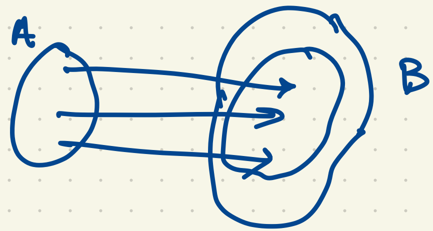
Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \le |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.

Bernstein-Cantor-Schröder Theorem uses ZF



Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?
There is an injection $(0,1) \to [0,1]$, $x \longmapsto x$. So $|(0,1)| \le |[0,1]|$.
There is an injection $[0,1] \to (0,1)$, $x \longmapsto \frac{1}{3}(x+1)$. So $|[0,1]| \le |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \to [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \to [0,1]$, $(x,y,z) \mapsto 0.x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 x_4 y_4 z_4 \cdots$

$$x = 0.x_1 x_2 x_3 x_4 \cdots$$
$$y = 0.y_1 y_2 y_3 y_4 \cdots$$
$$z = 0.z_1 z_2 z_3 z_4 \cdots$$

Theorem $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$|X| = |\mathbb{R}| = 2^{\aleph_0}$

And how many lines do we need to (partition) cover X?

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \to \mathbb{R}^3$ so $|\Sigma| \le |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \to \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} l \xrightarrow{1:1} \Sigma$

Let l be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \quad ?$$

hoping that "in the limit" we cover all of X.

$X = \mathbb{R}^3 - \{0\}$

$\Sigma_0 = \emptyset$.

$\Sigma_1 = \{l_0\}$

$\Sigma_2 = \{l_0, l_1\}$

$\Sigma_3 = \{l_0, l_1, l_2\}$

Well-order the points of X as P_α, $\alpha \in A$ where A is well-ordered. Actually we can take $A = \kappa$ the smallest ordinal such that $|\kappa| = 2^{\aleph_0}$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers P_α whenever $\alpha < \beta$.
- $|\Sigma_\beta| \leq |\beta| < |\kappa| = 2^{\aleph_0}$.
- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$

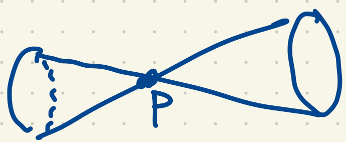
Take $\Sigma = \bigcup_{\beta \in A} \Sigma_\beta$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\mathbb{R}| = 2^{\aleph_0}$ with $P \in X$ not covered by Σ $\quad (P \notin \underbrace{\bigcup \Sigma}_{\text{union of lines in } \Sigma})$,

there exists line ℓ in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.



By the Pigeon hole Principle, ℓ exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizatizability ?
- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y, we have the disjoint union $X \sqcup Y$ which can be viewed as $\underbrace{(X \times \{0\})}_{\{(x,0)\,:\, x \in X\}} \cup \underbrace{(Y \times \{1\})}_{\{(y,1)\,:\, y \in Y\}}$

e.g. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$

$\longleftrightarrow$ $\mathbb{R} \times \{1\}$ = the line $y = 1$

$\longleftrightarrow$ $\mathbb{R} \times \{0\}$ = x-axis $(y = 0)$

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y, a <u>coproduct</u> of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \to X \sqcup Y$, $\iota_1 : Y \to X \sqcup Y$ such that whenever Z is a top. space and $f : X \to Z$, $g : Y \to Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \to Z$ such that this diagram commutes i.e. $(f \sqcup g) =$ see over

$$
\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dotted, "f \sqcup g"'] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"]
\end{tikzcd}
$$

$\iota_0(x) = (x, 0)$, $\iota_1(y) = (y, 1)$

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, "f \sqcup g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"]
\end{tikzcd}$$

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \sqcup g)(x, 0) = f(x) \in Z$$

$$(f \sqcup g)(y, 1) = g(y) \in Z$$

Any $X \sqcup Y$ together with ι_0, ι_1 satisfying this universal property is a (the) coproduct of X and Y. It exists by our construction; and it is unique. If W also satisfies the same universal property then

$$\begin{tikzcd}
 & W & \\
X \arrow[ur, "j_0"] & & Y \arrow[ul, "j_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & W \arrow[d, dashed, shift left] & \\
X \arrow[ur, "j_0"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, shift left] & Y \arrow[ul, "j_1"'] \arrow[l, "\iota_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & W \arrow[dd] & \\
X \arrow[ur, "j_0"] \arrow[dr, "j_0"'] & & Y \arrow[ul, "j_1"'] \arrow[dl, "j_1"] \\
 & W &
\end{tikzcd}$$

Given top. spaces X, Y, a <u>product</u> is a top. space $X \times Y$ together with morphisms (cont. maps) $\pi_0 : X \times Y \longrightarrow X$, $\pi_1 : X \times Y \longrightarrow Y$ such that for every top. space Z and morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique $h : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ X & X\times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

Existence of direct product: $X\times Y = \{(x,y) : x\in X,\ y\in Y\}$.

Topology: $U\times V \subseteq X\times Y$ ($U\subseteq X$, $V\subseteq Y$ open) are a basis for top. on $X\times Y$.

$$\pi_0 : (X,Y) \to X, \quad (x,y) \mapsto x$$

$$\pi_1 : X\times Y \to Y, \quad (x,y) \mapsto y$$

Given $\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$ we have $h(z) = (f(z), g(z))$.

The product topology $X\times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U\times Y$ to be open in $X\times Y$ whenever $U\subseteq X$ is open. Also

" " $\pi_1^{-1}(V) = X\times V$. - - - - - - - . $V\subseteq Y$ . . . . .

Then $U\times V = (U\times Y)\cap(X\times V)$ must be open in $X\times Y$.

Eg. $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ has topology generated by  $U\times V$ ($U, V \in \mathbb{R}$ open).
which is the standard topology.

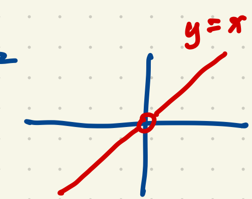
A *topological group* is a group G endowed with a topology such that the maps
$$G \longrightarrow G, \quad g \longmapsto g^{-1}$$
is continuous
and
$$G\times G \longrightarrow G, \quad (g,h) \longmapsto gh$$
is also continuous.

Eg. Consider $f: \underbrace{\mathbb{R}^2}_{\mathbb{R}\times\mathbb{R}} \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

The map $\mathbb{R}\to\mathbb{R}$, $x \mapsto f(x,b)$ is continuous for every $b\in\mathbb{R}$.
〃 〃 〃 $y \mapsto f(a,y)$ 〃 〃 〃 $a\in\mathbb{R}$.

But f is not continuous.
$$f^{-1}(1) = \left\{(x,y)\in\mathbb{R}^2 : f(x,y) = \frac{2xy}{x^2+y^2} = 1\right\}$$
$= \{(x,x)\in\mathbb{R}^2 : x\neq 0\}$. is not closed in $\mathbb{R}^2$.

$2xy = x^2+y^2$
$(x-y)^2 = 0$



$(\mathbb{R}, +)$ is a topological group.

$(\mathbb{R}^*, \times)$ " " " "

$+, \times$ are continuous maps $\mathbb{R}^2 \to \mathbb{R}$.

If $f, g : \mathbb{R} \to \mathbb{R}$ is continuous then so are $f+g$, fg.

One way to see this is

$$\begin{aligned} (f \times g) : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (f(x), g(y)) \end{aligned}$$

is continuous.

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (x, x) & \longmapsto & (f(x), g(x)) & \longmapsto & f(x) + g(x). \end{array}$$

diagonal embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

Similarly for multiplication.

Given a top. space X, is the diagonal embedding $X \to X \times X$, $x \mapsto (x, x)$ always continuous?

Given a metric space (X, d), $d: X \times X \longrightarrow [0,\infty]$,
d is continuous.

metric

This description of product spaces generalizes easily to $X_1 \times X_2 \times \cdots \times X_n$ including $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ as a special case.

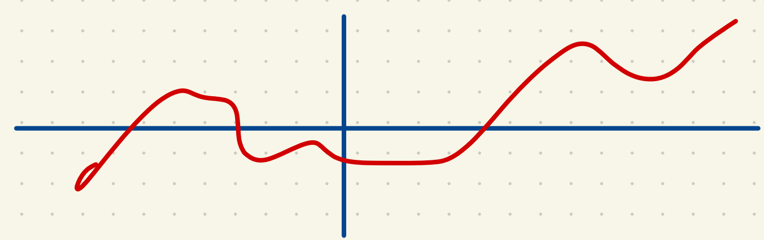
Infinite products are a little bit more subtle.

Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega = \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}\}$

Every function $\omega \longrightarrow \mathbb{R}$
$n \longmapsto a_n$

$\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$



The product topology for $\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R}\to\mathbb{R}\}$ is the coarsest topology for which the projections $f \mapsto f(a)$ $(a\in\mathbb{R})$ are continuous.
This means we require: for every $\underline{\varepsilon>0,\ b\in\mathbb{R}}$, $\{f\in\mathbb{R}^{\mathbb{R}} : f(a)\in B_\varepsilon(b)\}$ is open in $\mathbb{R}^{\mathbb{R}}$.

$B_\varepsilon(b)$ or any open set in $\mathbb{R}$.

$$(\underbrace{\cdot\ -\ -\ -}_{\text{no restriction}},\ f(a),\ \underbrace{\cdot\ -\ -\ -\ -}_{\text{no restriction}}) \in \mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}\times U\times\mathbb{R}\times\cdots$$

General product: Let X_α ($\alpha\in A$, some index set A) be top. spaces.
The <u>product space</u> $\prod_{\alpha\in A} X_\alpha$ has the Cartesian product as its underlying set.
As a set, an element $x=(x_\alpha)_{\alpha\in A}\in\prod_{\alpha\in A}X_\alpha$ is really a function
$A \longrightarrow \bigcup_{\alpha\in A} X_\alpha$ subject to $x_\alpha\in X_\alpha$ for all $\alpha\in A$.
(Special case: all X_α isomorphic to X; $x\mapsto x_\alpha$ is a map $A\to X_\alpha = X$).
If $X_\alpha\neq\emptyset$ for all $\alpha\in A$, then $\prod_{\alpha\in A}X_\alpha\neq\emptyset$. This uses AC = Axiom of Choice

If all $X_\alpha = X$ for all $\alpha \in A$ then $\prod_{\alpha\in A} X_\alpha = X^A = \{\text{functions } A \to X\} \neq \emptyset$ assuming $X \neq \emptyset$. This holds in ZF without requiring AC. Let $x \in X$ and consider the constant function $f(\alpha) = x$ for all $\alpha \in A$. This gives the diagonal embedding $X \to X^A$.

Topology on $\prod_{\alpha\in A} X_\alpha$: A subbasis consists of the open cylinders

$\{x = (x_\alpha)_\alpha : x_\alpha \in X_\alpha$ arbitrary for $\alpha \neq \beta$; $x_\beta \in U\}$ where $\beta \in A$, $U \subseteq X_\beta$ open

$$= \underbrace{U}_{\text{in coordinate } \beta} \times \prod_{\substack{\alpha\in A\\ \alpha\neq\beta}} X_\alpha = \pi_\beta^{-1}(U) \text{ where } \pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$$

$$x = (x_\alpha)_{\alpha\in A} \mapsto x_\beta .$$

Under finite intersections, these generate a basis for the topology on the product space. Basic open sets have the form

$\{x \in (x_\alpha)_{\alpha\in A} : x_{\alpha_i} \in U_{\alpha_i}$ for $i = 1, \dots, k\}$ where $k \geq 1$ is a positive integer; $\alpha_1, \dots, \alpha_k \in A$; $U_{\alpha_i} \subseteq X_{\alpha_i}$ for each $i = 1, \dots, k$ are open sets

Arbitrary open sets are arbitrary unions of basic open sets.
This is the <u>product topology</u> (or the Tychonoff topology).

If instead one takes as basic open sets $\prod_{\alpha\in A} U_\alpha$ ($U_\alpha \subseteq X_\alpha$ open), then one gets the <u>box</u> <u>topology</u>. ↖ Cartesian product

This is a refinement of the product topology.

Unless otherwise specified, the topology on $\prod_{\alpha\in A} X_\alpha$ is understood to be the product topology.

Eg. $\mathbb{R}^{\mathbb{R}} = \prod_{\alpha\in\mathbb{R}} \mathbb{R} = \{\text{functions } \mathbb{R}\to\mathbb{R}\}$

Each function $f: \mathbb{R}\to\mathbb{R}$ determines a point $(f(\alpha))_{\alpha\in\mathbb{R}}$ (a generalized sequence).

A basic open nbhd of $f\in\mathbb{R}^{\mathbb{R}}$ has the form

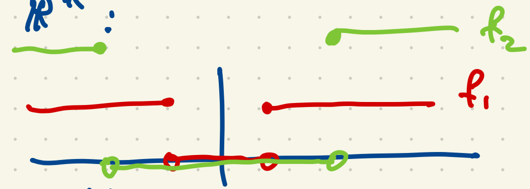
$\{g\in\mathbb{R}^{\mathbb{R}} : g(x_i)\in U_i,\ i=1,2,\cdots,k\}$, U_i is an open nbhd of $f(x_i)$ in $\mathbb{R}$ }.

or specifically

$\{g\in\mathbb{R}^{\mathbb{R}} : |g(x_i) - f(x_i)| < \varepsilon_i,\ i=1,\cdots,k\}$

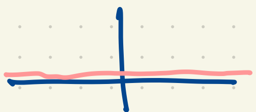
Varying $x_1,\cdots,x_k$, k, $\varepsilon_1,\cdots,\varepsilon_k$ we get a basis for the topology of $\mathbb{R}^{\mathbb{R}}$ in this way.

A convergent sequence of functions in $\mathbb{R}^{\mathbb{R}}$:



$$f_n(x) = \begin{cases} 0, & \text{if } |x| < n; \\ n, & \text{if } |x| \geq n. \end{cases}$$

$f_n \to \underbrace{0}_{\substack{\text{zero} \\ \text{function}}}$ i.e. for any basic open nbhd U of 0, $f_n \in U$ for all $n \gg 0$.



In usual language, $f_n \to 0$ <u>pointwise</u> meaning for all $x \in \mathbb{R}$,
$f_n(x) \to 0$.

In the box topology, $f_n \not\to 0$.

$\omega = \{0, 1, 2, 3, \dots\}$

Take $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{ \overbrace{(a_0, a_1, a_2, a_3, \dots)}^{x} : a_i \in \mathbb{R} \} = \mathbb{R}^{\omega}$
as a set (Cartesian product). Compare product topology, box topology, and topologies from a few norms including

$$\|x\|_1 = \sum_{i \in \omega} |a_i| = |a_0| + |a_1| + |a_2| + \cdots \qquad \|x\|_\infty = \sup |a_i| .$$

$$\|x\|_2 = \left(\sum |a_i|^2 \right)^{1/2}$$

$$\ell^1 = \{x \in \mathbb{R}^\omega : \|x\|_1 < \infty\}$$
$$\ell^2 = \{x \in \mathbb{R}^\omega : \|x\|_2 < \infty\}$$
$$\ell^\infty = \{x \in \mathbb{R}^\omega : \|x\|_\infty < \infty\}$$

$$x_1 = (1, 1, 1, 1, 1, 1, \dots)$$
$$x_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots)$$
$$x_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots)$$
$$\vdots$$
$$x_n = (\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \dots) \in \mathbb{R}^\omega$$

$x_n \to 0 = (0, 0, 0, \dots)$ in the product topology but not in the box topology

In the uniform norm topology, $x_n \to 0$ ($x_n \to 0$ in ℓ^∞).

In the box topology, $\prod_{n=0}^{\infty} \left(-\frac{1}{n+1}, \frac{1}{n+1}\right)$ is a basic open nbhd of 0 and it contains no terms of the sequence $(x_n)_{n \in \omega}$.

Now consider

$$y_1 = (1, 0, 0, 0, 0, \cdots)$$
$$y_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \cdots)$$
$$y_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \cdots)$$

etc.

$$\|y_n\|_1 = 1 < \infty$$
$$\|y_n\|_2 = \frac{1}{\sqrt{n}} < \infty$$
$$\|y_n\|_\infty = \frac{1}{n} < \infty$$

$$y_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \cdots, \tfrac{1}{n}}_{n}, 0, 0, \cdots) \longrightarrow 0$$

in ℓ^2, ℓ^∞, product topology but <u>not</u> in the box topology.

The box topology has $\prod_{n=0}^{\infty} \left(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right)$ as a basic open nbhd of 0 and it contains no term of the sequence of points $(y_n)_n$.

The product topology is sometimes called the topology of pointwise convergence.

The box topology is not usually as useful the other topologies.

A sequence f_n in $\mathbb{R}^A$ converges uniformly to f if for all $\varepsilon > 0$ there exists N such that $|f_n(a) - f(a)| < \varepsilon$ for all $a \in A$ whenever $n > N$.

Basic open sets in the uniform topology look like $U^A = \prod_{a \in A} U$, $U \subseteq \mathbb{R}$ is open.
(finer than the product topology but coarser than the box topology).

If $|A| < \infty$ then the product topology on $\prod_{\alpha \in A} X_\alpha$ agrees with the box topology.

If $|A| = |B|$ then products $\prod_{\alpha \in A} X_\alpha$ and $\prod_{\beta \in B} Y_\beta$ are essentially the same.
(The order of the ~~factors~~ does not alter the definition of the product or box topology.)

$\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R} \times \mathbb{R}}$ in the product topology

The Cantor Space (as a topological space)

$K_1 = [0,1]$

$K_2 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$K_3 = [0,\frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

etc.

$C = \bigcap_{n=1}^{\infty} K_n$ is a compact top. space. $C \subset \mathbb{R}$ and we take the standard topology. It is a metric space.

It is totally disconnected: given $x \neq y$ in C, there exists a partition $C = U \sqcup V$, $U, V \subset C$ open, $x \in U$, $y \in V$.

Equivalently, $C = \{0,1\}^{\omega} = 2^{\omega}$ with the product topology. ($\{0,1\}$ is discrete)

Points of C have the form $(a_0, a_1, a_2, a_3, \cdots)$ where $a_i \in \{0,1\}$.

$|C| = |\mathbb{R}| = 2^{\aleph_0}$

A set of basic open nbhds of $a=(a_0, a_1, a_2, a_3, \dots) \in C$ is the set of $\{ b \in C : \ b_i = a_i \text{ for } i \le n \}$.

A metric defining this topology is

$$d(a, b) = \begin{cases} 0, & \text{if } a=b \\ \frac{1}{2^n}, & \text{if } a_n \neq b_n \text{ for some } n \text{ and we take the smallest such } n. \end{cases}$$

This is really $\mathbb{Z}_2$ = 2-adic integers

$= \{ a \in \mathbb{Q} : \ \|a\|_2 \le 1 \}$.

A homeomorphism

$\{0,1\}^{\mathbb{N}} \longrightarrow$ Usual Cantor set $\bigcap_{n=1}^{\infty} K_n$ is

$$(a_1, a_2, a_3, \dots) \longmapsto \sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

The Cantor Space is the unique compact Hausdorff space without isolated points which is second countable having a countable base of clopen sets.

Second countable: having a countable base.

Separable: having a countable dense subset.

$A \subseteq X$ is dense if $A \cap U \neq \emptyset$ for every open $U \neq \emptyset$.

Tychonoff's Theorem A product of compact spaces is compact.

That is, if X_α $(\alpha\in A)$ is an indexed family of compact spaces, then

$\prod_{\alpha\in A} X_\alpha$ is compact. (NB: We are using the product topology here.)

NB means "take note"

eg. $[0,1]^\omega$ is compact in the product topology.

Not in the box topology eg. for every $a\in\{0,1\}^\omega$ i.e. $a=(a_0,a_1,a_2,\dots)$ $a_i\in\{0,1\}$

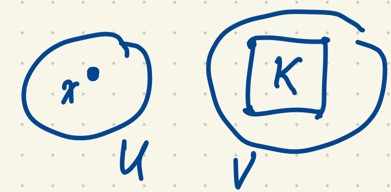
the sets $U_a=\prod_{i\in\omega}U_{a(i)}$ $U_0=[0,\frac{2}{3})$, $U_1=(\frac{1}{3},1]$ open in $[0,1]$

including $U_{(0,1,0,1,1,1,0,1,0,0,\dots)} = U_0\times U_1\times U_0\times U_1\times U_1\times\cdots$

covers $[0,1]^\omega$. No finite number of these U_a's cover $[0,1]^\omega$.

X is Hausdorff if for all $x\neq y$ in X, there are disjoint open nbhds of x and y.

X is regular if for every closed set K and every point $x\notin K$, there are open sets U, V with $U\cap V=\emptyset$, $x\in U$, $K\subseteq V$.



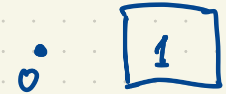
X is normal if

Warning normal spaces are not necessarily regular (unless points are closed)

Eg. $X = \{0,1\}$ This space is normal. It's not regular.

Open sets: $\emptyset, \{0\}, X$

Closed sets: $\emptyset, \{1\}, X$.



Urysohn's Lemma X is a normal top. space iff for every pair of disjoint closed sets, K, L, there exists a continuous function $X \to [0,1]$ such that $f(x) = 0$ for all $x \in K$, $f(x) = 1$ for all $x \in L$.

Metric spaces are Hausdorff, normal and regular.

In any metric space (X, d), $d : X \times X \to [0,\infty)$ is continuous.

If $A \subseteq X$, we can define distance from $x \in X$ to A: $d(x,A) = \inf\limits_{a \in A} d(x,a)$.

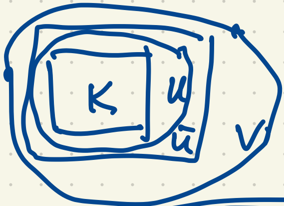
This is a continuous map $X \to [0,\infty)$. $d(x,A) = 0$ iff $x \in \overline{A}$ = closure of A.

$d(A,B) = \inf\limits_{a \in A} d(a,B)$. If A, B are disjoint closed sets then $d(A,B) > 0$.

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}$$

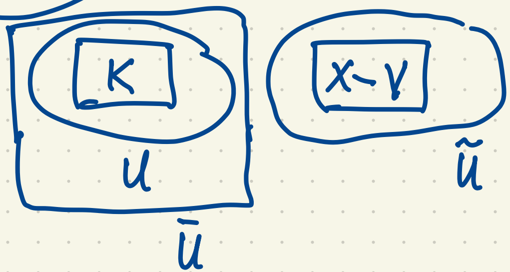
Wed Oct 19 } prerecorded lectures on Baire Category - see website (lecture
Fri Oct 21 } videos + pdfs)

<u>Lemma</u> X is normal iff whenever $K \subseteq V$ with K closed and V open, there exists an open set U such that $K \subseteq U \subseteq \bar{U} \subseteq V$.
($\bar{U}$ = closure of U = smallest closed set containing U).



$X \setminus V = \{x \in X : x \notin V\}$
closed

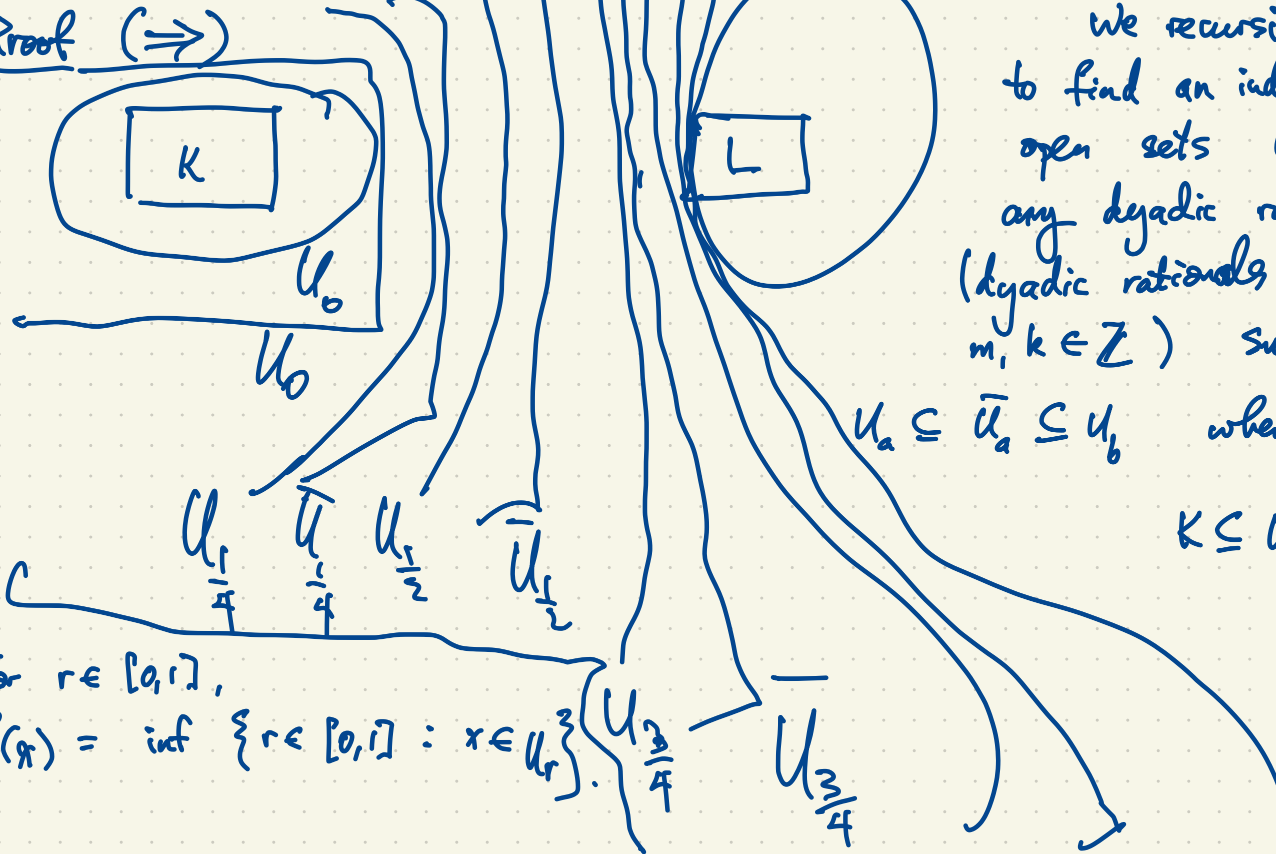
<u>Proof</u>



Proof of Urysohn's Lemma ($\Leftarrow$) Suppose K, L disjoint closed sets in a space X and $f: X \to [0,1]$ is continuous with $f|_K = 0$, $f|_L = 1$.
Let $U = f^{-1}([0,\frac{1}{3})) \subseteq X$ is open, $V = f^{-1}((\frac{2}{3},1]) \subseteq X$ is open.
$U \cap V = \emptyset$, $K \subseteq U$, $L \subseteq V$. So X is normal.

Urysohn's Lemma X is a normal top. space iff for every pair of disjoint closed sets K, L, there exists a continuous function $X \to [0,1]$ such that $f(x) = 0$ for all $x \in K$, $f(x) = 1$ for all $x \in L$.

Proof $(\Rightarrow)$



We recursively use the Lemma to find an indexed collection of open sets U_a where a is any dyadic rational in $[0,1]$ (dyadic rationals have the form $\frac{m}{2^k}$, $m, k \in \mathbb{Z}$) such that

$$U_a \subseteq \overline{U}_a \subseteq U_b \quad \text{whenever} \quad 0 \le a < b \le 1$$

$$K \subseteq U_a, \qquad \overline{U}_a \cap L = \emptyset.$$

For $r \in [0,1]$,

$$f(x) = \inf\{r \in [0,1] : x \in U_r\}.$$

$\square$

Question: If X is regular, i.e. $\dot{x}$ $\boxed{K}$ $x \notin K$

must there exist a continuous function $f: X \to [0,1]$ such that $f(x)=0$, $f|_K = 1$?

No! There is no analogue of Urysohn's Lemma for regularity.

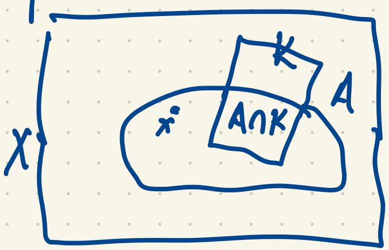
X is <u>completely regular</u> if whenever $\dot{x}$ $\boxed{K}$ K closed, $x \notin K$, there exist continuous $f: X \to [0,1]$, $f(x)=0$, $f|_K = 1$.

There exist top. spaces which are regular but not completely regular (e.g. Tychonoff corkscrew) but we will omit this.

X is <u>completely normal</u> if every subspace of X is normal.

<u>Remarks</u>: If X is completely regular then X is regular (easy) and every subspace of X is also completely regular.

In X:



There exist continuous $f: X \to [0,1]$ such that $f(x)=0$, $f|_K = 1$. Restricting f to $f|_A$, we see that A is also completely regular.

Is every subspace of a normal space normal? No: see Tychonoff's Plank.