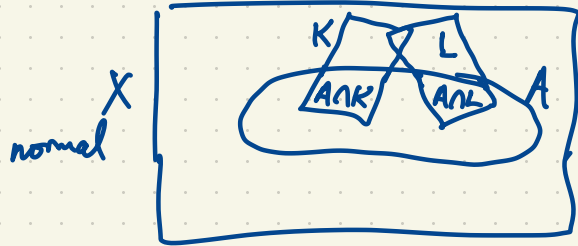
In X :



There exist continuous $f: X \rightarrow [0,1]$ such that $f(x)=0$, $f|_K=1$. Restricting f to $f|_A$, we see that A is also completely regular.

Is every subspace of a normal space normal?

No: see Tychonoff's Plank.



$$\omega = \{1, 2, 3, \dots\} \text{ discrete}$$

$$= \{0, 1, 2, 3, \dots\}$$

$$\omega+1 = \{0, 1, 2, \dots\} \cup \{\omega\}$$

$$= \{1, 2, 3, \dots\} \text{ in which } \{\omega\} \text{ is not open}$$

$$\omega^2 = \{1, 2, 3, \dots\} = \omega + \omega$$

$$\omega^2 = \underbrace{\omega + \omega + \omega + \dots}_{\omega \text{ times}} = \{1, 2, 3, \dots\} \cdot \{1, 2, 3, \dots\} \dots \approx \{m^{-1/n} : m, n \text{ positive integers}\} \subset \mathbb{R}$$

$$(0, 0) < (0, 1) < (0, 2) < (0, 3) < \dots$$

$$0 \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4}$$

$$< \begin{matrix} 1 \\ (1, 0) \end{matrix} < \begin{matrix} 1\frac{1}{2} \\ (1, 1) \end{matrix} < \begin{matrix} 1\frac{2}{3} \\ (1, 2) \end{matrix} < \begin{matrix} 1\frac{3}{4} \\ (1, 3) \end{matrix} \dots$$

$$< \begin{matrix} 2 \\ (2, 0) \end{matrix} < \begin{matrix} 2\frac{1}{2} \\ (2, 1) \end{matrix} < \begin{matrix} 2\frac{2}{3} \\ (2, 2) \end{matrix} < \begin{matrix} 2\frac{3}{4} \\ (2, 3) \end{matrix} \dots$$

$$(m-1, n-1)$$



For $\{a_\alpha : \alpha \in A\}$ any indexed set of positive real numbers,

$$\sum \{a_\alpha : \alpha \in A\} = \sum_{\alpha \in A} a_\alpha = \sup \{a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_k} : \begin{array}{l} \text{distinct} \\ \alpha_1, \dots, \alpha_k \in A \\ k \geq 1 \end{array}\}$$

$$\sum \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\} = 2$$

But if A is uncountable and $a_\alpha > 0$ (positive reals) then $\sum_{\alpha \in A} a_\alpha = \infty$ (always diverges)!

Why? In other words, if $\sum_{\alpha \in A} a_\alpha < \infty$, why must A be countable?

$$\sum_{n, \alpha \in \mathbb{Z}} \frac{1}{(n^4 + \alpha^4 + 1)}$$

i.e. there exists M real
such that $a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_k} < M$
for all $k \geq 1$; $\alpha_1, \dots, \alpha_k \in A$ distinct.

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup \dots \quad \text{where}$$

$$\begin{aligned} A_1 &= \{\alpha \in A : a_\alpha \in [1, \infty)\} \\ A_2 &= \{\alpha \in A : a_\alpha \in [\frac{1}{2}, 1)\} \\ A_3 &= \{\alpha \in A : a_\alpha \in [\frac{1}{4}, \frac{1}{2})\} \text{ etc.} \end{aligned}$$

$$|A_n| < \infty \text{ for all } n$$

So A is a countable union of finite sets so it's countable.

Recall order topology on a totally ordered set $(X, <)$

Sub-basic open sets $\{x \in X : x < \beta\}$ for $\beta \in X$

$\{x \in X : x > \alpha\}$ for $\alpha \in X$

Basic open sets $(\alpha, \beta) = \{x \in X : \alpha < x < \beta\}$ including the sub-basic sets above

Ordinals: well-ordered sets; ordinals are the canonical examples of well-ordered sets.

Given an ordinal γ , $[0, \gamma] = \{\text{ordinals } \alpha \leq \gamma\}$.

In γ , every nonempty subset $A \subseteq [0, \gamma]$ has a least element $\inf A$.

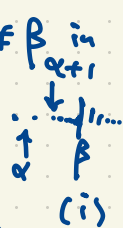
Also A has a supremum $\sup A$ which may or may not be an element of A .

($\sup A =$ least upper bound for A ; since γ is an upper bound for A , so $\{\text{upper bounds for } A\}$ is a nonempty subset of $[0, \gamma]$ so $\sup A$ exists by well-ordering).

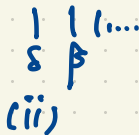
Moreover: $[0, \gamma]$ is compact Hausdorff. If $\alpha \neq \beta$ in $[0, \gamma]$, wlog $\alpha < \beta$ so

Case (i). β is a limit ordinal.

$\alpha \in [0, \alpha+1)$, $\beta \in (\alpha+1, \gamma]$.



(i)



(ii)

Case (ii) β is a successor ordinal, $\beta = \delta+1$, $\alpha \leq \delta$, $\alpha \in [0, \alpha+1)$, $\beta \in (\delta, \gamma]$.

Why is $[0, \gamma]$ compact?

(If γ is a limit ordinal then $\gamma = [0, \gamma)$ is not compact.)

Let \mathcal{O} be an open cover of $[0, \gamma]$.

A set $A \subseteq [0, \gamma]$ is finitely covered by \mathcal{O}

$$[0, \gamma] = \gamma \cup \{\gamma\}$$

if there exist $U_1, \dots, U_n \in \mathcal{O}$ such that $A \subseteq U_1 \cup U_2 \cup \dots \cup U_n$.

We must that $X = [0, \gamma]$ is finitely covered by \mathcal{O} .

Let $S = \{\alpha \in [0, \gamma] : [\alpha, \gamma] \text{ is finitely covered by } \mathcal{O}\}$.

$S \subseteq [0, \gamma]$ is nonempty, since $\gamma \in S$. So S has a least element m .

If $m = 0$ we're done. Otherwise $[m, \gamma] \subseteq U_1 \cup \dots \cup U_n$ for some $U_1, \dots, U_n \in \mathcal{O}$.

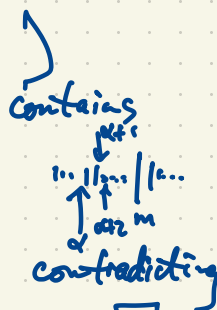
m must be a limit ordinal (if $m = \alpha + 1$ then some $U_0 \in \mathcal{O}$ covers α and

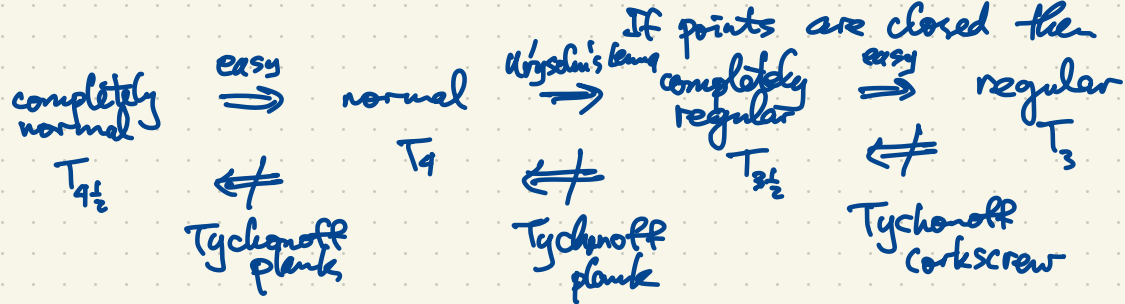
$[\alpha, \gamma] \subseteq U_0 \cup U_1 \cup \dots \cup U_n$, contradicting the minimality of $m \in S$.)

Now $m \in [0, \gamma]$ is a limit ordinal. $m \in U_0$ for some $U_0 \in \mathcal{O}$ and U_0 contains

some basic open nbhd of m i.e. $m \in (\alpha, \beta) \subseteq U_0$. Here $\alpha < m$

Now $U_0 \cup U_1 \cup U_2 \cup \dots \cup U_n$ covers $[\alpha + 1, \gamma]$. Again $\alpha + 1 \in S$, $\alpha + 1 < m$ contradicting the minimality of $m \in S$.



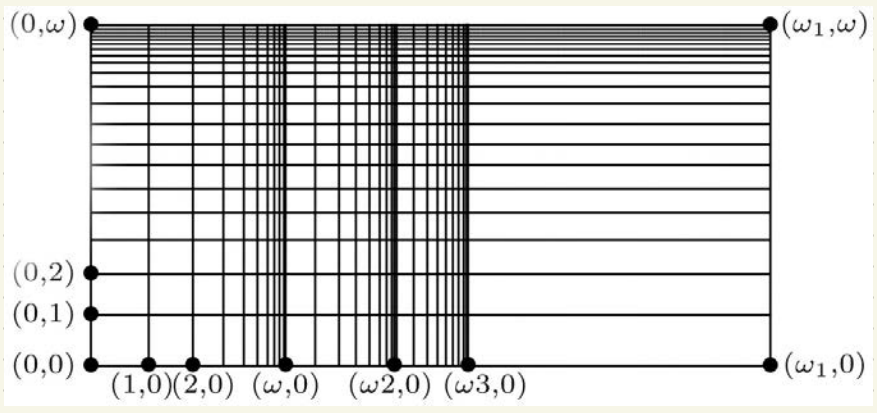


Prop 1.1 on Separation Properties

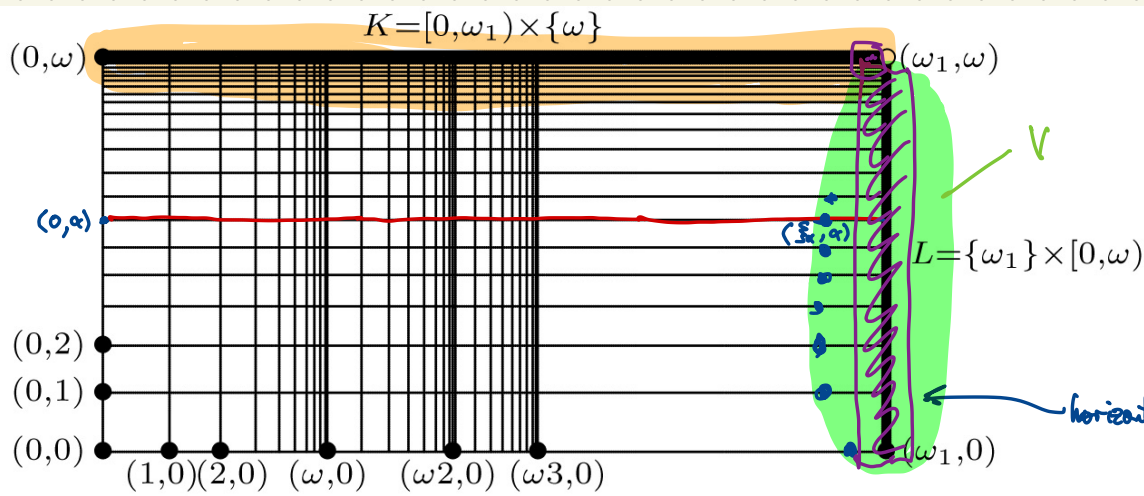
Tychonoff plank $X = [0, w_1] \times [0, w_2]$.

is compact Hausdorff. X is normal. But X is not completely normal i.e. it has a subspace that is not normal. Delete (w_1, w_2)

Warning: (α, β) can be either an interval or a point.



Deleted Tychonoff Plank $X = ([0, \omega_1] \times [0, \omega]) - \{(\omega_1, \omega)\} \subset$ Tychonoff Plank : X is not normal.



K = top edge
 L = right edge

$K, L \subset X$ closed subsets

$K \cap L = \emptyset$

(Claim: We can't find open sets $U, V \subset X$, $K \subset U$, $L \subset V$, $U \cap V = \emptyset$)

Suppose $V \subset X$ open, $L \subset V$. For each $\alpha \in \omega$, there is a minimal $\xi_\alpha \in [0, \omega_1)$ such that $[\xi_\alpha, \omega_1] \times \{\alpha\} \subset V$.
 ξ_α is countable (since $\xi_\alpha < \omega_1$). So $\xi = \sup_{\alpha < \omega} \xi_\alpha = \bigcup_{\alpha < \omega} \xi_\alpha$ is countable. So $[\xi, \omega_1] \times [0, \omega] \subset V$.

Now $K \subset U \subset X$. U contains points $[\xi, \alpha]$ for α sufficiently close to ω . (α large positive integers).

Then $[\xi, \omega] \in U \cap V$, contradiction.

what's next?

- Ultrafilters
- Nonstandard Reals
- Proof of Tychonoff's Theorem
- Construction of the Stone-Čech Compactification

But first let's review the real numbers.

One way to construct \mathbb{R} from \mathbb{Q} is via Dedekind cuts

Better: completion via Cauchy sequences.

Start with the set of all Cauchy sequences in \mathbb{Q} .

Introduce an equivalence relation \sim on X

$$X = \{x \in \mathbb{Q}^{\omega} : x \text{ is Cauchy}\}$$

$x = (x_0, x_1, x_2, x_3, \dots)$, $x_i \in \mathbb{Q}$
For all $\varepsilon > 0$ there exists N such
that $|x_i - x_j| < \varepsilon$ whenever $i, j > N$.

Here $|x_i - x_j|$ can be replaced by $d(x_i, x_j)$
in any metric space so the same notion
leads to a completion of a general
metric space.