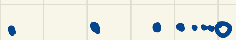
How do we judge a set as large or small?

(i) cardinality
(ii) measure
(iii) dimension
(iv) Baire Category : first category (small or meagre) or second category (large ; not meagre)
(v) ultrafilter sense (later)

<u>Warm up</u> Let $A \subseteq X$, X top. space. We say A is <u>dence</u> if every nonempty $U \subseteq X$ satisfies $A \cap U \neq \emptyset$ (A meets U). A set $A \subseteq X$ is <u>somewhere</u> <u>dense</u> if $\bar{A}$ contains some nonempty open set $U \subseteq X$. Eg. $\mathbb{Q}$ is somewhere dense since $\bar{\mathbb{Q}} = \mathbb{R}$. $\mathbb{Q} \cap [0,1]$ is somewhere dense since its closure is $[0,1]$. Note: $A \subseteq X$ is somewhere dense iff $\bar{A}^{\circ} \neq \emptyset$.

$A = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$ is nowhere dense. $\bar{A} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\} \cup \{1\}$ has empty interior.



A sequence (x_n) in a metric space (X, d) is <u>Cauchy</u> if for all $\varepsilon > 0$ there exist N such that for all $m, n > N$, $d(x_m, x_n) < \varepsilon$. Every convergent sequence in a metric space is Cauchy. X is <u>complete</u> (or d is complete) if every Cauchy sequence in X converges.

If $A \subseteq X$ $(A \neq \emptyset)$ where X is a metric space, then $\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\} \in [0, \infty]$. If $\operatorname{diam}(A) < \infty$ then A is bounded. If A is unbounded then $\operatorname{diam} A = \infty$.

<u>Lemma 1.1</u> Consider a chain $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$ of nonempty closed sets in a complete metric space X with $\operatorname{diam} K_n \to 0$. Then $\bigcap_n K_n \neq \emptyset$.

<u>Lemma 1.2</u> Every closed subspace of a complete metric space is complete. Proof: Exercise.

In a metric space (X, d), if $\delta < \varepsilon$ then $B_\delta(a) \subseteq \overline{B_\delta(a)} \subseteq B_\varepsilon(a)$.

Let X be a top. space. A subset $A \subseteq X$ is of <u>first category</u> if it's a countable union of nowhere dense sets i.e. $A = A_1 \cup A_2 \cup A_3 \cup \cdots$, A_n nowhere dense. Eg. $\mathbb{Q}$ is of first category. However $\mathbb{R}$ is not of first category. This follows from

<u>Baire Category Theorem</u> If X is a complete metric space then X is of second category i.e. not of first category.

<u>Corollary</u> (A.K.A Baire Category Theorem Part II) Let X be a complete metric space and suppose $D_1, D_2, D_3, \dots \subseteq X$ are dense open sets. Then $D_1 \cap D_2 \cap D_2 \cap D_3 \cap \cdots \neq \emptyset$.

Eg. Let $l_1, l_2, l_3, \cdots$ be lines in $\mathbb{R}^3$. Then $\bigcup_n l_n \neq \mathbb{R}^3$.

Proof Take $D_n = \mathbb{R}^3 \setminus l_n$. (dense open set in $\mathbb{R}^3$). $\mathbb{R}^3$ is a complete metric space.

Eg. Color the points of $\mathbb{R}^2$ using a countable set of colors. Then at least one of the color sets is somewhere dense.

Application There exist functions $\mathbb{R} \to \mathbb{R}$ which are continuous but nowhere differentiable.

Proof uses Baire Category Theory.

$V = \mathcal{C}([0,1]) = \{$continuous functions $[0,1] \to \mathbb{R}\}$. This vector space is a metric space using $\|f\| = \sup\{|f(x)| : x \in [0,1]\} < \infty$ for all $f \in V$.

$d(f,g) = \|f-g\|$.

Define subsets $A_{m,n} \subset V$ where m,n are pos. integers by

$A_{m,n} = \{f \in V : \left|\frac{f(y)-f(x)}{y-x}\right| < m$ whenever $0 < |x-y| < \frac{1}{n}\}$.

closed, nowhere dense. (see Lemma 4.1)

Also if $f \in V$ is differentiable then $f \in A_{m,n}$ for some $m,n \geq 1$.

Finite Permutation Groups

S_n = symmetric group of degree n = {all permutations of X}, $X = \{1, 2, \cdots, n\}$.

$|S_n| = n!$; nonabelian if $n \geq 3$.

A permutation group of degree n is a subgroup $H \leq S_n$.

For $i \in X$, the stabilizer of i in H is $H_i = \{h \in H : h(i) = i\} \leq H$.

The orbit of i is $i^H = \{h(i) : h \in H\} \subseteq X$.

Theorem (Orbit-Stabilizer Formula) $|i^H| = [H : H_i]$ = number of left (or right) cosets

index of subgroup $H_i \leq H$

$$[H : H_i] = \frac{|H|}{|H_i|}$$

$$|H| = |H_i|\,[H : H_i]$$

In the infinite case, $|A||B| = |A \times B| = \max\{|A|, |B|\}$ whenever at least one of A, B is infinite.

If $A \subseteq X$ then H_A = stabilizer of $A = \{h \in H : h(A) = A\}$ where $h(A) = \{h(a) : a \in A\}$.

The orbit of $A = A^H = \{h(A) : h \in A\}$.

$|A^H| = [H : H_A]$; $|H| = |A^H|\,|H_A|$.

Infinite Permutation Groups

$\mathrm{Sym}\,X = \{\text{permutations of } X\}$

$G = \mathrm{Sym}\,\mathbb{N} = \{\text{all permutations of } 1, 2, 3, 4, \dots\}$.

$|G| = |\mathbb{R}| = 2^{\aleph_0}$

Proof: G has a subgroup $H = \langle (1,2), (3,4), (5,6), (7,8), \dots \rangle$

$$= \{ (1,2)^{a_0} (3,4)^{a_1} (5,6)^{a_2} (7,8)^{a_3} \cdots \;:\; a_i \in \{0,1\} \}$$

$$|H| = |2^{\omega}| = 2^{\aleph_0}$$

So $|G| \geq |H| = 2^{\aleph_0}$.

There is an injection $G \to \mathbb{R}$, $f \xmapsto{1:1} (f(1), f(2), f(3), \dots) \xmapsto{1:1} f(1) + \cfrac{1}{f(2) + \cfrac{1}{f(3) + \cfrac{1}{f(4) + \cdots}}}$ infinite continued fraction

so $|G| \leq |\mathbb{R}| = 2^{\aleph_0}$.

A permutation group on $\mathbb{N}$ is a subgroup $H \leq G = \mathrm{Sym}\,\mathbb{N}$.

$|H|$ is $\underbrace{\text{countable}}_{\text{finite or countably infinite}}$ or $2^{\aleph_0}$

CH (Continuum Hypothesis): There do not exist sets A with $\aleph_0 < |A| < 2^{\aleph_0}$.

CH is independent of ZFC. But most everyday mathematics uses ZFC only, and does not depend on whether or not CH is true.

<u>Theorem</u> If $H \leq G = \mathrm{Sym}\,\mathbb{N}$ is a closed subgroup, then $[G:H]$ is either countable or $2^{\aleph_0}$.

This is proved using the Baire Category Theorem. G is a top. space (top. group) and its topology comes from a metric for which G is complete.

To define $d(f,g)$ for $f,g \in G$:

$$d_0(f,g) = \begin{cases} 0 & \text{if } f = g \\ \frac{1}{2^n} & \text{if } f(n) \neq g(n) \text{ for some } n, \text{ with } n \text{ minimal} \end{cases}$$

This is a metric. Replace it with a nicer metric

$d(f,g) = \max\{d_0(f,g),\ d_0(f^{-1},g^{-1})\}$. This <u>is</u> a metric.

d, d_0 define the same topology in G but (G,d) is complete; (G, d_0) is not complete.

e.g. $(1,2,3,4,5,6,\cdots)$
$(2,1,3,4,5,6,\cdots)$
$(2,3,1,4,5,6,\cdots)$
$(2,3,4,1,5,6,\cdots)$
etc.

converges in (G,d_0) to $(2,3,4,5,6,7,\cdots) \in \mathbb{N}^\omega$

But it does not converge in (G,d) (not Cauchy).

G is a top. group i.e. $(f,g) \longmapsto fg \in G$ (composition) is continuous $G\times G \to G$

$f \longmapsto f^{-1} \in G$ " " " " $G \to G$

- ~~Basic~~ Sub open sets in G: These are the stabilizers of finite sets, and their cosets.

Basic open sets: Given $i_1, \dots, i_k \in \mathbb{N}$, $G_{(i_1,i_2,\dots,i_k)} : \{f \in G : f(i_1) = i_1, \dots, f(i_k) = i_k\}$

$= G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_k}$

and $hG_{(i_1,\dots,i_k)}$ $h \in G$; $i_1, \dots, i_k \in \mathbb{N}$. $k \geq 1$.

<u>Theorem</u> Let H, K be closed subgroups of $G = \mathrm{Sym}\,\mathbb{N}$ with $K \leq H$. Then $[H:K] \leq \aleph_0$ or $[H:K] = 2^{\aleph_0}$. Moreover $[H:K] \leq \aleph_0$ iff $K \supseteq H_A$ for some $A \subseteq \mathbb{N}$, $|A| < \infty$.