

Point Set Topology

Book 3

A filter on X is a collection \mathcal{F} consisting of subsets of X such that

- $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$
- If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
- If $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.

Every ultrafilter is a filter, but not conversely.

A collection \mathcal{S} of subsets of X has the finite intersection property (f.i.p.) if for all $A_1, \dots, A_n \in \mathcal{S}$, $A_1 \cap A_2 \cap \dots \cap A_n \neq \emptyset$.

A filter has the f.i.p. If \mathcal{S} is any collection of subsets of X having f.i.p. then \mathcal{S} generates a filter: $\mathcal{F}_{\mathcal{S}} = \{ \text{supersets of finite intersections of sets in } \mathcal{S} \}$

$$= \{ B \subseteq X : A_1 \cap A_2 \cap \dots \cap A_n \subseteq B \text{ for some } A_1, A_2, \dots, A_n \in \mathcal{S} \}.$$

This is the (unique) smallest collection of subsets of X which contains \mathcal{S} and is a filter.

If $\mathcal{F}, \mathcal{F}'$ are filters on X , we say \mathcal{F}' refines \mathcal{F} if $\mathcal{F} \subseteq \mathcal{F}'$.

The collection of all filters on X is partially ordered by refinement.

Given a filter \mathcal{F}_0 on X , the collection of filters refining \mathcal{F}_0 has a maximal member by Zorn's Lemma. This is guaranteed to be an ultrafilter.

Assume we are given a nonprincipal ultrafilter \mathcal{U} on $\omega = \{0, 1, 2, 3, \dots\}$.

Construction of the nonstandard real numbers (hyperreals) ${}^*\mathbb{R}$ or \mathbb{R}^* or $\hat{\mathbb{R}}$.

$\hat{\mathbb{R}}$ and \mathbb{R} are examples of ordered fields. $\hat{\mathbb{R}}$ and \mathbb{R} are very similar from first appearances.

eg. If $f(x) \in \mathbb{R}[x]$ or $\hat{\mathbb{R}}[x]$ (polynomial in x) of degree ≥ 3 then f has a root (in \mathbb{R} or $\hat{\mathbb{R}}$ respectively).
If $f' > 0$ then this root is unique. Positive elements have a unique square root.

But: \mathbb{R} is an Archimedean field: it has no infinite or infinitesimal elements. More precisely, if $a \in \mathbb{R}$ satisfies $0 \leq a < \frac{1}{n}$ for all $n=1,2,3,4,\dots$ then $a=0$.

$\hat{\mathbb{R}}$ has infinitesimal elements (it is Non-Archimedean field).

Construction: Start with $\mathbb{R}^\omega = \{ (a_0, a_1, a_2, a_3, \dots) : a_i \in \mathbb{R} \}$ (all sequences of real numbers).

Given $a, b \in \mathbb{R}^\omega$ we can add/multiply/subtract pointwise

$$a \pm b = (a_0 \pm b_0, a_1 \pm b_1, a_2 \pm b_2, \dots)$$

$$ab = (a_0 b_0, a_1 b_1, a_2 b_2, \dots)$$

making \mathbb{R}^ω into a ^{commutative} ring with identity $1 = (1, 1, 1, 1, \dots)$. It's not a field; it has zero divisors e.g.

$$(1, 0, 1, 0, 1, 0, \dots) (0, 1, 0, 1, 0, 1, \dots) = (0, 0, 0, 0, 0, 0, \dots) = 0 \in \mathbb{R}^\omega.$$

But take an ultrafilter \mathcal{U} on ω (\mathcal{U} nonprincipal).

If $a_i = b_i$ for all $i \in U \in \mathcal{U}$ then $a_i \sim b_i$ (equivalence mod \mathcal{U}).

In this case $(0, 1, 0, 1, 0, 1, \dots) \sim (1, 1, 1, 1, 1, 1, \dots) = 1$
 $(1, 0, 1, 0, 1, 0, \dots) \sim (0, 0, 0, 0, 0, 0, \dots) = 0$

Given $a, b \in \mathbb{R}^\omega$, let $A = \{i \in \omega : a_i = b_i\}$. Either $A \in \mathcal{U}$ (in which case $a \sim b$) or $\omega - A \in \mathcal{U}$ (in which case $a \not\sim b$). $\hat{\mathbb{R}} = \mathbb{R}^\omega / \sim = \{ [a]_{\sim} : a \in \mathbb{R}^\omega \}$, $[a]_{\sim} =$ equiv. class of $a = \{x \in \mathbb{R}^\omega : x \sim a\}$.

$\hat{\mathbb{R}}$ is a field. If $a \neq 0$ then actually $a \neq 0$ ($[a]_{\sim} \neq [0]_{\sim}$) so $\{i \in \omega : a_i \neq 0\} \in \mathcal{U}$. (most coordinates of a are nonzero). Then $\frac{1}{a} = (\frac{1}{a_i} : i \in \omega)$

$a \cdot \frac{1}{a} = 1$
Anywhere that $a_i = 0$, ignore or replace by 1.

$\hat{\mathbb{R}}$ is an ordered field. Given $a, b \in \hat{\mathbb{R}}$, either $a < b$ or $a = b$ or $b < a$.

$$\omega = \{i \in \omega : a_i < b_i\} \sqcup \{i \in \omega : a_i = b_i\} \sqcup \{i \in \omega : b_i < a_i\}$$

Exactly one of these three sets is an ultrafilter set. Correspondingly, $a < b$ or $a = b$ or $b < a$.

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^\omega$$

Given $a \in \mathbb{R}$, identify with $(a, a, a, a, \dots) \in \mathbb{R}^\omega$. This way \mathbb{R} is embedded in \mathbb{R}^ω .

The ^{standard} topology on $\hat{\mathbb{R}}$ is the order topology: basic open sets are open intervals (a, b) , $a, b \in \hat{\mathbb{R}}$.

Eg. $\varepsilon = [(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)]_\omega \in \hat{\mathbb{R}}$ is an infinitesimal.

$\frac{1}{\varepsilon} = [(1, 2, 3, 4, 5, \dots)]_\omega \in \hat{\mathbb{R}}$ is infinite.

$$|\mathbb{Q}| = \aleph_0, |\mathbb{R}| = 2^{\aleph_0}, |\hat{\mathbb{R}}| = 2^{\aleph_0}$$

$$\mathbb{Q} \subset \mathbb{R} \subset \hat{\mathbb{R}}$$

$$|\mathbb{R}^\omega| = |\mathbb{R}|^{|\omega|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$$

Every hyperreal is either infinite ($a \in \hat{\mathbb{R}}$, $|a| > n$ for every positive integer n) or it's bounded in which case a has a unique standard part $st(a) \in \mathbb{R}$ (the closest real number to a).

To compute $f'(x)$ where $f(x) = x^2 + 3x + 7$ using nonstandard analysis, let $a \in \mathbb{R}$, and we want to compute $f'(a) \in \mathbb{R}$.

Pick $\hat{a} \in \hat{\mathbb{R}}$, $st(\hat{a}) = a$, $\hat{a} - a = \varepsilon$ is an infinitesimal. $f(\hat{a}) - f(a) = f(a + \varepsilon) - f(a) = (a + \varepsilon)^2 + 3(a + \varepsilon) + 7 - (a^2 + 3a + 7) = 2\varepsilon a + \varepsilon^2 + 3\varepsilon$

$$\frac{f(a + \varepsilon) - f(a)}{\varepsilon} = 2a + 3 + \varepsilon, \quad st(2a + 3) = 2a = f'(a).$$

Warm-up to the proof of Tychonoff's Theorem.

Let \mathcal{S} be a collection of subsets of X . \mathcal{S} has the finite intersection property (f.i.p.) if every finite intersection of sets in \mathcal{S} is nonempty i.e.

$$S_1, S_2, \dots, S_n \in \mathcal{S} \Rightarrow S_1 \cap S_2 \cap \dots \cap S_n \neq \emptyset.$$

(Recall: if \mathcal{S} has f.i.p. then supersets of finite intersections of sets in \mathcal{S} is a filter.)

Lemma 1.1 Let X be a top. space. Then the following are equivalent.

(i) X is compact. (Every open cover of X has a finite subcover.)

(ii) If \mathcal{S} is any collection of closed sets with f.i.p. then $\bigcap \mathcal{S} \neq \emptyset$.

← via complementation (use de Morgan's laws)

←

Proof: exercise.

← filter such that for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, not both.

An ultrafilter \mathcal{U} on X converges to a point $x \in X$ if every ^(open) nbhd of x is in \mathcal{U} .

(A nbhd is a superset of an open nbhd.)

We write $\mathcal{U} \searrow x$ in this case. (Recall: The nbhds of x form a filter.)

Much topology is readily formulated in the language of ultrafilters e.g.

- X is Hausdorff iff every ultrafilter converges to at most one point.
- X is compact iff every ultrafilter converges to at least one point.
- A function $f: X \rightarrow Y$ is continuous iff it maps convergent ultrafilters to convergent ultrafilters.

Theorem 2.1(a) Let X be a top. space. Then X is Hausdorff iff every ultrafilter on X converges to at most one point of X .

Proof (\Rightarrow) Suppose X is Hausdorff. Suppose \mathcal{U} is an ultrafilter on X converging to two different points $x \neq y$ in X . There exist $U, V \subseteq X$ disjoint open sets with $x \in U, y \in V$. $U \circledast x$ $V \circledast y$

Since $\mathcal{U} \searrow x$, $U \in \mathcal{U}$. Similarly $V \in \mathcal{U}$. Then $U \cap V = \emptyset \in \mathcal{U}$, contradiction.

(\Leftarrow) Suppose every ultrafilter on X converges to at most one point of X .

(\Leftarrow) Suppose every ultrafilter on X converges to at most one point of X . Let $x \neq y$ in X . By way of contradiction, suppose that $U \cap V \neq \emptyset$ for every open nbhd U of x and every open nbhd V of y . Then

$\{\text{open nbhds of } x\} \cup \{\text{open nbhds of } y\}$ has f.i.p.

This generates a filter which in turn refines to an ultrafilter \mathcal{U} . $\mathcal{U} \ni x, \mathcal{U} \ni y$, a contradiction. So X must be Hausdorff.

Theorem 2.1 (b) Let X be a top. space. Then X is compact \iff every ultrafilter on X converges to at least one point of X .

Proof (\Rightarrow) Suppose X is compact. Let \mathcal{U} be an ultrafilter on X . Suppose \mathcal{U} does not converge to any point of X . So for each $x \in X$, there exists an open nbhd U_x of x such that $U_x \notin \mathcal{U}$. So $\{U_x : x \in X\}$ is an open cover of X . So there is a finite subcover

$$X = U_{x_1} \cup U_{x_2} \cup U_{x_3} \cup \dots \cup U_{x_n} \quad \text{for some } n \geq 1; x_1, \dots, x_n \in X.$$

So $U_{x_i} \in \mathcal{U}$ for some i , contradiction.

(\Leftarrow) Suppose every ultrafilter on X converges to at least one point of X . We must show that X is compact. Let \mathcal{S} be a collection of closed subsets of X with f.i.p.; we must show $\bigcap \mathcal{S} \neq \emptyset$. Now \mathcal{S} generates a filter which refines to an ultrafilter $\mathcal{U} \supseteq \mathcal{S}$. By assumption, $\mathcal{U} \ni x$ for some point $x \in X$. We will show $x \in \bigcap \mathcal{S}$. If not, then there exists $K \in \mathcal{S}$ such that $x \notin K$. Then $X - K$ is an open nbhd of x . So $X - K \in \mathcal{U}$. But also $K \in \mathcal{S} \subseteq \mathcal{U}$, contradiction. \square

Ultrafilters gives the following characterization of open sets.

Theorem 2.2 Let X be a top. space, and let $U \subseteq X$. The following are equivalent:



(i) U is open.

(ii) Whenever an ultrafilter converges to a point $x \in U$, we have $U \in \mathcal{U}$.

Proof (\Rightarrow) Trivial. Suppose U is open. Suppose also \mathcal{U} is an ultrafilter converging to a point $x \in U$.

Then $U \ni x \in U$ so $U \in \mathcal{U}$.

(\Leftarrow) Suppose (ii) holds. We must prove U is open. If not, then there is some $x \in U$ such that every open nbhd of x meets $X - U$ (i.e. has points outside U). The collection

$\{\text{open nbhds of } x\} \cup \{X - U\}$ has f.i.p.

It generates a filter which refines to an ultrafilter $\mathcal{U} \ni x \in U$. By (ii), $U \in \mathcal{U}$.

Also $X - U \in \mathcal{U}$, contradiction. \square

Let $f: X \rightarrow Y$. Given an ultrafilter \mathcal{U} on X , f pushes \mathcal{U} forward to an ultrafilter $f_*\mathcal{U}$ on Y . This works just like for measures. If μ was a measure on X then for each measurable subset $A \subseteq X$, $\mu(A) \in [0, \infty]$. We'll be interested in probability measures so $\mu(A) \in [0, 1]$, $\mu(\emptyset) = 0$, $\mu(X) = 1$, $\mu(A \cup B) = \mu(A) + \mu(B)$. "Measure" usually require countable additivity (stronger than finite additivity) so when it's only finitely additive we call μ a finitely additive measure. Ultrafilters can be viewed as finitely additive measures. But $\mu(A) = \begin{cases} 1 & \text{if } A \in \mathcal{U} \\ 0 & \text{if } A \notin \mathcal{U} \end{cases}$

In general, measures on X give rise to measures on Y : For every $B \subseteq Y$, $\mu_*(B) = \mu(f^{-1}(B))$.

Check: μ_* is a measure on Y ; it's the push-forward of μ via f .

Special case: Let $f: X \rightarrow Y$, \mathcal{U} ultrafilter on X . Then the pushforward of \mathcal{U} via f is $f_*\mathcal{U} = \{V \subseteq Y: f^{-1}(V) \in \mathcal{U}\}$. Check: this is an ultrafilter on Y .

Theorem 3.2 Let X and Y be top. spaces and let $f: X \rightarrow Y$. Then the following are equivalent:

(i) f is continuous.

(ii) f maps convergent ultrafilters to convergent ultrafilters; more precisely if $\mathcal{U} \downarrow x \in X$ (\mathcal{U} ultrafilter in X) then $f_*\mathcal{U} \downarrow f(x) \in Y$.

Proof (\Rightarrow) Suppose f is continuous, and let \mathcal{U} be an ultrafilter on X such that $\mathcal{U} \downarrow x \in X$. We must show that $f_*\mathcal{U} \downarrow f(x) \in Y$. Given an open nbhd V of $f(x)$ in Y , we must show that $V \in f_*\mathcal{U}$, i.e. show $f^{-1}(V) \in \mathcal{U}$. Since f is continuous, $f^{-1}(V)$ is an open nbhd of x , so $f^{-1}(V) \in \mathcal{U}$.

(\Leftarrow) Suppose (ii). We must show f is continuous. Let $V \subseteq Y$ be open; we must show that $f^{-1}(V)$ is open in X . Let $x \in f^{-1}(V)$ and \mathcal{U} be an ultrafilter converging to x : $\mathcal{U} \downarrow x \in f^{-1}(V)$. By assumption (ii), $f_*\mathcal{U} \downarrow f(x) \in V$. Since V is an open nbhd of $f(x)$ in Y , $V \in f_*\mathcal{U}$ i.e. $f^{-1}(V) \in \mathcal{U}$. By Thm 2.2, $f^{-1}(V)$ is open. \square

Theorem 4.2 Let \mathcal{U} be an ultrafilter on $X = \prod X_\alpha$, and let $x = (x_\alpha)_\alpha \in X$. Then $\mathcal{U} \downarrow x$ iff $(\pi_\alpha)_*\mathcal{U} \downarrow x_\alpha \in X_\alpha$ for all α . ($\pi_\alpha: X \rightarrow X_\alpha$).

Proof (\Rightarrow) Suppose $\mathcal{U} \downarrow x = (x_\alpha)_\alpha \in X$. Since π_α is continuous, $(\pi_\alpha)_*\mathcal{U} \downarrow x_\alpha$ by Theorem 3.2.

Theorem 4.2 Let \mathcal{U} be an ultrafilter on $X = \prod X_\alpha$, and let $x = (x_\alpha)_\alpha \in X$. Then $\mathcal{U} \ni x$ iff $(\pi_\alpha)_* \mathcal{U} \ni x_\alpha \in X_\alpha$ for all α . ($\pi_\alpha: X \rightarrow X_\alpha$, $x \mapsto x_\alpha$).

Proof (\Leftarrow) Suppose $(\pi_\alpha)_* \mathcal{U} \ni x_\alpha \in X_\alpha$ for all α and let $x = (x_\alpha)_\alpha \in X$. We must show $\mathcal{U} \ni x$. Given an arbitrary open nbhd U of x in X , we must show $U \in \mathcal{U}$.

Without loss of generality, U is a subbasic open set of the form

$$U = \pi_\alpha^{-1}(U_\alpha) = \left(\prod_{\beta \neq \alpha} X_\beta \right) \times U_\alpha.$$

(some α)

This follows from Thm 3.2 because π_α is continuous. \square

Theorem 5.1 (Tychonoff) If each X_α is compact then so is $X = \prod X_\alpha$.

Proof Let X_α be compact. Let \mathcal{U} be any ultrafilter on $X = \prod X_\alpha$; we must show that \mathcal{U} converges to at least one point of X . But $(\pi_\alpha)_* \mathcal{U} \ni x_\alpha \in X_\alpha$ for some point x_α since X_α is compact. Let $x = (x_\alpha)_\alpha$ and show $\mathcal{U} \ni x$. This follows from Theorem 4.2. \square

Typical application:

Let V be a normed vector space e.g. $(\mathbb{R}[0,1])$ or $l_\infty = \{ \text{bounded sequences } a \in \mathbb{R}^\omega \}$,
i.e. $\|\cdot\|: V \rightarrow \mathbb{R}$ satisfies

(i) $\|v\| \geq 0$, and equality holds iff $v=0$;

(ii) $\|v+w\| \leq \|v\| + \|w\|$

(iii) $\|cv\| = |c|\|v\|$ for all $c \in \mathbb{R}, v \in V$.

$d(v,w) = \|v-w\|$. A bounded linear functional on V is a map $f: V \rightarrow \mathbb{R}$ such that

• $f(av+bv) = af(u)+bf(v)$ for all $a,b \in \mathbb{R}; u,v \in V$

• there exists $C \in \mathbb{R}$ such that $|f(v)| \leq C\|v\|$ for all $v \in V$.

$V^* = \{ \text{bounded linear functionals on } V \}$ is a normed vector space (but larger than V)
unless $\dim V < \infty$

For $f \in V^*$, $\|f\| = \sup \{ |f(v)| : v \in B \}$, $B = \text{unit ball in } V = \{ v \in V : \|v\| \leq 1 \}$.

$d(f,g) = \|f-g\|$.

$B^* = \{ f \in V^* : \|f\| \leq 1 \} = \{ f \in V^* : |f(v)| \leq \|v\| \text{ for } v \in V \}$.

We can regard $B^* \subseteq [-1,1]^B = \{ \text{functions } B \rightarrow [-1,1] \}$ (B^* consists of all functions $B \rightarrow [-1,1]$
which extend to a linear function on V).

B^* is not compact in the $\|\cdot\|$ topology (unless $\dim V = \dim V^* < \infty$)

E.g. $V = l_\infty$, $B = \{ (a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}, |a_i| \leq 1 \}$ is covered by open balls of radius $\frac{1}{2}$ but
no finite number of these cover B . The point set $\{ (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \dots) \} \subset B$

The product topology in $[-1, 1]^{\mathbb{B}}$ is really the topology of "pointwise convergence" which is weaker than the norm topology i.e. if $f, f_1, f_2, f_3, \dots \in V^*$

then saying $f_n \rightarrow f$ in the norm topology ($\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$) is a much stronger statement than saying $f_n \rightarrow f$ pointwise (for all $v \in V$, $f_n(v) \rightarrow f(v)$ as $n \rightarrow \infty$) which is weaker. In this topology, B^* is embedded as a closed (topological) subspace of $[-1, 1]^{\mathbb{B}}$, hence B^* is compact.

Given a top. space X , we would like to embed X in a "nice" space that we think we understand. An embedding of X in Y is an injection $\iota: X \rightarrow Y$ such that the image $\iota(X) \subseteq Y$ is homeomorphic to X via ι . (ι is continuous and $\iota|_{\iota(X)}: \iota(X) \rightarrow X$ is continuous. In this case X is identified as a subspace of Y as a dense subspace

Eg. a completion of a metric space (X, d) is an embedding of (X, d) in a complete metric space (Y, d') . If moreover ι preserves distances i.e. $d'(\iota(x), \iota(x')) = d(x, x')$ for all $x, x' \in X$ then ι is an isometric embedding.

(Y, d') is complete means that Cauchy sequences converge i.e. if $(y_n)_n$ is a Cauchy sequence in Y (for all $\varepsilon > 0$ there exists N such that $d'(y_m, y_n) < \varepsilon$ whenever $m, n > N$) then there exists $y \in Y$ such that $y_n \rightarrow y$ (i.e. $d(y_n, y) \rightarrow 0$).

Eg. $(\mathbb{Q}, \text{usual distance})$ is a metric space which is not complete. It can be embedded in a complete metric space; there are many ways to do this. eg. $\mathbb{Q} \rightarrow \mathbb{C}$



We regard $\mathbb{Q} \hookrightarrow \mathbb{R}$ as "the" completion of \mathbb{Q} : it is unique up to equivalence.

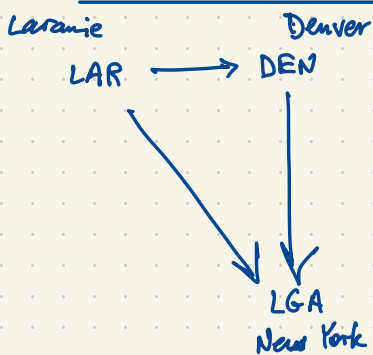
$$a \mapsto a$$

We'll use: If $f: X \rightarrow Y$ is a continuous map, and if Y is Hausdorff, then f is determined by its values on a dense subset of X . ($S \subseteq X$ is dense if every nonempty open subset of X meets S). This says: if $f, g: X \rightarrow Y$ are continuous and they agree on a dense subset $S \subseteq X$, then $f = g$.

Proof Suppose $f \neq g$, i.e. there exists $x \in X$ such that $f(x) \neq g(x)$ in Y . Then there exist open nbhds $f(x) \in U, g(x) \in V, U \cap V = \emptyset$.



Then $f^{-1}(U), g^{-1}(V)$ are open nbhds of $x \in X$. So there exists $s \in S \cap f^{-1}(U) \cap g^{-1}(V)$, so $f(s) = g(s) \in U \cap V$, contradiction. \square



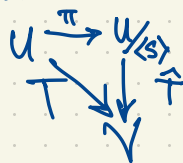
Denver is a "universal hub" for most flights out of Laramie.

for every $T: U \rightarrow V$ vanishing on S , there is a unique $\hat{T}: U/\langle S \rangle \rightarrow V$ making this diagram commute i.e. $T = \hat{T} \circ \pi$.

Go to the category of real vector spaces (objects are real vector spaces; arrows (morphisms) are linear transformations).

Given: U, V vector spaces; $S \subseteq U$ any set of vectors.

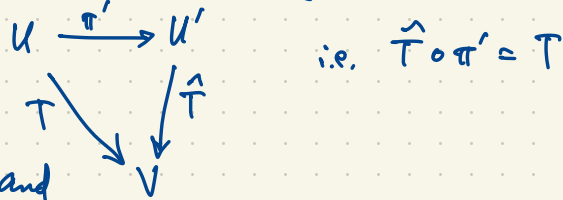
You are looking for a linear transformation $T: U \rightarrow V$ vanishing on S ($Tv = 0$ for all $v \in S$).



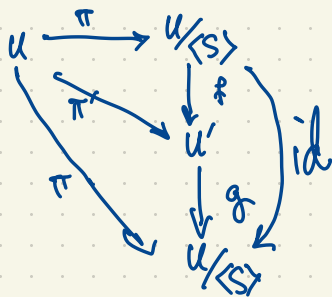
$\langle S \rangle =$ subspace spanned by S
 $\pi: U \rightarrow U/\langle S \rangle$ is the canonical map $u \mapsto u + \langle S \rangle$

The quotient $U/\langle S \rangle$, and in fact the map $\pi: U \rightarrow U/\langle S \rangle$ is the unique such morphism making the above universal property hold.

(up to equivalence). If $\pi': U \rightarrow U'$ also had this universal property i.e. for every $T: U \rightarrow V$ vanishing on S , there is a unique \hat{T} making the diagram



then $U' \cong U/\langle S \rangle$ and more



By uniqueness for π , $g \circ f = \text{id}: U/\langle S \rangle \rightarrow U/\langle S \rangle$

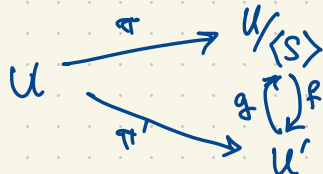
Also $f \circ g = \text{id}: U' \rightarrow U'$

$$f \circ \pi = \pi'$$

$$g \circ \pi' = \pi$$

$$g \circ f \circ \pi = \pi$$

$$\text{id} \circ \pi = \pi$$



This is what we mean when we say $U \xrightarrow{\pi} U/\langle S \rangle$ is "the" universal domain for maps on U vanishing on S .

(Existence requires a construction; uniqueness follows from the universal property.)

Ex. in the category of groups ... objects are groups; arrows (morphisms) are group homomorphisms. Every group G comes with an "abelianization" $G/[G,G]$,

actually $\pi: G \rightarrow G/[G,G]$ (the canonical homomorphism) which makes this into the universal domain for morphisms $G \rightarrow A$ (A abelian). I.e.

given any $f: G \rightarrow A$ (A abelian) there exists a unique $\hat{f}: G/[G,G] \rightarrow A$ making the diagram

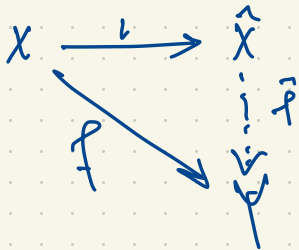


In the category Top whose objects are top. spaces and arrows (morphisms) are continuous maps:

Ex. Let X be a metric space. Then a completion of X is a map $\iota: X \rightarrow \hat{X}$ (continuous) \hat{X} is a complete metric space such that for every $f: X \rightarrow Y$ there is a unique \hat{f} making this

diagram commute:

i.e. $f = \hat{f} \circ \iota$.



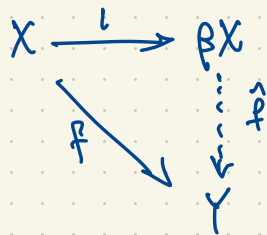
Theorem Every metric space has a completion and it is essentially unique.

It requires moreover that ι be an embedding. Moreover $\iota(X) \subseteq \hat{X}$ must be dense.

Ex. $X = (\mathbb{Q}, \text{usual metric})$ has $\hat{X} = \mathbb{R}$ as its completion.

Ex. Compactification Given a top. space X , we want to define a kind of universal compactification of X , $\iota: X \rightarrow \beta X$ which is Compact Hausdorff and which is the universal object having this property i.e. for every $f: X \rightarrow Y$ where Y is compact Hausdorff, there exists $\hat{f}: \beta X \rightarrow Y$ making the following commute:

i.e. $\hat{f} \circ \iota = f$



Theorem X has such a universal compactification $\iota: X \rightarrow \beta X$ iff X is completely regular and Hausdorff. In this case $\iota: X \rightarrow \beta X$ is unique and it's an embedding of X in βX as a dense subspace. (Stone-Ćech)

Compare: the one-point compactification $X \cup \{\infty\}$ of a space X (where $\infty \notin X$) is a compact space containing X as a subspace i.e. $X \rightarrow X \cup \{\infty\}$, $x \mapsto x$ is an embedding, constructed as follows:

In the new subset $X \subset X \cup \{\infty\}$, basic open nbhds of $x \in X$ are same as in the original space X . Basic open nbhds of ∞ are the complements of the compact subsets of X .

Theorem Given X which is Hausdorff and not compact, the construction above gives a compact extension $X \cup \{\infty\}$ iff X is locally compact i.e. every point of X has a compact nbhd (i.e. every $x \in X$ has an open nbhd contained in a compact subset of X). It's easy to see that local compactness of X is necessary; here we're saying it's also sufficient.

Eg. \mathbb{R}^2 has a one-point compactification $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ (the 2-sphere).



$L: \mathbb{R}^2 \rightarrow S^2$ embedding

Stereographic projection

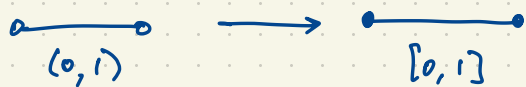
For $n \geq 1$, the one-point compactification of \mathbb{R}^n is S^n .

$\mathbb{R}^1 \cong \text{---} \xrightarrow{L} \text{---}$ is the one-point compactification.

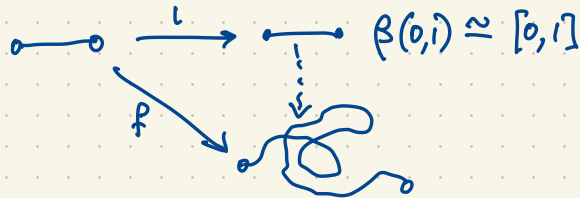
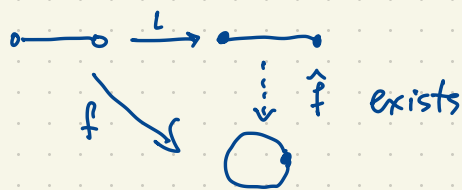
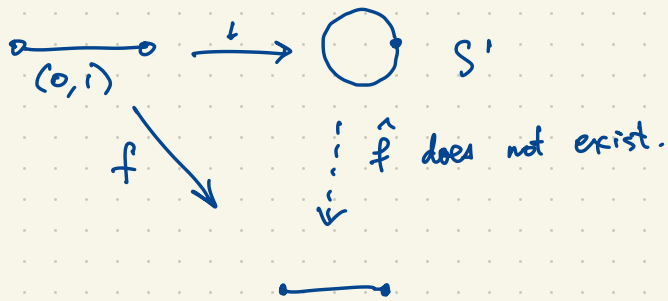
$(0,1)$



Compare: The universal (Stone-Čech) compactification of $\mathbb{R} \simeq (0,1)$ is $[0,1]$.



The one-point compactification lacks the universal property.



What is the Stone-Ćech compactification of \mathbb{R}^2 ?
(universal)

$\mathbb{R}^2 \simeq$ open disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

Some different ways to compactify \mathbb{R}^2 :

- one-point compactification $\mathbb{R}^2 \rightarrow \mathbb{R}^2 \cup \{\infty\} \simeq S^2$
- $\mathbb{P}^2\mathbb{R}$ = real projective plane ($\simeq D \cup \partial D$ where we identify antipodal points on $\partial D \simeq S^1$)
- \bar{D} = closed disk. This is $\beta\mathbb{R}^2 \simeq \beta D$.

The Stone-Ćech compactification theorem is proved in 2 parts:

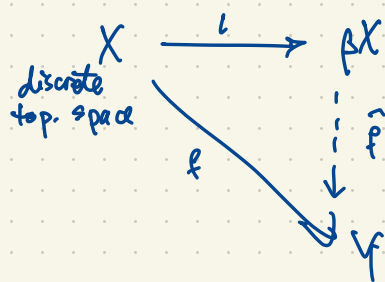
(i) special case X is discrete;

Note: Discrete spaces are completely regular and Hausdorff.

(ii) general case.

Case (i) is not trivial! But it is the key to (ii). From (i), we get (ii) by taking a quotient. To prove (i),

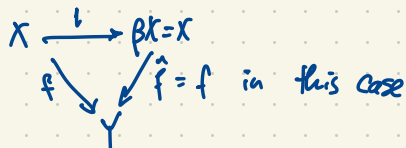
$\beta X = \{ \text{all ultrafilters on } X \}$. = Stone-Ćech compactification of X



Given $x \in X$, $i(x) = \mathcal{F}_x \in \beta X$ $\mathcal{F}_x = \{A \subseteq X : x \in A\}$.
principal ultrafilter.

If X is a finite discrete set then $\beta X = X$, $L: X \rightarrow \beta X$, $x \mapsto x$.

(X finite $\Rightarrow X$ already compact).



What if X is infinite discrete? $\beta X = \{\text{ultrafilters on } X\}$. How big is βX ?

$|\beta X| = 2^{2^{|X|}}$ (doubly exponential in $|X|$). Proof: not hard, see e.g. Keisler.

Eg. X countably infinite, $|X| = \aleph_0$, $\beta X = 2^{2^{\aleph_0}} = |\text{functions } \mathbb{R} \rightarrow \mathbb{R}| = \aleph_2$

Caution: If X is infinite then $2^{|X|} = |\mathcal{P}(X)| = \text{number of subsets of } X > |X|$.

The number of topologies on X is $2^{2^{|X|}}$.

Every ultrafilter \mathcal{U} on X is $\mathcal{U} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{U} \in \mathcal{P}(\mathcal{P}(X))$

Every topology \mathcal{T} on X is $\mathcal{T} \subseteq \mathcal{P}(X) \Rightarrow \mathcal{T} \in \mathcal{P}(\mathcal{P}(X))$.

$\left. \begin{array}{l} \text{no. of ultrafilters;} \\ \text{no. of topologies} \end{array} \right\} \Rightarrow |\mathcal{P}(\mathcal{P}(X))| = 2^{2^{|X|}}$

After proving $|\beta X| = 2^{2^{|X|}}$, it follows that

Corollary If X is infinite then the number of topologies on X is $2^{2^{|X|}}$.

Proof The number of topologies is $\leq 2^{2^{|X|}}$; see above. For the lower bound, if \mathcal{U} is an ultrafilter

on X then $\mathcal{U} \cup \{\emptyset\}$ is a topology on X . This gives an injection

$$\{\text{ultrafilters on } X\} \rightarrow \{\text{topologies on } X\}.$$

So $|\{\text{topologies on } X\}| \geq |\{\text{ultrafilters on } X\}| = 2^{2^{|X|}}$. \square

Remark: For n small, the number of topologies on $\{1, 2, \dots, n\}$ is

n	no. of topologies on $\{1, 2, \dots, n\}$
0	1
1	1
2	4
3	29
4	355
5	6942
6	209527

The topology on $\beta X = \{ \text{ultrafilters on } X \}$ has basis $\{ \langle A \rangle : A \subseteq X \}$ where $\langle A \rangle = \{ \mathcal{U} \in \beta X : A \in \mathcal{U} \}$

The map $A \mapsto \langle A \rangle$, $\mathcal{P}(X) \rightarrow \mathcal{P}(\beta X)$ satisfies

- $\langle \emptyset \rangle = \emptyset$
- $\langle X \rangle = \beta X$
- If $A \subseteq B$ then $\langle A \rangle \subseteq \langle B \rangle$
- $\langle A \cup B \rangle = \langle A \rangle \cup \langle B \rangle$
- $\langle A \cap B \rangle = \langle A \rangle \cap \langle B \rangle$
- $\langle X - A \rangle = \beta X - \langle A \rangle$
- The map $A \mapsto \langle A \rangle$ is injective.

The proof of the Stone-Čech theorem in the case X is discrete is easy.

$\iota: X \rightarrow \iota(X) \subseteq \beta X$ embeds X as a dense discrete subspace of βX . If $x \neq x'$ in X

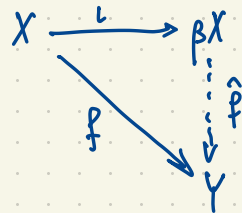
then

$$\iota(x) = \bigcirc_x$$

$$\iota(x') = \bigcirc_{x'}$$

singleton open nbhds.

$$\langle \{x\} \rangle = \bigcirc_x$$



$$\begin{aligned} \iota(x) &= \bigcirc_x \\ &= \langle \{x\} \rangle \\ &= \{ \text{all subsets of } X \\ &\quad \text{containing } x \} \end{aligned}$$

If $f: X \rightarrow Y$ is continuous where Y is compact Hausdorff, then there is a unique continuous $\hat{f}: \beta X \rightarrow Y$ such that $\hat{f} \circ L = f$.