Point Set Topology

Book 1

Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X (called the *open* sets) such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.

if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

(So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

__Example__

The *standard* topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon\}$.

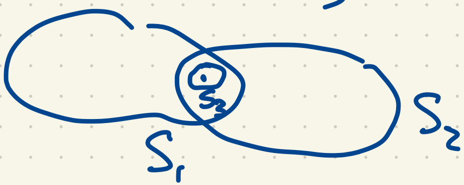
$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$$

(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let $\mathcal{S}$ be a collection of subsets of X which cover X, i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \ldots, S_k \in \mathcal{S}$ is a topology on X. The members of $\mathcal{S}$ are called a sub-basis for this topology and the topology is said to be <u>generated</u> by $\mathcal{S}$.

$\mathcal{S}$ is called a <u>base</u> (or a <u>basis</u>) for the topology if the topology is the collection of arbitrary unions of elements of $\mathcal{S}$. This holds iff



for all $S_1, S_2 \in \mathcal{S}$,
and all $u \in S_1 \cap S_2$,
there exists $S_3 \in \mathcal{S}$ such that

$$u \in S_3 \subseteq S_1 \cap S_2.$$