Point Set Topology

Book 1

Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X (called the **open** sets) such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.

if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

(So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

Example

The standard topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{ x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon \}$.

$d(x,u) = \sqrt{(x_1-u_1)^2 + \dots + (x_n-u_n)^2}$

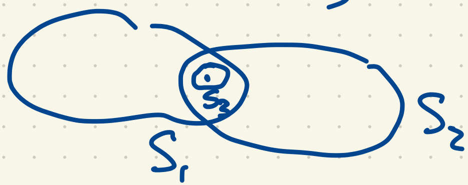
(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let $\mathcal{S}$ be a collection of subsets of X which cover X, i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \dots, S_k \in \mathcal{S}$ is a topology on X. The members of $\mathcal{S}$ are called a sub-basis for this topology and the topology is said to be <u>generated</u> by $\mathcal{S}$.

$\mathcal{S}$ is called a <u>base</u> (or a <u>basis</u>) for the topology if the topology is the collection of arbitrary unions of elements of $\mathcal{S}$. This holds iff for all $S_1, S_2 \in \mathcal{S}$, and all $u \in S_1 \cap S_2$, there exists $S_3 \in \mathcal{S}$ such that

$$u \in S_3 \subseteq S_1 \cap S_2 .$$



<u>Eg.</u> Let X be any set. The <u>discrete</u> <u>topology</u> on X is the collection of <u>all</u> subsets of X. (2^X)
The <u>indiscrete</u> topology on X is $\{\emptyset, X\}$.

If $X = \{0, 1\}$ then there are four possible topologies on X: $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let $\mathcal{T}$ be the collection of complements of finite sets, and $\emptyset$

i.e. $\mathcal{T} = \{\emptyset\} \cup \{X \setminus A : A \subseteq X, |A| < \infty\}$, $X \setminus A = \{x \in X : x \notin A\}$.

set difference

This is a topology on X, called the <u>finite complement topology</u>.

$X - A$, $X \setminus A$, $X \backslash A$

$\emptyset$, ϕ, $\varnothing$, 0

\varnothing

A <u>topological space</u> is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on a set X.

Note: $\bigcup \mathcal{T} = X$. By abuse of language, we often say that X is a topological space.

Let X be a set. A <u>distance function</u> (or <u>metric</u>) on X is a function $d : X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

$d(x,y) = d(y,x)$

$d(x,y) \geq 0$ and equality holds iff $x = y$.

$d(x,z) \leq d(x,y) + d(y,z)$

The standard topology on $\mathbb{R}^n$ is a metric topology.

The metric $d_2(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$ (the Euclidean metric)

$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$d_\infty(x,y) = \max\{|x_1-y_1|, \ldots, |x_n-y_n|\}$ all give the standard topology on $\mathbb{R}^n$.

In $\mathbb{R}^2$, open balls with respect to d_2, d_1, d_∞ look like

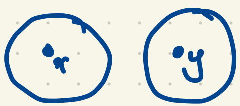
respectively. These three metrics define the same topology.

The metric $d(x,y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ defines the discrete topology.

A topological space is <u>metrizable</u> if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.

A topology is <u>Hausdorff</u> if for any two points $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

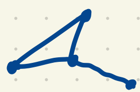


Every metric space is Hausdorff since if $x \neq y$, $\delta = d(x,y) > 0$. Take $U = B_{\delta/3}(x)$, $V = B_{\delta/3}(y)$

An **open neighbourhood** of a point $x \in X$ is an open set containing x.

nbhd

A **basic open nbhd** of a point $x \in X$ is an open nbhd of x which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.

Consider $X = \mathbb{Q}$. A **norm** on $\mathbb{Q}$ is a function $\mathbb{Q} \to [0,\infty)$, $x \mapsto \|x\|$ satisfying

(i) $\|x\| \geq 0$; equality holds iff $x = 0$.

(ii) $\|xy\| = \|x\| \cdot \|y\|$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

From any norm on $\mathbb{Q}$, we obtain a metric $d(x,y) = \|x-y\|$.

One way to do this is with the usual absolute value $\|x\| = |x| = |x|_\infty = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

This gives the standard topology on $\mathbb{Q}$.

An alternative is: given $x \in \mathbb{Q}$, if $x = 0$ define $\|0\|_2 = 0$.

If $x \neq 0$, write $x = 2^k \cdot \frac{a}{b}$, $a, b, k \in \mathbb{Z}$, $b \neq 0$; a, b odd. Then define $\|x\|_2 = 2^{-k}$.

This is the 2-adic norm on $\mathbb{Q}$. In fact it satisfies a stronger form of (iii), the ultrametric inequality $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$.

E.g. $\left\|\frac{20}{21}+\frac{5}{14}\right\|_2 = \left\|\frac{40+15}{42}\right\|_2 = \left\|\frac{55}{42}\right\|_2 = 2. = \max\Big\{\underbrace{\left\|\frac{20}{21}\right\|_2}_{\frac{1}{4}}, \underbrace{\left\|\frac{5}{14}\right\|_2}_{2}\Big\} = 2$

$\left\|\frac{20}{21}\right\|_2 = \frac{1}{4}$, $\left\|\frac{5}{14}\right\|_2 = 2$

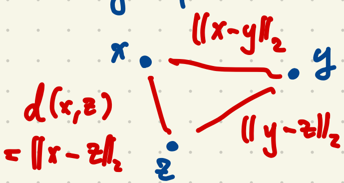
Compare: $\left\|\frac{20}{21}\right\|_2 + \left\|\frac{5}{14}\right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

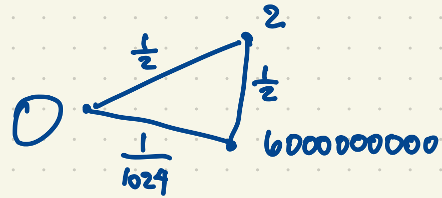
$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \{\frac{a}{b} : a, b \in \mathbb{Z},\ a \text{ even},\ b \text{ odd}\}.$

Every point in the ball is a centre of the ball i.e. if $c \in B_1(0)$ then $B_1(c) = B_1(0)$.



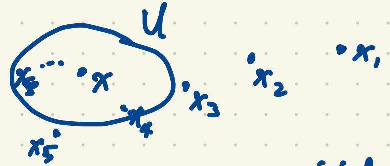
Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1+2+4+8+16+32+64+\cdots = -1$$

The partial sums $1, 3, 7, 15, 31, 63, \ldots$ converge to -1 in the 2-adic norm.

Note: If $(x_n)_n$ is a sequence of points in a top. space X, we say $(x_n)_n$ <u>converges</u> to $x \in X$ if for every open nbhd U of x, $x_n \in U$ for all n sufficiently large. (This means: for all U open nbhd of x, the exists N such that $x_n \in U$ whenever $n > N$.)



In place of arbitrary open nbhds of x, it suffices to check <u>basic</u> open nbhds.
For metric topology, it suffices to check open balls. In this case,
$x_n \to x$ provided that for all $\varepsilon > 0$, there exists N such that

$$\left.\begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array}\right\} \text{ whenever } n > N.$$

In our example above, $d(x_n, x) = 2^n \to 0$ as $n \to \infty$.

$$\|2^n\| = \frac{1}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Find the inverse of 5 mod 64.

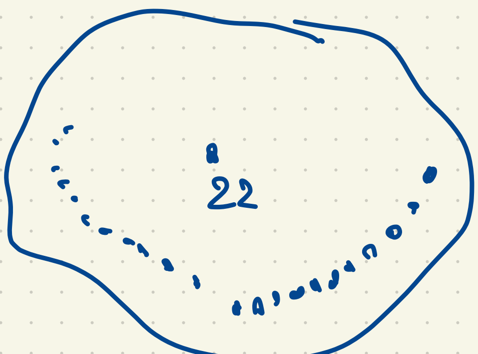
In $\mathbb{Z}/64\mathbb{Z}$, $\quad \frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - \underbrace{64 + 256 - 1024}_{\text{zero}} + \cdots$

$$= 1 - 4 + 16$$

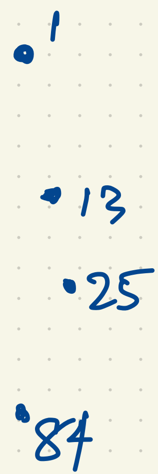
$$= 13.$$

Eg. in $\mathbb{Z}$ with the finite complement topology, the sequence $(n)_n = (1,2,3,\dots)$ converges. It converges to 22.

$(n)_n \longrightarrow 22.$



1, 13, 25, 84



In fact for every $a \in \mathbb{Z}$, $(n)_n \to a$.

<u>Theorem</u> If X is Hausdorff, then every sequence in X has at most one limit. (it converges to at most one point.)

<u>Proof</u> Suppose $a \neq b$ in a Hausdorff space X where a sequence $(x_n)_n \to a$ and $(x_n)_n \to b$. Choose disjoint open nbhds U, V of a, b respectively.

There exists N_1 such that $x_n \in U$ for all $n > N_1$; also N_2 such that $x_n \in V$ for all $n > N_2$.

Then pick $n > \max\{N_1, N_2\}$ to obtain a contradiction.

We prefer to write $(x_n)_n \to a$ rather than $\lim\limits_{n\to\infty} x_n = a$ in general.

In any top. space, closed sets are the complements of open sets.
$\emptyset, X$ are closed
If K, K' are closed then $K \cup K'$ is closed. (So finite unions of closed sets are closed.)
Arbitrary intersections of closed sets are closed.
De Morgan laws:
$$X \smallsetminus \Big(\bigcup_{\alpha\in I} A_\alpha\Big) = \bigcap_{\alpha\in I} (X \smallsetminus A_\alpha)$$
$$X \smallsetminus \Big(\bigcap_{\alpha\in I} A_\alpha\Big) = \bigcup_{\alpha\in I} (X \smallsetminus A_\alpha)$$

Given an infinite set X, the finite complement topology has as its closed sets the finite sets and X itself.

Let X be a top. space. Given $A \subseteq X$, the <u>closure</u> of A is the (unique) smallest closed set containing A i.e. $\bar{A} = \bigcap \{K \subseteq X : K \text{ closed}, K \supseteq A\}$.
The <u>interior</u> of A is the largest open set contained in A, <u>i.e.</u>
$A^\circ = \bigcup \{U \subseteq A : U \text{ open in } X\}$. $\quad (X \smallsetminus A)^\circ = X \smallsetminus \bar{A}$; $\quad \overline{X \smallsetminus A} = X \smallsetminus A^\circ$.

__Theorem__ There are infinitely many primes.

Known proofs: Euclid's proof (elementary)

Euler's proof (analytic proof: $\sum \frac{1}{p}$ diverges)

This proof is topological.

__Proof__ Form a topology on $X = \mathbb{Z}$ whose basic open sets are the (infinite) arithmetic progressions

$\dots, -6, -1, 4, 9, 14, 19, \dots$ for example.

Every nonempty open set is infinite.

Suppose there are only finitely many primes: $|P| < \infty$ is the set of all primes.

$$\{-1, 1\} = \{a \in \mathbb{Z} : a \text{ is not divisible by any prime}\}.$$

$$= \bigcap_{p \in P} \{a \in \mathbb{Z} : a \text{ is not divisible by } p\}$$

$$= \bigcap_{p \in P} (U_{1,p} \cup U_{2,p} \cup \dots \cup U_{p-1,p}) \qquad U_{a,p} = \{x \in \mathbb{Z} : x \equiv a \bmod p\}$$

is open. However it has only 2 elements, a contradiction. $\square$

More generally, let G be a group. Consider the topology on G whose basic open sets are cosets of subgroups $H \leq G$ of finite index, i.e. $gH = \{gh : h \in H\}$, $[G:H] < \infty$.

T_2 : Hausdorff 

T_1 : Points are closed 

If $x \in X$ and $y \neq x$, then there is an open nbhd U of x with $y \notin U$.

$T_2 \Rightarrow T_1$. Exercise: Give an example of a top. space which is T_1 but not T_2.

One answer: the finite complement topology for an infinite set.

Let $f: X \to Y$ be any function. For any $B \subseteq Y$, the preimage of B in X under f is $f^{-1}(B) = \{x \in X : f(x) \in B\}$. Similarly if $A \subseteq X$, the image of A in Y is $f(A) = \{f(a) : a \in A\}$. In general

$$f(f^{-1}(A)) \subseteq A \subseteq f^{-1}(f(A)).$$

Now let X and Y be top. spaces, i.e. $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$.

A function $f: X \to Y$ is <u>continuous</u> if the preimage of every open set (in Y) is open (in X); i.e. for every $U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

Exercise: Convince yourself that the "standard" definition of continuity for functions $\mathbb{R}^m \to \mathbb{R}^n$ is just a special case of this. (for the standard topologies on $\mathbb{R}^m$ and $\mathbb{R}^n$).

Theorem If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $g \circ f : X \to Z$.

Proof If $U \subseteq Z$ is open then $g^{-1}(U) \subseteq Y$ is open so $\underbrace{f^{-1}(g^{-1}(U))}_{= (g\circ f)^{-1}(U)} \subseteq X$ is open. $\square$

When are two topological spaces X, Y "the same"? ($X \simeq Y$: X, Y are homeomorphic)
This means there is a bijection $X \to Y$ taking one topology to the other.
I.e. there is a bijection $f: X \to Y$ such that f, f^{-1} are continuous.

Eg. X is $\mathbb{R}$ with the standard topology;
Y is $\mathbb{R}$ with the finite complement topology;
Z is $\mathbb{R}$ with the discrete topology;
W is $\mathbb{R}$ with the indiscrete topology $\{\emptyset, \mathbb{R}\}$.

$$Z \xrightarrow{\iota} X \xrightarrow{\iota} Y \xrightarrow{\iota} W \qquad \text{where } \iota(x) = x.$$

Z: finest topology on $\mathbb{R}$

W: coarsest topology on $\mathbb{R}$

If $\mathcal{T}, \mathcal{T}'$ are two topologies on X, we say
$\mathcal{T}'$ is <u>finer than</u> $\mathcal{T}$ if $\mathcal{T}' \supset \mathcal{T}$
($\mathcal{T}'$ is a <u>refinement</u> of $\mathcal{T}$)
$\mathcal{T}'$ is coarser than $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$.

Eg. The finite complement topology on X is the coarsest topology for which points are closed.

i.e. any topology in which points are closed is a refinement of the finite complement topology.

Subspace Topology

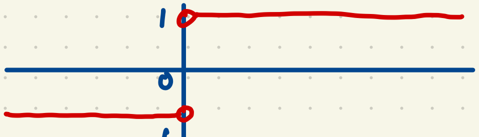
Let $A \subseteq X$ where X is a topological space $X = (X, \mathcal{T})$.
The topology A inherits from X in the most natural way is the subspace topology $\mathcal{T}_A = \{U \cap A \ : \ U \in \mathcal{T}\}$.

Eg. $[0,1) = \{a \in \mathbb{R} : 0 \leq a < 1\}$ is neither open nor closed in $\mathbb{R}$ but it is closed in $[0,1]$ and in $[0,\infty)$ since

$$[0,1) = (-1,1) \cap [0,1] = (-1,1) \cap [0,\infty).$$

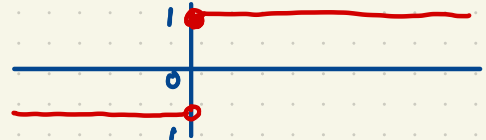
If $f: A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$, we say f is continuous if it is continuous relative to the standard topology of $\mathbb{R}^m$ and the subspace topology on $A \subseteq \mathbb{R}^n$.

Eg.



continuous

$f: (-\infty,0)\cup(0,\infty) \to \mathbb{R}$



not continuous

$f: \mathbb{R} \to \mathbb{R}$

If $f: A \to \mathbb{R}^m$ has $f(A) \subseteq B$ we might as well think of f as $f: A \to B$. To say $f: A \to \mathbb{R}^m$ is continuous is equivalent to saying $f: A \to B$ is continuous.

Suppose $f: A \to B$ is continuous and let $U \subseteq \mathbb{R}^m$. Then $f^{-1}(U) = f^{-1}(U \cap B)$ is open in A. Similarly one proves the converse.

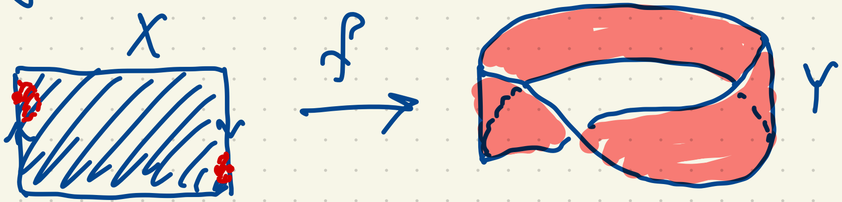
Given $A \subseteq X$ where X is a top. space, there is an inclusion map $\iota: A \to X$ $\iota(a) = a$. (one-to-one; not onto in general).

The subspace topology on A is the coarsest topology for which the inclusion map ι is continuous.

Given $U \subseteq X$ open. $i^{-1}(U) = U \cap A$.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

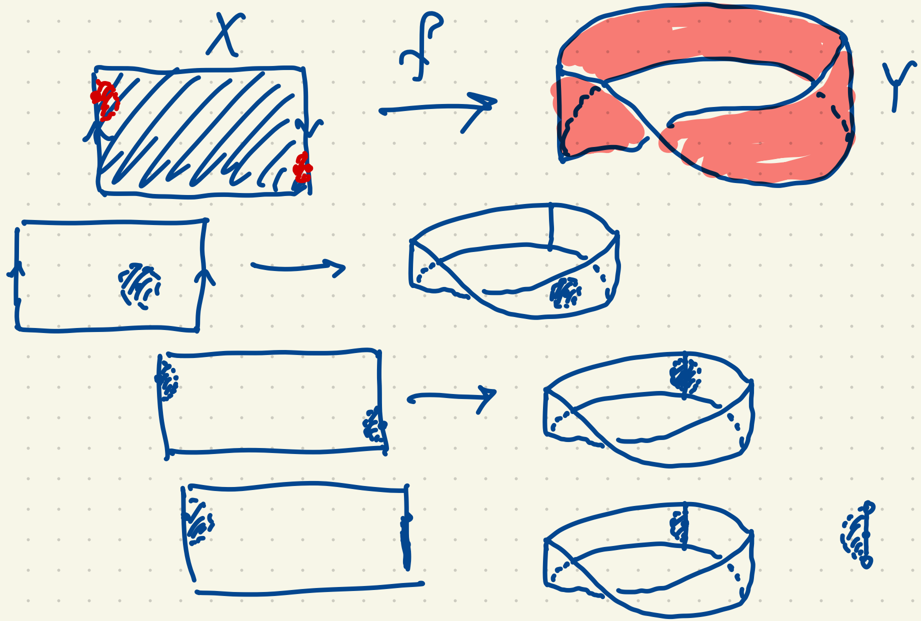
<u>Quotient Topology</u> Suppose $f: X \to Y$ is onto. Given a topology on $X = (X, \mathcal{T})$, the most natural way this gives a topology on Y is by taking the finest topology on Y for which f is continuous.

A Möbius strip 

There are three ways to think of this situation.
(i) Identify (collapse) certain points of X together.
(ii) We have an equivalence relation on X.
(iii) A partition of X.

The quotient topology on Y is the finest topology on Y for which the map $f: X \to Y$ is continuous.

The quotient topology on $Y = X/f$ or $X/\sim$ is $\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$.



To show this is a topology, use

$$\bigcup_\alpha f(A_\alpha) = f\left(\bigcup_\alpha A_\alpha\right)$$

$$\bigcap_\alpha f(A_\alpha) \supseteq f\left(\bigcap_\alpha A_\alpha\right)$$

$$\bigcup_\alpha f^{-1}(A_\alpha) = f^{-1}\left(\bigcup_\alpha A_\alpha\right)$$

$$\bigcap_\alpha f^{-1}(A_\alpha) = f^{-1}\left(\bigcap_\alpha A_\alpha\right)$$

$\sin((-\infty,0)) = [-1,1]$

$\sin((0,\infty)) = [-1,1]$

$\sin((-\infty,0) \cap (0,\infty)) = \sin\varnothing = \varnothing$

$\sin((-\infty,0) \cup (0,\infty)) = [-1,1]$

(closed) annulus

closed disk

Möbius strip

torus

≃ Klein bottle

not embeddable in $\mathbb{R}^3$.

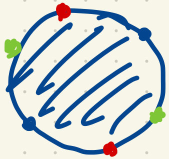
≃ $P^2\mathbb{R}$ = real projective plane

identify all boundary points

$\longrightarrow$

S^2 (2-sphere)

No two of the examples listed here are homeomorphic.

 $\cong \mathbb{P}^2\mathbb{R} \cong$  $+$ 

boundary $\cong S^1$ boundary $\cong S^1$

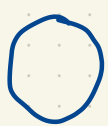
In $\mathbb{R}^3$ consider the following two subspaces :

X

Y

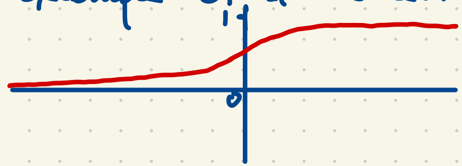
Is $X \cong Y$? Yes.

S^n = n-sphere $\cong$ unit sphere in $\mathbb{R}^{n+1}$ $= \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$.

$S^0 = \cdot \quad \cdot$ $S^1 =$  $S^2 =$  $S^3 = \mathbb{R}^3 \cup \{\infty\}$

$\mathbb{R} \cong (0,1) \cong (a,b) \cong (0,\infty)$ for $a<b$
(open interval)

An example of a homeomorphism $f: \mathbb{R} \to (0,\infty)$ is $f(x) = \dfrac{e^x}{1+e^x}$.



$\mathbb{R} \cong (0,1) \not\cong \begin{cases} [0,1] \\ [0,1) \end{cases}$ Why is $(0,1) \not\cong [0,1)$?

If we remove any point of $(0,1)$, what's left is disconnected. This is not true in $[0,1)$ which has a point 0 whose removal leaves a connected set $(0,1)$. $(0,\frac{1}{2}) \cup (\frac{1}{2},1)$ is disconnected since it is a disjoint union of two open sets.

<u>Def</u>. A top. space X is <u>disconnected</u> if $X = U \sqcup V$ where U, V are nonempty open sets in X. If X is not disconnected, then it is connected. In other words, X is connected iff its only clopen sets are $\emptyset$ and X ("clopen" means both open and closed).

$[0,1]$ is connected. This is a theorem in analysis.

Outline of argument: Suppose $[0,1] = U \sqcup V$ where U, V are nonempty open. $0 \in U$ without loss of generality. So $[0,\varepsilon) \subseteq U$ for some $\varepsilon > 0$. How large can ε be?

$\{\varepsilon : [0,\varepsilon) \subseteq U\}$ is a nonempty set with upper bound 1.

So there is a least upper bound. (supremum)

Is this supremum in U or in V? Either way leads to a contradiction.

If we remove any point from $(0,1) \simeq \mathbb{R} \sqcup \mathbb{R}$, we get a subspace $\simeq \mathbb{R} \sqcup \mathbb{R}$ which is disconnected.

This is not true for $[0,1)$.

$\mathbb{Q}$ is disconnected (in the standard topology in $\mathbb{R}$)

$\mathbb{Q} = U \sqcup V$ where $U = \{x \in \mathbb{Q} : \dots \; x < \sqrt{2}\} = \mathbb{Q} \cap (-\infty, \sqrt{2})$

$V = \{x \in \mathbb{Q} : \dots \; x > \sqrt{2}\} = \mathbb{Q} \cap (\sqrt{2}, \infty)$

$\mathbb{Q}$ is totally disconnected:

An <u>interval</u> in $\mathbb{R}$ is the same thing as a connected subset of $\mathbb{R}$.

<u>Theorem</u> $\mathbb{R}$ is connected.

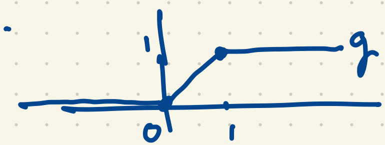
We'll talk about the foundations of $\mathbb{R}$ a little later, including completeness.

<u>Theorem</u> A continuous image of a connected space is connected.

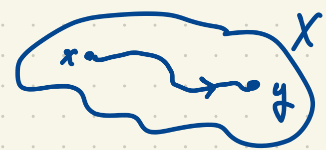
In other words if $f: X \to Y$ is $\left\{\begin{array}{l}\text{surjective} \\ \text{and continuous}\end{array}\right.$ and X is connected, then Y is connected.

<u>Proof</u> Suppose $Y = U \sqcup V$ where $U, V \subseteq Y$ are open. Then $X = f^{-1}(U) \sqcup f^{-1}(V)$ where $f^{-1}(U)$, $f^{-1}(V)$ are open in X. So one of these, say $f^{-1}(U)$, is empty. So $U = \emptyset$. This means Y is connected. $\square$

In a video I sent you, we showed $\mathbb{R}$ is connected.



<u>Corollary</u> $[0,1]$ is connected. Define $g: \mathbb{R} \to [0,1]$ which is a continuous surjection. $\square$



<u>Definition</u> A <u>path</u> from x to y in X is a continuous function $\gamma: [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$.

X is <u>path-connected</u> if for any $x, y \in X$, there is a path from x to y in X.

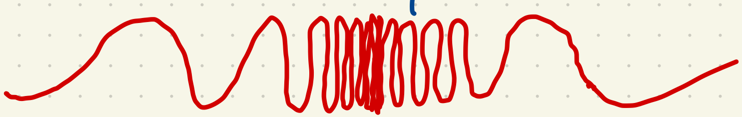
Theorem If X is path-connected then X is connected.

Proof Suppose $X = U \sqcup V$ where $U, V \subseteq X$ are nonempty open. Let $x \in U$, $y \in V$. If X is path-connected, there is a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$. Then

$$[0,1] = \gamma^{-1}(U) \sqcup \gamma^{-1}(V) = \gamma^{-1}(X),$$

a contradiction since $[0,1]$ is connected and $\gamma^{-1}(U), \gamma^{-1}(V)$ are disjoint nonempty open. $\square$.

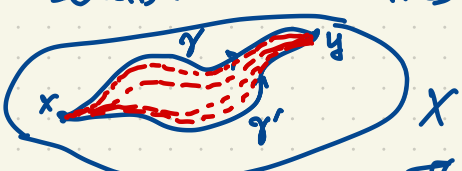
The converse of the theorem is false. An example of a space that is connected but not path-connected:



$X \subset \mathbb{R}^2$

$$X = \{(x, \sin\tfrac{1}{x}) : x \neq 0\} \cup \underbrace{(\{0\} \times [-1,1])}_{\text{interval on } y\text{-axis}}$$

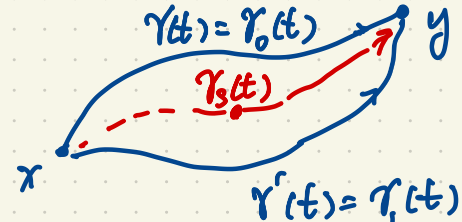
Details: See Munkres.



Let γ, γ' be two paths in X from x to y i.e. $\gamma, \gamma' : [0,1] \to X$, $\gamma(0) = \gamma'(0) = x$, $\gamma(1) = \gamma'(1) = y$.

Then γ, γ' are <u>homotopic</u> if there is a continuous map $[0,1]\times[0,1] \longrightarrow X$

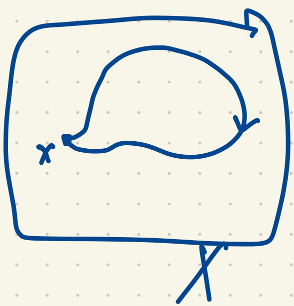
$$(s,t) \longmapsto \gamma_s(t)$$



such that $\gamma_s(0) = x$, $\gamma_s(1) = y$ for all $s \in [0,1]$

$$\left.\begin{array}{l} \gamma_0(t) = \gamma(t) \\ \gamma_1(t) = \gamma'(t) \end{array}\right\} \text{ for all } t \in [0,1].$$

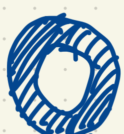
We think of $\gamma_s(t)$ as a "continuous deformation" from $\gamma(t)$ to $\gamma'(t)$.
(homotopy)



A closed curve based at $x \in X$ is a curve from x to x.
The null curve based at $x \in X$ is the curve

$$[0,1] \longrightarrow \{x\}.$$

If every closed curve in X is homotopic to a null curve, then X is simply connected.

Eg.  is connected but not simply connected. So this is not homeomorphic to a closed disk .

annulus

Let $(x_n)_n$ be a sequence in X.

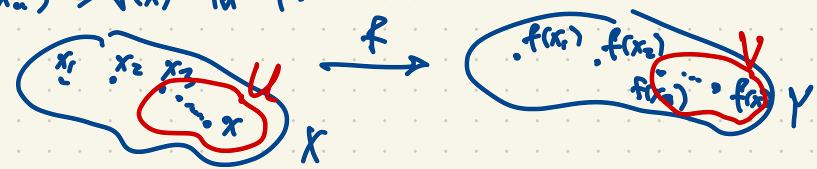
" $(x_1, x_2, x_3, \dots)$

We say $x_n \to x \in X$ if for every open nbhd U of x in X, beyond some point in the sequence all remaining terms are in U i.e. there exists N such that $x_n \in U$ whenever $n > N$. (We say $x_n \in U$ for all sufficiently large n, i.e. $x_n \in U$ whenever $n \gg 1$.)

The full definition of $x_n \to x$ is:

For every open nbhd U of x in X, there exists N such that $x_n \in U$ whenever $n > N$.

<u>Theorem</u> Let $f : X \to Y$ be continuous where X, Y are top. spaces. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.



<u>Proof</u> Let V be an open nbhd of $f(x)$ in Y. Let $U = f^{-1}(V)$ which is open in X since f is continuous. Note that $x \in U$. There exists N such that $x_n \in U$ for all $n > N$. So $f(x_n) \in V$ for all $n > N$. $\square$

Is the converse true? Namely if $f: X \to Y$ maps convergent sequences to convergent sequences, does this mean f is continuous?
In other words, suppose $f: X \to Y$ such that whenever $x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y. Must f be continuous?

Yes for metrizable spaces; no in general.

Metrizable spaces are <u>first countable</u>: there is a countable basis of open nbhds at every point. Given $a \in X$ where X is a metric space,

$B_\varepsilon(a) = \{x \in X : d(x,a) < \varepsilon\}$ is a collection of basic open nbhds at a.

There are uncountably many of these. The open nbhds $B_{\frac{1}{n}}(a)$ $(n = 1, 2, 3, \ldots)$ suffice for doing topology.

$x_n \to x$ iff for all $m \geq 1$ there exists N such that $x_n \in B_{\frac{1}{m}}(x)$ for all $n > N$.

The balls $B_{\frac{1}{m}}(a)$, $a \in X$, $m \geq 1$ generate all the open sets as a basis.

<u>First countability</u> of a top. space says that we have a countable collection of basic open nbhds at each point (a <u>local</u> condition).

Metric spaces are first countable.

$\mathbb{R}^n$ has a stronger property: it is <u>second countable</u> meaning it has a countable basis for the entire topology $\{B_{\frac{1}{m}}(a) : a \in \mathbb{Q}^n\}$.

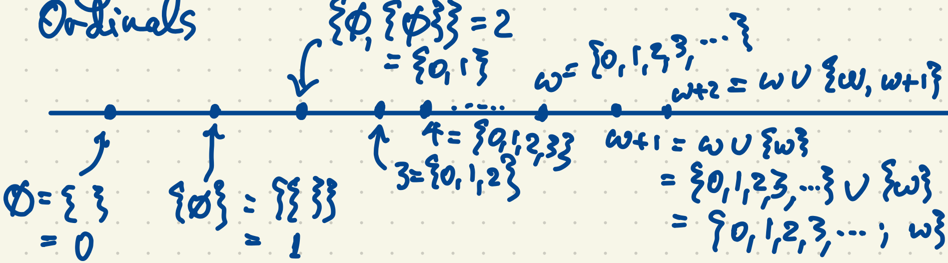
Theorem For first countable spaces, a function is continuous iff it maps convergent sequences to convergent sequences.

This is an inevitable result of the fact that sequences are inherently countable.

Remark: Second countability is strictly stronger than first countability.

Beyond countable:

Ordinals



Recursive construction: Each ordinal is the set of all the smaller ordinals.

A <u>totally ordered set</u> $(S, <)$ is a set S with a binary relation '$<$' on S satisfying

- Given $x, y \in S$, exactly one of the statements $x<y$, $x=y$, $\underbrace{y<x}_{\text{i.e. } x>y}$ is true ("trichotomy property");
- If $x<y<z$ then $x<z$ ("transitivity").

A <u>well-ordered set</u> is a totally ordered set in which every nonempty subset has a least element. Eg. for the usual order, $\underbrace{(\mathbb{N}, <)}_{=\omega}$ is well-ordered; $(\mathbb{Z}, <)$ is not; $[0, \infty)$ is not well-ordered.

Every well-ordered set is order-isomorphic to a unique ordinal. So the ordinals are the canonical representatives of the well-ordered sets.

Well-ordered sets are exactly the sets on which we can do induction.

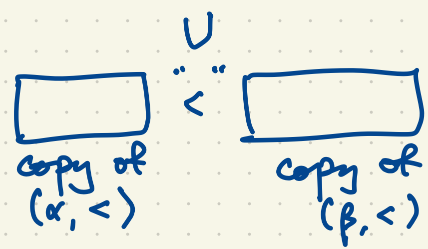
Every set can be well-ordered (the <u>well-ordering principle</u>).

In ZFC = Zermelo-Fraenkel + Axiom of Choice, the Well-Ordering Principle is a theorem. So is Zorn's Lemma.

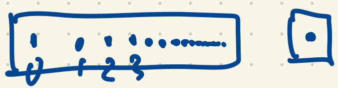
In ZF, the following are equivalent:

- Axiom Choice
- Well-Ordering Principle
- Zorn's Lemma
- Transfinite Induction

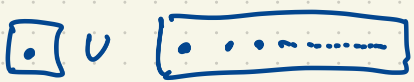
If α and β are ordinals, then $\alpha+\beta$ =



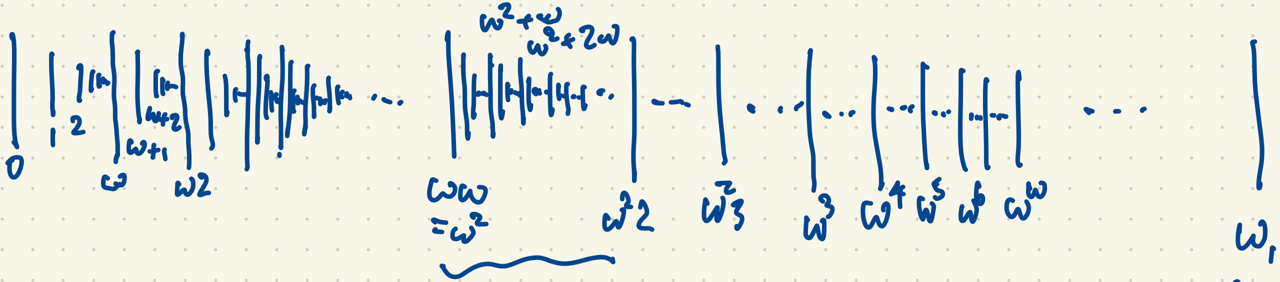
$\omega+1$ =



$1+\omega$ =



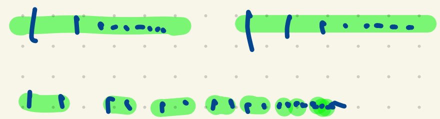
$= \omega$



= first uncountable ordinal

$\omega 2 = \omega + \omega$

$2\omega = \omega$



$\omega = \omega_0$

$|\omega_\alpha| = \aleph_\alpha$

The order topology on an ordinal λ has subbasis $\{x<\alpha : x\in\lambda\}$ for each $\alpha\in\lambda$
$\{x>\beta : x\in\lambda\}$ ... ... $\beta\in\lambda$

Compare: $(-\infty,\alpha)\subset\mathbb{R}$
$(\beta,\infty)\subset\mathbb{R}$

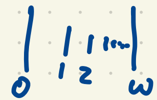
Eg. $\{x\in\lambda : \beta<x<\alpha\} = (\beta,\alpha)$

Eg. $\omega+1 = \{0,1,2,\cdots\}\cup\{\omega\}$ has open sets (β,α) $(\beta,\alpha\in\omega+1)$ $[0,\alpha) = \{x\in\lambda : x<\alpha\}$
$(\beta,\omega]$ $(\beta\in\omega+1)$
and unions of these i.e. these sets form a basis.

Eg. $X = \omega_1+1 = [0,\omega_1]$. In X, ω_1 does not have a countable local basis

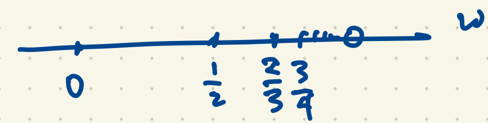
$\omega = \{0, 1, 2, \cdots\}$

$\omega + 1 = [0, \omega] = \{0, 1, 2, 3, \cdots\} \cup \{\omega\}$



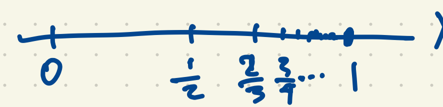
$X = \omega + 1 \cong \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\} \cup \{1\} \subset \mathbb{R}$ (subspace topology)

X is the "one-point compactification" of ω.



S^1 is the one-point compactification of $\mathbb{R} \simeq (0,1)$

S^2 . – – – ... ... $\mathbb{R}^2$



S^n . - - - - - - - - - $\mathbb{R}^n$ $(n \geq 1)$

A topological space X is <u>compact</u> if every open cover of X has a finite subcover, ie. if $X = \bigcup_{\alpha \in A} U_\alpha$, $U_\alpha \subseteq X$ open, there exist $k \geq 1$; $\alpha_1, \dots, \alpha_k \in A$ such that $X = U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_k}$. (See my video on compactness for review).

(*) <u>Remark</u>: The compact subspaces of $\mathbb{R}^n$ are the closed bounded subsets.
But this statement depends on the choice of metric.

The new metric $\tilde{d}(x,y) = \min\{\underbrace{d(x,y)}_{\text{standard metric}}, 1\}$ on $\mathbb{R}^n$ defines the same topology on $\mathbb{R}^n$ (the standard topology)

$[0,\infty)$ is a closed bounded subset of $\mathbb{R}^n$ with respect to $\tilde{d}$ but it is not compact.

ω (with the order topology) is a discrete topological space. You can also see this by thinking of $\omega \cong \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\} \subset \mathbb{R}$. Every point is isolated.

$X = [0,\omega] = \omega \cup \{\omega\} \cong$ 

is compact. This can be seen from (*) above or from the definition of compactness. If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of $X = [0,\omega]$ then there exists $\alpha_0 \in A$ such that $\omega \in U_{\alpha_0}$ which also covers $(\beta,\omega]$ for some $\beta < \omega$ (β is finite). There are only finitely many elements $\alpha \leq \omega$ and these points are covered by finitely many U_α's. Together with U_{α_0} we have a finite subcover of X.

Every ordinal is either a "successor ordinal" or a "limit ordinal"

$\alpha+1 = \alpha \cup \{\alpha\}$ eg. $\omega, \omega 2, \dots, \omega^2, \omega^\omega, \omega_1, \dots$

$1, 2, 3, \dots, \omega+1, \omega+2, \dots$ 0 is also a limit ordinal

If λ is a limit ordinal, then λ is compact.

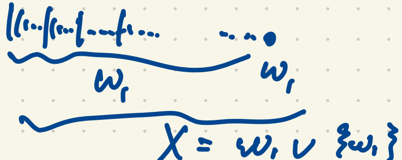
If λ is any ordinal, $[0,\lambda]$ is compact.

Let X be a top. space and let $A \subseteq X$. A <u>limit</u> <u>point</u> or <u>accumulation point</u> or <u>cluster point</u> of A is a point $x \in X$ such that every open nbhd of x has a point of A other than x itself i.e. every "deleted" nbhd $U - \{x\}$ has a point of A.

Eg. the limit points of $(0,1) \subset \mathbb{R}$ in $\mathbb{R}$ are $[0,1]$.
The Limit . . . . . . $[0,1]$ . . . . . $[0,1]$.

In $X = [0, \omega] = \omega \cup \{\omega\}$, ω is a limit point of ω.

In $X = [0, \omega_1]$, ω_1 is a limit point of ω_1 but there is no <u>sequence</u> in ω_1 converging to ω_1.



An example of a discontinuous map $f: X \to Y$ which maps convergent sequences to convergent sequences?

The **Axiom of Choice** (AC) says that if $\{A_\alpha\}_{\alpha\in I}$ is a collection of nonempty sets, then there exists a set $S = \{a_\alpha : \alpha\in I\}$ where $a_\alpha \in A_\alpha$.

ZFC = ZF + AC

Zorn's Lemma Let $(S,\leqslant)$ be a partially ordered set. So '$\leqslant$' is a binary relation on the elements of S such that for all $a,b,c\in S$,

- $a\leq a$ (reflexive)
- If $a\leq b$ and $b\leq c$ then $a\leq c$.
- If $a\leq b$ and $b\leq a$ then $a=b$.

'Partial' (as distinct from 'total') order means we may have incomparable elements a,b where neither $a\leq b$ nor $b\leq a$.

Eg. Subsets of a given set under inclusion form a partial order.

Linear Algebra V is a vector space over a field F. This is not a normed space. Given a set $S\subset V$, $\operatorname{Span} S = \{$linear combinations of vectors in $S\}$ $= \{a_1v_1+\cdots+a_kv_k : a_i\in F,\ v_i\in S,\ k\geqslant 1\}$. S is linearly independent if the only solution of $a_1v_1+\cdots+a_kv_k=0$ $(a_i\in F,\ v_i\in S)$ is all $a_i=0$.

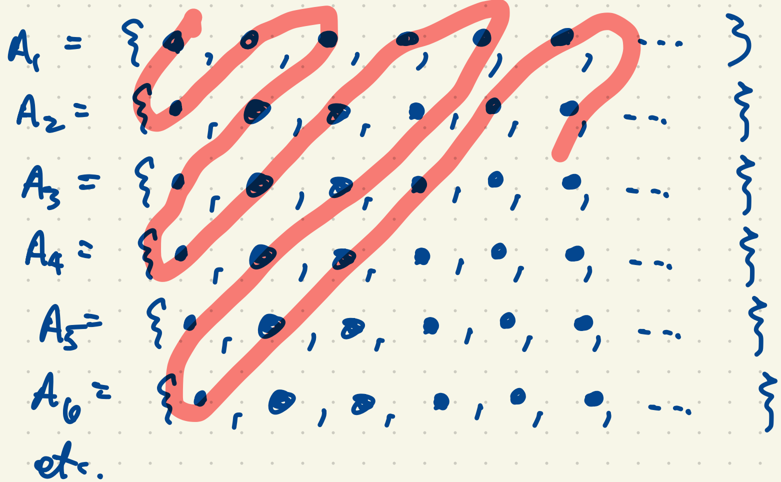
Theorem Every vector space has a basis $B\subset V$, i.e. B is linearly independent and $\operatorname{Span} B = V$.

Proof Let S be the collection of all linearly independent subsets of V. Let $C \subseteq S$ be a chain. Then $M = \bigcup C$ is linearly independent. (If $v_1, \dots, v_k \in M$ then $v_i \in A_i \in C$, $A_1 \cup \dots \cup A_k \in C$; in fact WLOG $A_1 \subseteq A_2 \subseteq \dots \subseteq A_k$ and $A_1 \cup \dots \cup A_k = A_k \in C$.) By Zorn's Lemma, S has a maximal element B. This is a basis. (Since $B \in S$, B is lin. indep. To show $\operatorname{Span} B = S$, suppose $v \in V$, $v \notin \operatorname{Span} B$; then $B \cup \{v\}$ is lin. indep., a contradiction.) $\square$

Eg. $\mathbb{R}$ is a vector space over $\mathbb{Q}$. So it has a basis B. But this cannot be written down explicitly. B is uncountable. (Use: If $A_1, A_2, A_3, \dots$ are countable sets then $\bigcup_{k=1}^{\infty} A_k$ is countable.)

Zorn's Lemma If $(S, <)$ is a nonempty partially ordered set in which every chain is bounded above, then S has a maximal element.

A chain is a totally ordered subset. If C is a chain then an <u>upper bound</u> for C is an element $m \in S$ such that $x \leq m$ for all $x \in C$. (Note: We do not require m to be in C.)



A countable union of countable sets is countable.

<u>Theorem</u> $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.