Point Set Topology

Book 1

Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X (called the open sets) such that

  (i) $\emptyset, X \in \mathcal{T}$

  (ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.
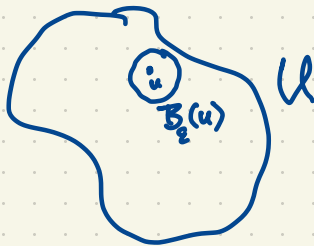
     if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

     if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

     (So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

Example

The standard topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{x \in \mathbb{R}^n : d(x, u) < \varepsilon\}$.
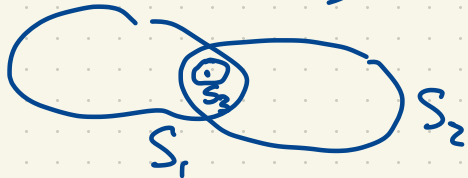
Euclidean distance

$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$

(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let S be a collection of subsets of X which cover X, i.e. $\cup S = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \ldots, S_k \in S$ is a topology on X. The members of S are called a sub-basis for this topology and the topology is said to be $\underline{\text{generated}}$ by S.

S is called a $\underline{\text{base}}$ (or a $\underline{\text{basis}}$) for the topology if the topology is the collection of arbitrary unions of elements of S. This holds iff for all $S_1, S_2 \in S$, and all $u \in S_1 \cap S_2$, there exists $S_3 \in S$ such that $u \in S_3 \subseteq S_1 \cap S_2$.

S_1 S_2

Eg. Let X be any set. The discrete $\underline{\text{topology}}$ on X is the collection of all $\underline{\text{all}}$ subsets of X. (2^X)
The $\underline{\text{indiscrete}}$ topology on X is $\{\emptyset, X\}$.

If $X = \{0, 1\}$ then there are four possible topologies on X: $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let $\mathcal{J}$ be the collection of complements of finite sets, and $\emptyset$

i.e. $\mathcal{J} = \{\emptyset\} \cup \{X \sim A : A \subseteq X, |A| < \infty\}$, $X \sim A = \{x \in X : x \notin A\}$.

set difference

This is a topology on X, called the
finite complement topology.

$X - A, \quad X \sim A, \quad X \backslash A$

A topological space is a pair $(X, \mathcal{J})$ where

$\emptyset, \varnothing, \emptyset, 0$

$\mathcal{J}$ is a topology on a set X.

\varnothing

Note: $\bigcup \mathcal{J} = X$. By abuse of language, we often say that X is a topological
space.

Let X be a set. A distance function (or metric) on X is a function

$d : X \times X \longrightarrow [0, \infty]$ such that for all $x, y, z \in X$,

$d(x, y) = d(y, x)$

$d(x, y) \geq 0$ and equality holds iff $x = y$.

$d(x, z) \leq d(x, y) + d(y, z)$

The standard topology on $\mathbb{R}^n$ is a metric topology.

The metric $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ (the Euclidean metric)

$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$d_\infty(x, y) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$ all give the standard topology on $\mathbb{R}^n$.

In $\mathbb{R}^2$, open balls with respect to d_2, d_1, d_∞ look like

 respectively.

These three metrics define the same topology.

The metric $d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ defines the discrete topology.

A topological space is <u>metrizable</u> if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.

A topology is <u>Hausdorff</u> if for any two points $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Every metric space is Hausdorff since if $x \neq y$, $d = d(x,y) > 0$. Take $U = B_{\delta/3}(x)$, $V = B_{\delta/3}(y)$

An open <u>neighbourhood</u> of a point $x \in X$ is an open set containing x.

A <u>basic</u> <u>open</u> nbhd of a point $x \in X$ is an open nbhd of x which is basic (i.e. it's in the basis).

Even metric spaces can be rather surprising.

Consider $X = \mathbb{Q}$. A <u>norm</u> on $\mathbb{Q}$ is a function $\mathbb{Q} \to [0, \infty)$,
$$x \mapsto \|x\| \quad \text{satisfying}$$

(i) $\|x\| \geq 0$; equality holds iff $x = 0$.

(ii) $\|xy\| = \|x\| \cdot \|y\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

From any norm on $\mathbb{Q}$, we obtain a metric $d(x,y) = \|x - y\|$.

One way to do this is with the usual absolute value $\|x\| = |x| = |x|_\infty = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0 \end{cases}$

This gives the standard topology on $\mathbb{Q}$.

An alternative is : given $x \in \mathbb{Q}$, if $x = 0$ define $\|0\|_2 = 0$.

If $x \neq 0$, write $x = 2^k \cdot \frac{a}{b}$, $a, b, k \in \mathbb{Z}$, $b \neq 0$; a, b odd. Then define $\|x\|_2 = 2^{-k}$.

This is the 2-adic norm on $\mathbb{Q}$. In fact it satisfies a stronger form of (iii), the ultrametric inequality $\|x + y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$.

E.g. $\left\|\frac{20}{21} + \frac{5}{14}\right\|_2 = \left\|\frac{40+15}{42}\right\|_2 = \left\|\frac{55}{42}\right\|_2 = 2 . = \max\left\{\left\|\frac{20}{21}\right\|_2, \left\|\frac{5}{14}\right\|_2\right\} = 2$

$$\underbrace{}_{\frac{1}{4}} \qquad \underbrace{}_{2}$$

$\left\|\frac{20}{21}\right\|_2 = \frac{1}{4}$, $\left\|\frac{5}{14}\right\|_2 = 2$
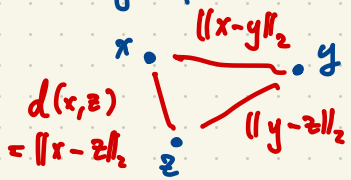
Compare: $\left\|\frac{20}{21}\right\|_2 + \left\|\frac{5}{14}\right\|_2 = 2\frac{1}{4} = 2.25$.
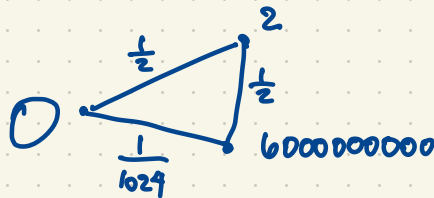
A basic open nbhd of zero looks like

$\qquad B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$\qquad B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \{\frac{a}{b} : a, b \in \mathbb{Z}, \quad a \text{ even}, b \text{ odd}\}$.

Every point in the ball is a centre of the ball i.e. if $c \in B_1(0)$ then $B_1(c) = B_1(0)$.

$x \bullet \overset{\|x-y\|_2}{\underset{}{}} \bullet y$

$d(x,z)$
$= \|x - z\|_2 \quad \bullet z \quad \|y-z\|_2$

Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.

$O \qquad \frac{1}{2} \qquad 2 \qquad \frac{1}{2}$

$\frac{1}{1024} \qquad 6000000000$

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots = -1$$

The partial sums $1, 3, 7, 15, 31, 63, \ldots$ converge to -1 in the 2-adic norm.

Note: If $(x_n)_n$ is a sequence of points in a top. space X, we say $(x_n)_n$ converges to $x \in X$ if for every open nbhd U of x, $x_n \in U$ for all n sufficiently large. (This means: for all U open nbhd of x, the exists N such that $x_n \in U$ whenever $n > N$.)

In place of arbitrary open nbhds of x, it suffices to check basic open nbhds. For metric topology, it suffices to check open balls. In this case, $x_n \to x$ provided that for all $\varepsilon > 0$, there exists N such that

$$\left.\begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array}\right\} \text{ whenever } n > N.$$

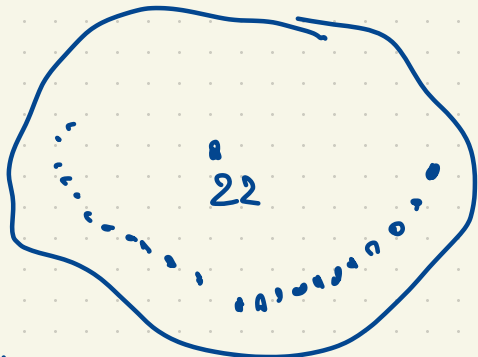In our example above, $d(x_n, x) = 2^n \to 0$ as $n \to \infty$.

$$\|2^n\| = \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Find the inverse of 5 mod 64.

In $\mathbb{Z}/64\mathbb{Z}$, $\quad \frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - 64 + 256 - 1024 + \cdots$

$$= 1 - 4 + 16 \underbrace{\quad\quad}_{\text{zero}}$$

$$= 13.$$

Eg. in $\mathbb{Z}$ with the finite complement topology, the sequence $(n)_n = (1, 2, 3, \cdots)$ converges. It converges to 22.

$$(n)_n \longrightarrow 22.$$

In fact for every $a \in \mathbb{Z}$,

$$(n)_n \longrightarrow a.$$

1, 13, 25, 84

•1

•13

•25

•84

then pick $n > \max\{N_1, N_2\}$ to obtain a contradiction.

Theorem If X is Hausdorff, then every sequence in X has at most one limit. (it converges to at most one point.)

Proof Suppose $a \neq b$ in a Hausdorff space X where a sequence $(x_n)_n \to a$ and $(x_n)_n \to b$. Choose disjoint open nbhds U, V of a, b respectively.

$\textcircled{a}$ $\textcircled{b}$
U V

There exists N_1 such that $x_n \in U$ for all $n > N_1$; also N_2 such that $x_n \in V$ for all $n > N_2$.

We prefer to write $(x_n)_n \longrightarrow a$ rather than $\lim_{n \to \infty} x_n = a$ in general.

In any top. space, closed sets are the complements of open sets.

$\emptyset, X$ are closed

If K, K' are closed then $K \cup K'$ is closed. (So finite unions of closed sets are closed.)

Arbitrary intersections of closed sets are closed.

De Morgan laws:
$$X \smallsetminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \smallsetminus A_\alpha)$$

$$X \smallsetminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \smallsetminus A_\alpha)$$

Given an infinite set X, the finite complement topology has as its closed sets the finite sets and X itself.

Let X be a top. space. Given $A \subseteq X$, the closure of A is the (unique) smallest closed set containing A i.e. $\overline{A} = \bigcap \{ K \subseteq X : K \text{ closed}, K \supseteq A \}$.

The interior of A is the largest open set contained in A, i.e.
$$A^\circ = \bigcup \{ U \subseteq A : U \text{ open in } X \}. \qquad (X \smallsetminus A)^\circ = X \smallsetminus \overline{A} \; ; \quad \overline{X \smallsetminus A} = X \smallsetminus A^\circ .$$

<u>Theorem</u> There are infinitely many primes.

Known proofs :   Euclid's proof (elementary)

   Euler's proof (analytic proof : $\sum \frac{1}{p}$ diverges)

This proof is topological.

<u>Proof</u> Form a topology on $X = \mathbb{Z}$ whose basic open sets are the (infinite) arithmetic progressions

 ..., $-6, -1, 4, 9, 14, 19, ...$ for example.

 Every nonempty open set is infinite.

Suppose there are only finitely many primes : $|P| < \infty$ is the set of all primes.

$\{-1, 1\} = \{a \in \mathbb{Z} : a$ is not divisible by any prime$\}$.

$\quad = \bigcap_{p \in P} \{a \in \mathbb{Z} : a$ is not divisible by $p\}$

$\quad = \bigcap_{p \in P} \left(U_{1,p} \cup U_{2,p} \cup \cdots \cup U_{p-1, p} \right)$ $U_{a,p} = \{x \in \mathbb{Z} : x \equiv a \mod p\}$

 is open.   However it has only 2 elements, a contradiction. $\square$

More generally, let G be a group. Consider the ~~topology~~
on G whose basic open sets are cosets of subgroups $H \leqslant G$ of finite
  index, i.e. $gH = \{gh : h \in H\}$, $[G : H] < \infty$.

T_2 :   Hausdorff

T_1 :   Points are closed

If $x \in X$ and $y \neq x$, then there is an open nbhd U of x with $y \notin U$.

$T_2 \Rightarrow T_1$.   Exercise: Give an example of a top. space which is T_1 but not T_2.

One answer: the finite complement topology for an infinite set.

Let $f : X \longrightarrow Y$ be any function. For any $B \subseteq Y$, the preimage of B in X under f is $f^{-1}(B) = \{x \in X : f(x) \in B\}$.   Similarly if $A \subseteq X$, the image of A in Y is $f(A) = \{f(a) : a \in A\}$.   In general

$$f(f^{-1}(A)) \subseteq A \subseteq f^{-1}(f(A)).$$

Now let X and Y be top. spaces, ie. $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$.

A function $f : X \longrightarrow Y$ is continuous if the preimage of every open set (in Y) is open (in X); ie. for every $U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

Exercise: Convince yourself that the "standard" definition of continuity for functions $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ is just a special case of this. (For the standard topologies on $\mathbb{R}^m$ and $\mathbb{R}^n$).

Theorem If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $g \circ f: X \to Z$.

Proof If $U \subseteq Z$ is open then $g^{-1}(U) \subseteq Y$ is open so $f^{-1}(g^{-1}(U)) \subseteq X$ is open. $\quad \square$

$$\underbrace{\quad}_{(g \circ f)^{-1}(U)}$$

When are two topological spaces X, Y "the same"? ($X \cong Y$: X, Y are homeomorphic) This means there is a bijection $X \to Y$ taking one topology to the other. I.e. there is a bijection $f: X \to Y$ such that f, f^{-1} are continuous.

Eg. X is $\mathbb{R}$ with the standard topology;
$\quad Y$ is $\mathbb{R}$ with the finite complement topology;
$\quad Z$ is $\mathbb{R}$ with the discrete topology;
$\quad W$ is $\mathbb{R}$ with the indiscrete topology $\{\emptyset, \mathbb{R}\}$.

$$Z \xrightarrow{\iota} X \xrightarrow{\iota} Y \xrightarrow{\iota} W \qquad \text{where } \iota(x) = x.$$

$\quad\;\; \wedge \qquad\qquad\qquad \wedge$
finest $\qquad\quad$ coarsest
topology $\qquad\;$ topology
on $\mathbb{R}$ $\qquad\qquad$ on $\mathbb{R}$

If $\mathcal{J}, \mathcal{J}'$ are two topologies on X, we say $\mathcal{J}'$ is <u>finer</u> than $\mathcal{J}$ if $\mathcal{J}' \supset \mathcal{J}$
($\mathcal{J}'$ is a <u>refinement</u> of $\mathcal{J}$)
$\boxed{\;\mathcal{J}' \text{ is coarser than } \mathcal{J} \text{ if } \mathcal{J}' \subset \mathcal{J}.}$

Eg. The finite complement topology on X is the coarsest topology for which points are closed.

i.e. any topology in which points are closed is a refinement of the finite complement topology.

Subspace Topology

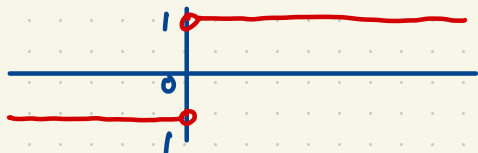Let $A \subseteq X$ where X is a topological space $X = (X, \mathcal{T})$.

The topology A inherits from X in the most natural way is the subspace topology $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$.

Eg. $[0,1) = \{a \in \mathbb{R} : 0 \leq a < 1\}$ is neither open nor closed in $\mathbb{R}$ but it is closed in $[0,1]$ and in $[0,\infty)$ since

$$[0,1) = (-1,1) \cap [0,1] = (-1,1) \cap [0,\infty).$$

If $f: A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$, we say f is continuous if it is continuous relative to the standard topology of $\mathbb{R}^m$ and the subspace topology on $A \subseteq \mathbb{R}^n$.

Eg.

continuous

$f: (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$
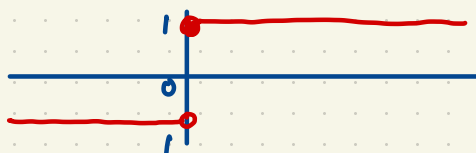
not continuous

$f: \mathbb{R} \to \mathbb{R}$

If $f: A \to \mathbb{R}^m$ has $f(A) \subseteq B$ we might as well think of f as $f: A \to B$. To say $f: A \to \mathbb{R}^m$ is continuous is equivalent to saying $f: A \to B$ is continuous.

Suppose $f: A \to B$ is continuous and let $U \subseteq \mathbb{R}^m$. Then $f^{-1}(U) = f^{-1}(U \cap B)$ is open in A. Similarly one proves the converse.

Given $A \subseteq X$ where X is a top. space, there is an inclusion map $\iota: A \to X$ $\iota(a) = a$. (one-to-one; not onto in general). The subspace topology on A is the coarsest topology for which the inclusion map ι is continuous.
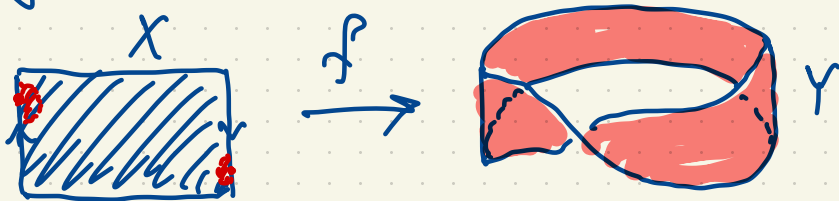
Given $U \subseteq X$ open. $i^{-1}(U) = U \cap A$.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Quotient Topology   Suppose $f: X \longrightarrow Y$ is onto.   Given a topology on $X = (X, \mathcal{T})$,   the most natural way this gives a topology on Y is by taking the finest topology on Y for which f is continuous.

A Möbius strip

There are three ways to think of this situation.
(i) Identify (collapse) certain points of X together.
(ii) We have an equivalence relation on X.
(iii) A partition of X.

The quotient topology on Y is the finest topology on Y for which the map $f: X \longrightarrow Y$ is continuous.

The quotient topology on $Y = X/f$ or $X/\sim$ is $\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$.
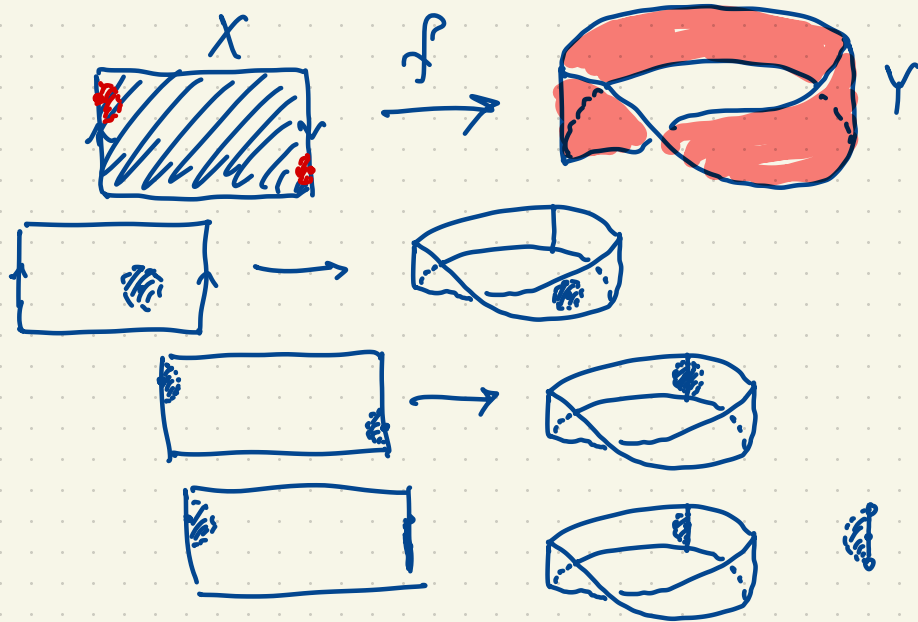

X   f   Y

To show this is a topology, use

$$\bigcup_\alpha f(A_\alpha) = f(\bigcup_\alpha A_\alpha)$$

$$\bigcap_\alpha f(A_\alpha) \supseteq f(\bigcap_\alpha A_\alpha)$$

$$\bigcup_\alpha f^{-1}(A_\alpha) = f^{-1}(\bigcup_\alpha A_\alpha)$$

$$\bigcap_\alpha f^{-1}(A_\alpha) = f^{-1}(\bigcap_\alpha A_\alpha)$$

$$\sin((-\infty,0)) = [-1,1]$$
$$\sin((0,\infty)) = [-1,1]$$

$$\sin((-\infty,0) \cap (0,\infty)) = \sin \emptyset = \emptyset$$
$$\sin((-\infty,0) \cup (0,\infty)) = [-1,1]$$

(closed) annulus

closed disk

Möbius strip

torus

Klein bottle
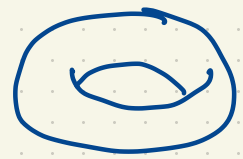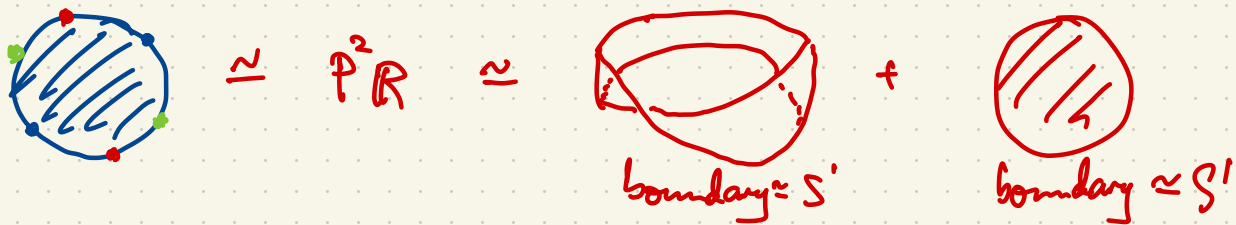not embeddable in $\mathbb{R}^3$

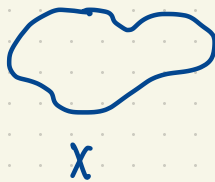$\mathbb{P}^2\mathbb{R}$ = real projective plane

identify all boundary points $\longrightarrow$ S^2 (2-sphere)

No two of the examples listed here are homeomorphic.

$$\simeq \quad P^2 R \quad \simeq$$

boundary $\simeq S'$ +   boundary $\simeq S'$

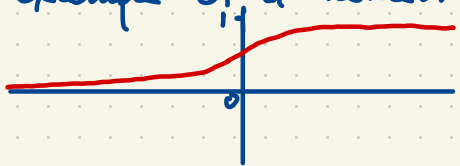In $\mathbb{R}^3$ consider the following two subspaces :

X

Y

Is $X \simeq Y$?   Yes.

$S^n = n\text{-sphere} \simeq \text{unit sphere in } \mathbb{R}^{n+1} = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : \Sigma x_i^2 = 1\}.$

$S^0 = \bullet \ \bullet \qquad S' = \bigcirc \qquad S^2 = \bigcirc \qquad S^3 = \mathbb{R}^3 \cup \{\infty\}$

$\mathbb{R} \simeq (0,1) \simeq (a,b) \overset{\simeq \ (0,\infty)}{\text{for}} \ a < b$

(open interval)

An example of a homeomorphism $f: \mathbb{R} \to (0,\infty)$ is $f(x) = \dfrac{e^x}{1+e^x}$.

$\mathbb{R} \simeq (0,1) \not\simeq \begin{cases} [0,1] \\ [0,1) \end{cases}$ Why is $(0,1) \not\simeq [0,1)$?

If we remove any point of $(0,1)$, what's left is disconnected. This is not true in $[0,1)$ which has a point 0 whose removal leaves a connected set $(0,1)$. $(0,\frac{1}{2}) \cup (\frac{1}{2},1)$ is disconnected since it is a disjoint union of two open sets.

<u>Def</u>. A top. space X is <u>disconnected</u> if $X = U \sqcup V$ where U, V are nonempty open sets in X. If X is not disconnected, then it is connected. In other words, X is connected iff its only clopen sets are $\emptyset$ and X ("clopen" means both open and closed).

$[0,1]$ is connected. This is a theorem in analysis.

Outline of argument: Suppose $[0,1] = U \sqcup V$ where U, V are nonempty open. $0 \in U$ without loss of generality. So $[0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$. How large can ε be?

$$\{ \varepsilon : [0, \varepsilon) \subseteq U \} \text{ is a nonempty set with upper bound } 1.$$

So there is a least upper bound. (supremum)

Is this supremum in U or in V? Either way leads to a contradiction.

If we remove any point from $(0,1) \cong \mathbb{R} \sqcup \mathbb{R}$, we get a subspace $\cong \mathbb{R} \sqcup \mathbb{R}$ which is disconnected.

This is not true for $[0,1)$.

$\mathbb{Q}$ is disconnected (in the standard topology in $\mathbb{R}$)

$\mathbb{Q} = U \sqcup V$ where $U = \{ x \in \mathbb{Q} : \ldots \ x < \sqrt{2} \} = \mathbb{Q} \cap (-\infty, \sqrt{2})$

$\qquad\qquad\qquad\qquad\quad V = \{ x \in \mathbb{Q} : \ldots \ x > \sqrt{2} \} = \mathbb{Q} \cap (\sqrt{2}, \infty)$

$\mathbb{Q}$ is totally disconnected:

An interval in $\mathbb{R}$ is the same thing as a connected subset of $\mathbb{R}$.

Theorem $\mathbb{R}$ is connected.

We'll talk about the foundations of $\mathbb{R}$ a little later, including completeness.

Theorem A continuous image of a connected space is connected.

In other words if $f: X \longrightarrow Y$ is $\{$surjective (and continuous) and X is connected, then Y is connected.

Proof Suppose $Y = U \sqcup V$ where $U, V \subseteq Y$ are open. Then $X = f^{-1}(U) \sqcup f^{-1}(V)$ where $f^{-1}(U), f^{-1}(V)$ are open in X. So one of these, say $f^{-1}(U)$, is empty. So $U = \emptyset$. This means Y is connected. $\square$

In a video I sent you, we showed $\mathbb{R}$ is connected.

Corollary $[0,1]$ is connected. Define $g: \mathbb{R} \longrightarrow [0,1]$

which is a continuous surjection. $\square$

Definition A path from x to y in X is a continuous function $\gamma: [0,1] \longrightarrow X$ such that $\gamma(0) = x$, $\gamma(1) = y$.

X is path-connected if for any $x, y \in X$, there is a path from x to y in X.

Theorem If X is path-connected then X is connected.

Proof Suppose $X = U \sqcup V$ where $U, V \subseteq X$ are nonempty open. Let $x \in U$, $y \in V$. If X is path-connected, there is a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$. Then

$$[0,1] = \gamma^{-1}(U) \sqcup \gamma^{-1}(V) = \gamma^{-1}(X), \quad \text{a contradiction since } [0,1] \text{ is}$$

connected and $\gamma^{-1}(U)$, $\gamma^{-1}(V)$ are disjoint nonempty open. $\qquad \square$.

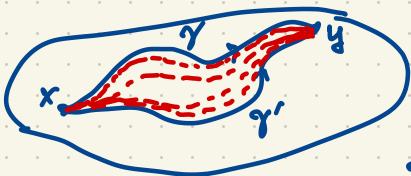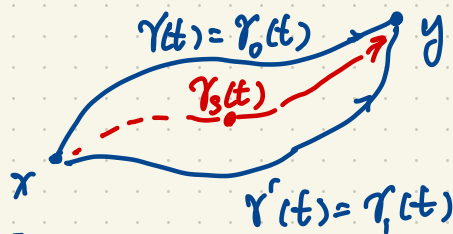The converse of the theorem is false. An example of a space that is connected but not path-connected :

$$X \subset \mathbb{R}^2$$

$$X = \left\{ (x, \sin\tfrac{1}{x}) : x \neq 0 \right\} \cup \left(\{0\} \times [-1, 1] \right)$$

$\underline{\text{interval on } y\text{-axis}}$

Details : See Munkres.

Let γ, γ' be two paths in X from x to y i.e. $\gamma, \gamma' : [0,1] \to X$, $\gamma(0) = \gamma'(0) = x$, $\gamma(1) = \gamma'(1) = y$.

Then γ, γ' are $\underline{\text{homotopic}}$ if

there is a $\overset{\text{continuous}}{\text{map}}$ $[0,1] \times [0,1] \longrightarrow X$

$$(s,t) \longmapsto \gamma_s(t)$$

$\gamma(t) = \gamma_0(t)$

$\gamma_s(t)$

x

y

$\gamma'(t) = \gamma_1(t)$

such that $\gamma_s(0) = x$, $\gamma_s(1) = y$ for all $s \in [0,1]$

$\left. \begin{array}{l} \gamma_0(t) = \gamma(t) \\ \gamma_1(t) = \gamma'(t) \end{array} \right\}$ for all $t \in [0,1]$.

We think of $\gamma_s(t)$ as a "continuous deformation" from $\gamma(t)$ to $\gamma'(t)$.

(homotopy)

x

X
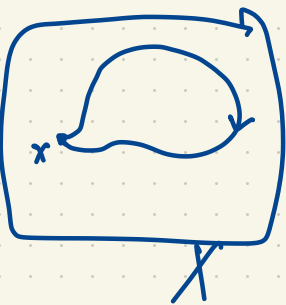
A <u>closed</u> <u>curve</u> based at $x \in X$ is a curve from x to x.

The <u>null</u> <u>curve</u> based at $x \in X$ is the curve

$$[0,1] \longrightarrow \{x\}.$$

If every closed curve in X is homotopic to a null curve, then X is simply connected.

Eg.

annulus

is connected but not simply connected. So this is not homeomorphic

to a closed disk .

Let $(x_n)_n$ be a sequence in X. We say $x_n \to x \in X$ if for every open nbhd U of x in X, beyond some point in the sequence all remaining terms are in U i.e. there exists N such that $x_n \in U$ whenever $n > N$. (We say $x_n \in U$ for all sufficiently large n, i.e. $x_n \in U$ whenever $n \gg 1$.)

$$(x_1, x_2, x_3, \ldots)$$

The full definition of $x_n \to x$ is:
For every open nbhd U of x in X, there exists N such that $x_n \in U$ whenever $n > N$.

$x_1 \quad x_2 \quad \ldots$

Theorem Let $f: X \to Y$ be continuous where X, Y are top. spaces. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.
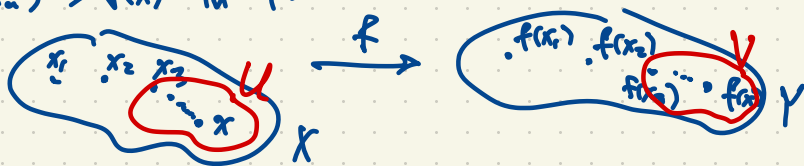
Proof Let V be an open nbhd of $f(x)$ in Y. Let $U = f^{-1}(V)$ which is open in X since f is continuous. Note that $x \in U$. There exists N such that $x_n \in U$ for all $n > N$. So $f(x_n) \in V$ for all $n > N$. $\square$

Is the converse true? Namely if $f: X \to Y$ maps convergent sequences
to convergent sequences, does this mean f is continuous?
In other words, suppose $f: X \to Y$ such that whenever $x_n \to x$ in X,
we have $f(x_n) \to f(x)$ in Y. Must f be continuous?
  Yes for metrizable spaces; no in general.
Metrizable spaces are <u>first countable</u>: there is a countable basis of open
nbhds at every point. Given $a \in X$ where X is a metric space,
  $$B_\varepsilon(a) = \{x \in X : d(x,a) < \varepsilon\}$$ is a collection of basic open nbhds at a.
There are uncountably many of these. The open nbhds $B_{\frac{1}{n}}(a)$ $(n=1,2,3,\dots)$
suffice for doing topology.
  $x_n \to x$ iff for all $m \geq 1$ there exists N such that $x_n \in B_{\frac{1}{m}}(x)$. for all $n > N$.
The balls $B_{\frac{1}{m}}(a)$, $a \in X$ generate all the open sets as a basis.
  $m \geq 1$

<u>First</u> countability of a top. space says that we have a countable collection
of basic open nbhds at each point (a <u>local</u> condition).
Metric spaces are first countable.
$\mathbb{R}^n$ has a stronger property: it is <u>second countable</u> meaning it has a countable
  basis for the entire topology $\{B_{\frac{1}{m}}(a) : a \in \mathbb{Q}^n\}$.
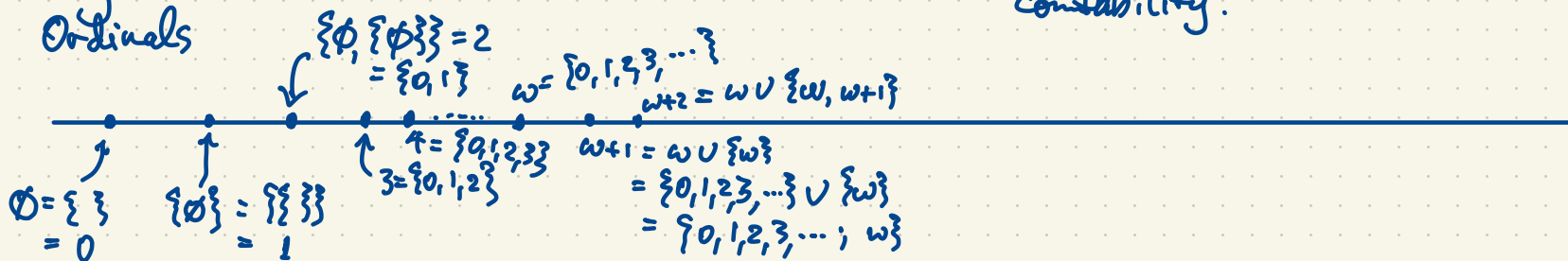
Theorem For first countable spaces, a function is continuous iff it maps convergent sequences to convergent sequences.

This is an inevitable result of the fact that sequences are inherently countable.

Remark: Second countability is strictly stronger than first countability.

Beyond countable:

Ordinals

$$\{\emptyset, \{\emptyset\}\} = 2$$
$$= \{0, 1\}$$

$$\omega = \{0, 1, 2, 3, \ldots\}$$

$$\omega + 2 = \omega \cup \{\omega, \omega+1\}$$

$$4 = \{0, 1, 2, 3\}$$
$$3 = \{0, 1, 2\}$$

$$\omega + 1 = \omega \cup \{\omega\}$$
$$= \{0, 1, 2, 3, \ldots\} \cup \{\omega\}$$
$$= \{0, 1, 2, 3, \ldots ; \omega\}$$

$$\emptyset = \{\ \} = 0 \qquad \{\emptyset\} = \{\{\ \}\} = 1$$

Recursive construction: Each ordinal is the set of all the smaller ordinals.

A ~~totally~~ ordered set $(S, <)$ is a set S with a binary relation '$<$' on S satisfying

- Given $x, y \in S$, exactly one of the statements $x < y$, $x = y$, $y < x$ is true ("trichotomy property"); i.e. $x > y$
- If $x < y < z$ then $x < z$ ("transitivity").

A well-ordered set is a totally ordered set in which every nonempty subset has a least element. Eg. for the usual order, $(\mathbb{N}, <)_{=\omega}$ is well-ordered; $(\mathbb{Z}, <)$ is not $[0, \infty)$ is not well-ordered.

Every well-ordered set is order-isomorphic to a unique ordinal. So the ordinals are the canonical representatives of the well-ordered sets.

Well-ordered sets are exactly the sets on which we can do induction.

Every set can be well-ordered (the well-ordering principle).

In ZFC = Zermelo-Fraenkel + Axiom of Choice, the Well-Ordering Principle is a theorem. So is Zorn's Lemma.

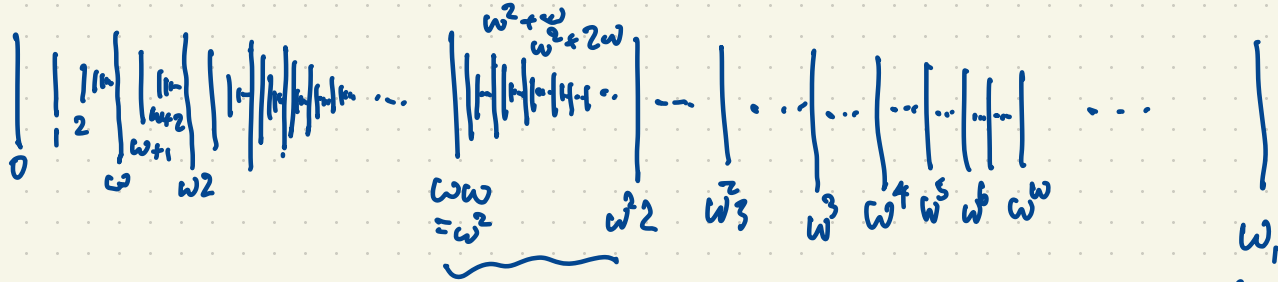In ZF, the following are equivalent:

- Axiom Choice
- Well-Ordering Principle
- Zorn's Lemma
- Transfinite Induction

If α and β are ordinals, then $\alpha + \beta = $

$$\boxed{} \overset{\cup}{\underset{<}{\cdots}} \boxed{}$$

copy of   copy of
$(\alpha, <)$ $(\beta, <)$

$\omega + 1 = $
$$\boxed{\begin{array}{c} \vdots \;\; \vdots \;\; \vdots \;\; \vdots \cdots \\ 0 \;\; 1 \;\; 2 \;\; 3 \end{array}} \quad \boxed{\cdot}$$

$1 + \omega = $ $\boxed{\cdot} \cup \boxed{\cdot \;\; \cdots \cdots} = \omega$
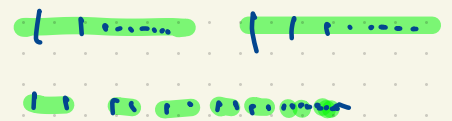
$$\omega 2 = \omega + \omega$$

$$2\omega = \omega$$

$$\omega = \omega_0$$

$$|\omega_\alpha| = \aleph_\alpha$$

The order topology on an ordinal λ has subbasis $\{x < \alpha : x \in \lambda\}$ for each $\alpha \in \lambda$
$\{x > \beta : x \in \lambda\}$ $\cdots$ $\cdots$ $\beta \in \lambda$

Compare: $(\infty, \alpha) \subset \mathbb{R}$
$(\beta, \infty) \subset \mathbb{R}$

Eg. $\{x \in \lambda : \beta < x < \alpha\} = (\beta, \alpha)$

Eg. $\omega + 1 = \{0, 1, 2, \cdots\} \cup \{\omega\}$ has open sets (β, α) $(\beta, \alpha \in \omega + 1)$ $[0, \alpha) = \{x \in \lambda : x < \alpha\}$
$(\beta, \omega]$ $(\beta \in \omega + 1)$

and unions of these i.e. these sets form a basis.

Eg. $X = \omega_1 + 1 = [0, \omega_1]$. In X, ω_1 does not have a countable local basis
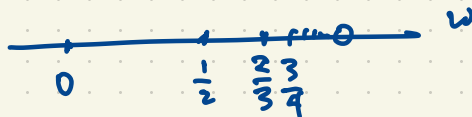
$\omega = \{0, 1, 2, \cdots\}$
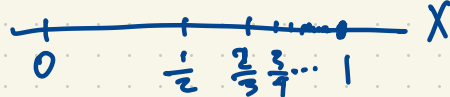
$\omega + 1 = [0, \omega] = \{0, 1, 2, 3, \cdots\} \cup \{\omega\}$

$X = \omega + 1 \simeq \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\} \cup \{1\} \subset \mathbb{R}$ (subspace topology)

X is the "one-point compactification" of ω.

S^1 is the one-point compactification of $\mathbb{R} \simeq (0,1)$

S^2 - - - - - - - - $\mathbb{R}^2$

S^n - - - - - - - - - $\mathbb{R}^n$ $(n \geq 1)$

A topological space X is <u>compact</u> if every open cover of X has a finite subcover, ie. if $X = \bigcup_{\alpha \in A} U_\alpha$, $U_\alpha \subseteq X$ open, there exist $k \geq 1$; $\alpha_1, \cdots, \alpha_k \in A$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_k}$. (See my video on compactness for review).

(*) <u>Remark</u>: The compact subspaces of $\mathbb{R}^n$ are the closed bounded subsets. <u>But</u> this statement depends on the choice of metric.

The new metric $\tilde{d}(x,y) = \min\{d(x,y), 1\}$ on $\mathbb{R}^n$ defines the same topology on $\mathbb{R}^n$ (the standard topology.)

standard metric

$[0,\infty)$ is a closed bounded subset of $\mathbb{R}^n$ with respect to $\tilde{d}$ but it is not compact.

ω (with the order topology) is a discrete topological space. You can also see this by thinking of $\omega \simeq \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \subset \mathbb{R}$. Every point is isolated.

$X = [0,\omega] = \omega \cup \{\omega\} \simeq$

is compact. This can be seen from $(*)$ above or from the definition of compactness. If $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $X = [0,\omega]$ then there exists $\alpha_0 \in A$ such that $\omega \in U_{\alpha_0}$ which also covers $(\beta, \omega]$ for some $\beta < \omega$ (β is finite). There are only finitely many elements $\alpha \leq \omega$ and these points are covered by finitely many U_α's. Together with U_{α_0} we have a finite subcover of X.

Every ordinal is either a "successor ordinal" or a "limit ordinal"

$$\alpha + 1 = \alpha \cup \{\alpha\}$$

$$1, 2, 3, \ldots, \quad \omega+1, \omega+2, \ldots$$

eg. $\omega, \omega 2, \ldots, \omega^2, \omega^\omega, \omega_1, \ldots$

0 is also a limit ordinal!

If λ is a limit ordinal, then λ is compact.

If λ is any ordinal, $[0,\lambda]$ is compact.

Let X be a top. space and let $A \subseteq X$. A limit point or cluster point or accumulation point of A is a point $x \in X$ such that every open nbhd of x has a point of A other than x itself i.e. every "deleted" nbhd $U \sim \{x\}$ has a point of A.

Eg. the limit points of $(0,1) \subset \mathbb{R}$ in $\mathbb{R}$ are $[0,1]$.

The limit ..... $[0,1]$ . . ... $[0,1]$.

In $X = [0, \omega] = \omega \cup \{\omega\}$, ω is a limit point of ω.

In $X = [0, \omega_1]$, ω_1 is a limit point of ω_1, but there is no sequence in ω_1 converging to ω_1.
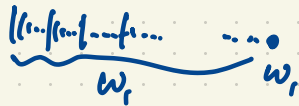
$$X = \omega_1 \cup \{\omega_1\}$$

An example of a discontinuous map $f : X \rightarrow Y$ which maps convergent sequences to convergent sequences ?

The Axiom of Choice $^{(AC)}$ says that if $\{A_\alpha\}_{\alpha \in I}$ is a collection of nonempty sets, then there exists a set $S = \{a_\alpha : \alpha \in I\}$ where $a_\alpha \in A_\alpha$.

ZFC = ZF + AC

Zorn's Lemma   Let $(S, \leq)$ be a partially ordered set. So '$\leq$' is a binary relation on the elements of S such that for all $a, b, c \in S$,

- $a \leq a$ (reflexive)
- If $a \leq b$ and $b \leq c$ then $a \leq c$.
- If $a \leq b$ and $b \leq a$ then $a = b$.

'Partial' (as distinct from 'total') order means we may have incomparable elements a, b where neither $a \leq b$ nor $b \leq a$.

Eg. Subsets of a given set under inclusion form a partial order.

Linear Algebra V is a vector space over a field F.   This is not a normed space.   Given a set $S \subset V$, $\text{Span } S = \{$ linear combinations of vectors in $S\}$
$= \{ a_1 v_1 + \cdots + a_k v_k : a_i \in F, v_i \in S, k \geq 1 \}$. S is linearly independent if the only solution of $a_1 v_1 + \cdots + a_k v_k = 0$ ($a_i \in F$, $v_i \in S$) is all $a_i = 0$.

Theorem   Every vector space has a basis $B \subset V$, i.e. B is linearly independent and $\text{Span } B = V$.

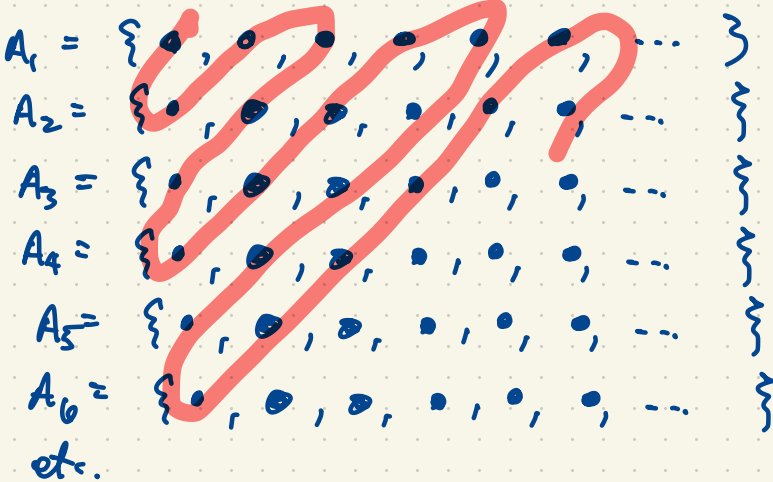**Proof** Let S be the collection of all linearly independent subsets of V. Let $C \subseteq S$ be a chain. Then $M = \cup C$ is linearly independent.

(If $v_1, \ldots, v_k \in M$ then $v_i \in A_i \in C$, $A_1 \cup \cdots \cup A_k \in C$; infact WLOG $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$ and $A_1 \cup \cdots \cup A_k = A_k \in C$.) By Zorn's Lemma, S has a maximal element B. This is a basis. (Since $B \in S$, B is lin. indep. To show $\operatorname{Span} B = S$, suppose $v \in V$, $v \notin \operatorname{Span} B$; then $B \cup \{v\}$ is lin. indep., a contradiction.) □

Eg. $\mathbb{R}$ is a vector space over $\mathbb{Q}$. So it has a basis B. But this cannot be written down explicitly. B is uncountable. (Use: If $A_1, A_2, A_3, \ldots$ are countable sets then $\bigcup_{k=1}^{\infty} A_k$ is countable.)

Zorn's Lemma If $(S, <)$ is a nonempty partially ordered set in which every chain is bounded above, then S has a maximal element.

A chain is a totally ordered subset. If C is a chain then an upper bound for C is an element $m \in S$ such that $x \le m$ for all $x \in C$. (Note: We do not require m to be in C.)

$A_1 = \{\ \bullet\ ,\ \bullet\ ,\ \bullet\ ,\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\cdots\quad\}$

$A_2 = \{\ \quad,\quad\bullet\ ,\quad\bullet\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\}$

$A_3 = \{\ \bullet\ ,\quad\bullet\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\cdots\quad\}$

$A_4 = \{\ \bullet\ ,\quad\bullet\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\bullet\quad,\quad\cdots\quad\}$

$A_5 = \{\ \bullet\ ,\quad\bullet\quad,\quad\bullet\ ,\quad\bullet\ ,\quad\bullet\quad,\quad\cdots\quad\}$

$A_6 = \{\ \bullet\quad,\quad\bullet\quad,\quad\bullet\ ,\quad\bullet\ ,\quad\bullet\ ,\quad\bullet\ ,\quad\cdots\quad\}$

etc.

A countable union of countable sets is countable.

__Theorem__ $X = \mathbb{R}^3 \smallsetminus \{0\}$ can be partitioned into lines.
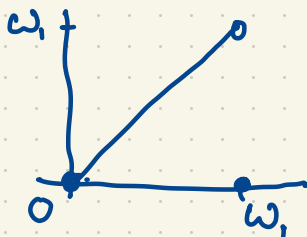
Use transfinite induction.

$|X| = ?$

And how many lines do we need to cover X?   (partition

$f: [0, \omega_1] \longrightarrow [0, \omega_1]$

$$f(\alpha) = \begin{cases} \alpha, & \text{if } \alpha < \omega_1 \\ 0, & \text{if } \alpha = \omega_1 \end{cases}$$

Whenever $(x_n)_{n=0,1,2,3,\ldots} = (x_0, x_1, x_2, x_3, \ldots) \longrightarrow x$ in $[0, \omega_1]$,

$f(x_n) \longrightarrow f(x)$ (f maps every convergent sequence to a convergent sequence). Yet f is not continuous.

$\{0\}$ is open in $X = [0, \omega_1]$ since $\{0\} = \{x \in X : x < 1\}$.

$f^{-1}(\{0\}) = \{0, \omega_1\}$ is not open in X. Any basic open nbhd

of X is $(\alpha, \omega_1] = \{x \in X : x > \alpha\}$.

In general $A^B = \{ \text{functions } B \to A \}$.

eg. $A^2 = A \times A$

$A^\omega = \{ \text{functions } \omega \to A \}$
$= \{ (a_0, a_1, a_2, a_3, \dots) : a_i \in A \}$

BUT for ordinals

$\omega^\omega = \{ (a_0, a_1, a_2, \dots) : a_i \in \omega ; \text{ but } a_i = 0 \text{ for } i \gg 0 \}$

$= \bigcup_{k \in \omega} \omega^k$

$2 = \{0, 1\}$

$A^2 = A^{\{0,1\}} = \{ \text{functions } \{0,1\} \to A \}$
$= \{ (a_0, a_1) : a_0, a_1 \in A \}$

$A^k = \underbrace{A \times A \times \dots \times A}_{k}$
$= \{ (a_0, a_1, \dots, a_{k-1}) : a_i \in A \}$
$= \{ \text{functions } \underbrace{\{0, 1, \dots, k-1\}}_{k} \to A \}$

$|\mathbb{R}| = |\omega|^{|\omega|} = |2^\omega| = 2^{\aleph_0} = | \{ \text{functions } \omega \to 2 \} |$
$= | \{ (a_0, a_1, a_2, a_3, \dots) : a_i = 0 \text{ or } 1 \} |$

$\aleph_0 < 2^{\aleph_0}$
$\overset{\shortparallel}{|\mathbb{Z}|} \quad \overset{\shortparallel}{|\mathbb{R}|}$

$\aleph_1 = $ smallest cardinality $> |\mathbb{Z}|$
$\aleph_1 = |\omega_1|$

i.e. $a_i \in 2$

$\aleph_0 < \aleph_1 \leq 2^{\aleph_0}$

Question of equality is independent of ZFC