Point Set Topology

Book 1

Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X (called the open sets) such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.

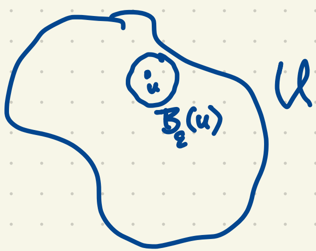
if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

(So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

Example

The standard topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that



$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{ x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon \}$.

$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$$

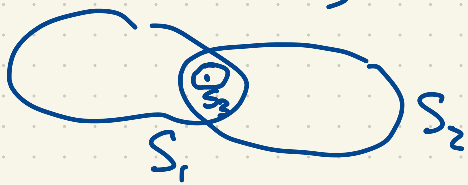
(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let $\mathcal{S}$ be a collection of subsets of X which cover X, i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \dots, S_k \in \mathcal{S}$ is a topology on X. The members of $\mathcal{S}$ are called a sub-basis for this topology and the topology is said to be <u>generated</u> by $\mathcal{S}$.

$\mathcal{S}$ is called a <u>base</u> (or a <u>basis</u>) for the topology if the topology is the collection of arbitrary unions of elements of $\mathcal{S}$. This holds iff for all $S_1, S_2 \in \mathcal{S}$, and all $u \in S_1 \cap S_2$, there exists $S_3 \in \mathcal{S}$ such that

$$u \in S_3 \subseteq S_1 \cap S_2 .$$



<u>Eg.</u> Let X be any set. The <u>discrete topology</u> on X is the collection of <u>all</u> subsets of X. (2^X)
The <u>indiscrete</u> topology on X is $\{\emptyset, X\}$.

If $X = \{0, 1\}$ then there are four possible topologies on X: $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let $\mathcal{T}$ be the collection of complements of finite sets, and $\emptyset$

i.e. $\mathcal{T} = \{\emptyset\} \cup \{X \setminus A : A \subseteq X, |A| < \infty\}$, $X \setminus A = \{x \in X : x \notin A\}$.

set difference

This is a topology on X, called the <u>finite complement topology</u>.

$X - A$, $X \setminus A$, $X \backslash A$

$\emptyset$, ϕ, $\varnothing$, 0

\varnothing

A <u>topological space</u> is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on a set X.

Note: $\bigcup \mathcal{T} = X$. By abuse of language, we often say that X is a topological space.

Let X be a set. A <u>distance function</u> (or <u>metric</u>) on X is a function $d : X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

$$d(x,y) = d(y,x)$$

$d(x,y) \geq 0$ and equality holds iff $x = y$.

$$d(x,z) \leq d(x,y) + d(y,z)$$

The standard topology on $\mathbb{R}^n$ is a metric topology.

The metric $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ (the Euclidean metric)

$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$ all give the standard topology on $\mathbb{R}^n$.

In $\mathbb{R}^2$, open balls with respect to d_2, d_1, d_∞ look like

respectively.

The metric $d(x,y) := \begin{cases} 0, & \text{if } x=y \\ 1, & \text{if } x\neq y \end{cases}$ defines the discrete topology.

A topological space is <u>metrizable</u> if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.