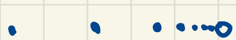
How do we judge a set as large or small?

(i) cardinality
(ii) measure
(iii) dimension
(iv) Baire Category : first category (small or meagre) or second category (large ; not meagre)
(v) ultrafilter sense (later)

<u>Warm up</u> Let $A \subseteq X$, X top. space. We say A is <u>dense</u> if every nonempty $U \subseteq X$ satisfies $A \cap U \neq \emptyset$ (A meets U). A set $A \subseteq X$ is <u>somewhere</u> <u>dense</u> if $\bar{A}$ contains some nonempty open set $U \subseteq X$. Eg. $\mathbb{Q}$ is somewhere dense since $\bar{\mathbb{Q}} = \mathbb{R}$. $\mathbb{Q} \cap [0,1]$ is somewhere dense since its closure is $[0,1]$. Note: $A \subseteq X$ is somewhere dense if $\bar{A}^{\circ} \neq \emptyset$.

$A = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}$ is nowhere dense. $\bar{A} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\} \cup \{1\}$ has empty interior.



A sequence (x_n) in a metric space (X,d) is <u>Cauchy</u> if for all $\varepsilon>0$ there exist N such that for all $m,n>N$, $d(x_m,x_n)<\varepsilon$. Every convergent sequence in a metric space is Cauchy. X is <u>complete</u> (or d is complete) if every Cauchy sequence in X converges.

If $A\subseteq X$ $(A\neq\emptyset)$ where X is a metric space, then $\operatorname{diam}(A)=\sup\{d(a,b) : a,b\in A\}\in[0,\infty]$. If $\operatorname{diam}(A)<\infty$ then A is bounded. If A is unbounded then $\operatorname{diam} A=\infty$.

<u>Lemma 1.1</u> Consider a chain $K_1\supseteq K_2\supseteq K_3\supseteq K_4\supseteq\dots$ of nonempty closed sets in a complete metric space X with $\operatorname{diam} K_n\to 0$. Then $\bigcap_n K_n\neq\emptyset$.

<u>Lemma 1.2</u> Every closed subspace of a complete metric space is complete. Proof: Exercise.

In a metric space (X,d), if $\delta<\varepsilon$ then $B_\delta(a)\subseteq\overline{B_\delta(a)}\subseteq B_\varepsilon(a)$.

Let X be a top. space. A subset $A\subseteq X$ is of <u>first category</u> if it's a countable union of nowhere dense sets i.e. $A=A_1\cup A_2\cup A_3\cup\cdots$, A_n nowhere dense. Eg. $\mathbb{Q}$ is of first category. However $\mathbb{R}$ is not of first category. This follows from

<u>Baire Category Theorem</u> If X is a complete metric space then X is of second category i.e. not of first category.

<u>Corollary</u> (A.K.A Baire Category Theorem Part II) Let X be a complete metric space and suppose $D_1,D_2,D_3,\dots\subseteq X$ are dense open sets. Then $D_1\cap D_2\cap D_2\cap D_3\cap\cdots\neq\emptyset$.

Eg. Let $l_1, l_2, l_3, \cdots$ be lines in $\mathbb{R}^3$. Then $\bigcup_n l_n \neq \mathbb{R}^3$.

Proof Take $D_n = \mathbb{R}^3 \setminus l_n$. (dense open set in $\mathbb{R}^3$). $\mathbb{R}^3$ is a complete metric space.

Eg. Color the points of $\mathbb{R}^2$ using a countable set of colors. Then at least one of the color sets is somewhere dense.

Application There exist functions $\mathbb{R} \to \mathbb{R}$ which are continuous but nowhere differentiable.

Proof uses Baire Category Theory.

$V = \mathcal{C}([0,1]) = \{$continuous functions $[0,1] \to \mathbb{R}\}$. This vector space is a metric space using $\|f\| = \sup\{|f(x)| : x \in [0,1]\} < \infty$ for all $f \in V$.

$d(f,g) = \|f-g\|$.

Define subsets $A_{m,n} \subset V$ where m,n are pos. integers by

$$A_{m,n} = \left\{ f \in V : \left| \frac{f(y)-f(x)}{y-x} \right| < m \text{ whenever } 0 < |x-y| < \frac{1}{n} \right\}.$$

closed, nowhere dense. (see Lemma 4.1)

Also if $f \in V$ is differentiable then $f \in A_{m,n}$ for some $m,n \geq 1$.