

Point Set Topology

Book 1

Let X be a set. A topology on X is a collection \mathcal{J} of subsets of X (called the open sets) such that

(i) $\emptyset, X \in \mathcal{J}$

(ii) \mathcal{J} is closed under finite intersection and arbitrary union, i.e.

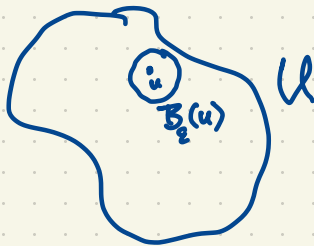
if $U, V \in \mathcal{J}$ then $U \cap V \in \mathcal{J}$;

if $\mathcal{U} \subseteq \mathcal{J}$ then $\bigcup \mathcal{U} \in \mathcal{J}$.

(So for $U, V \in \mathcal{J}$, $U \cup V \in \mathcal{J}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$.)

Example

The standard topology on \mathbb{R}^n : $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open if (standard open set)
for all $u \in U$, there exists $\varepsilon > 0$ such that



$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon\}$.

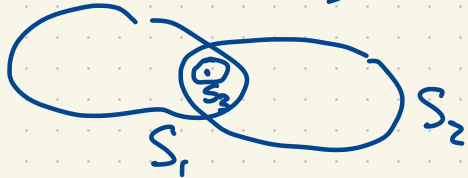
Euclidean distance

(the open ε -ball centered at u .)
$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$$

In other words, a standard open set in \mathbb{R}^n is a union of open balls.

Eg. (More generally) Let X be any set and let \mathcal{S} be a collection of subsets of X which cover X , i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \dots \cap S_k$, $S_1, \dots, S_k \in \mathcal{S}$ is a topology on X . The members of \mathcal{S} are called a sub-basis for this topology and the topology is said to be generated by \mathcal{S} .

\mathcal{S} is called a base (or a basis) for the topology if the topology is the collection of arbitrary unions of elements of \mathcal{S} . This holds iff



for all $S_1, S_2 \in \mathcal{S}$,
and all $u \in S_1 \cap S_2$,
there exists $S_3 \in \mathcal{S}$ such that
 $u \in S_3 \subseteq S_1 \cap S_2$.

Eg. Let X be any set. The discrete topology on X is the collection of all subsets of X . (2^X)

The indiscrete topology on X is $\{\emptyset, X\}$.

If $X = \{0, 1\}$ then there are four possible topologies on X : $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let \mathcal{J} be the collection of complements of finite sets, and \emptyset
 i.e. $\mathcal{J} = \{\emptyset\} \cup \{X - A : A \subseteq X, |A| < \infty\}$, $X - A = \{x \in X : x \notin A\}$.
 set difference

This is a topology on X , called the finite complement topology.

$X - A, X - A, X \setminus A$
 $\emptyset, \emptyset, \emptyset, \emptyset$
 \varnothing nothing

A topological space is a pair (X, \mathcal{J}) where \mathcal{J} is a topology on a set X .

Note: $\bigcup \mathcal{J} = X$. By abuse of language, we often say that X is a topological space.

Let X be a set. A distance function (or metric) on X is a function

$d : X \times X \rightarrow [0, \infty]$ such that for all $x, y, z \in X$,

$$d(x, y) = d(y, x)$$

$d(x, y) \geq 0$ and equality holds iff $x = y$.

$$d(x, z) \leq d(x, y) + d(y, z)$$

The standard topology on \mathbb{R}^n is a metric topology.

The metric $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$ (the Euclidean metric)

$$d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

$d_\infty(x, y) = \max \{|x_1 - y_1|, \dots, |x_n - y_n|\}$ all give the standard topology on \mathbb{R}^n .

In \mathbb{R}^2 , open balls with respect to d_2 , d_1 , d_∞ look like



respectively.

These three metrics define the same topology.

The metric $d(x,y) = \begin{cases} 0, & \text{if } x=y \\ 1, & \text{if } x \neq y \end{cases}$ defines the discrete topology.

A topological space is metrizable if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.

A topology is Hausdorff if for any two points $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.



Every metric space is Hausdorff since if $x \neq y$, $d = d(x,y) > 0$. Take $U = B_{d/3}(x)$, $V = B_{d/3}(y)$

An open neighbourhood of a point $x \in X$ is an open set containing x .

A basic open nbhd of a point $x \in X$ is an open nbhd of x which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.

Consider $X = \mathbb{Q}$. A norm on \mathbb{Q} is a function $\mathbb{Q} \rightarrow [0, \infty)$, $x \mapsto \|x\|$ satisfying

- (i) $\|x\| \geq 0$; equality holds iff $x=0$.
- (ii) $\|xy\| = \|x\| \cdot \|y\|$.
- (iii) $\|x+y\| \leq \|x\| + \|y\|$.

From any norm on \mathbb{Q} , we obtain a metric $d(x,y) = \|x-y\|$.

One way to do this is with the usual absolute value $\|x\| = |x| = |x|_{\infty} = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

This gives the standard topology on \mathbb{Q} .

An alternative is: given $x \in \mathbb{Q}$, if $x=0$ define $\|0\|_2 = 0$.

If $x \neq 0$, write $x = 2^k \cdot \frac{a}{b}$, $a, b, k \in \mathbb{Z}$, $b \neq 0$; a, b odd. Then define $\|x\|_2 = 2^{-k}$.

This is the 2-adic norm on \mathbb{Q} . In fact it satisfies a stronger form of (iii), the ultrametric inequality $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$.

E.g. $\left\| \frac{20}{21} + \frac{5}{14} \right\|_2 = \left\| \frac{10+15}{42} \right\|_2 = \left\| \frac{55}{42} \right\|_2 = 2. = \max \left\{ \underbrace{\left\| \frac{20}{21} \right\|_2}_{\frac{1}{4}}, \underbrace{\left\| \frac{5}{14} \right\|_2}_2 \right\} = 2$

$\left\| \frac{20}{21} \right\|_2 = \frac{1}{4}, \left\| \frac{5}{14} \right\|_2 = 2$

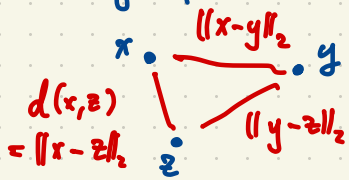
Compare: $\left\| \frac{20}{21} \right\|_2 + \left\| \frac{5}{14} \right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

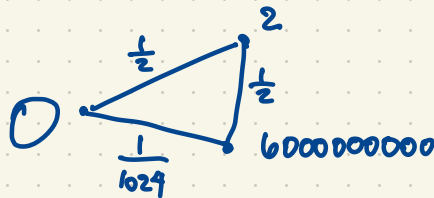
$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, a \text{ even}, b \text{ odd} \right\}.$

Every point in the ball is a centre of the ball i.e. if $c \in B_1(0)$ then $B_1(c) = B_1(0)$.



Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots = -1$$

The partial sums $1, 3, 7, 15, 31, 63, \dots$ converge to -1 in the 2-adic norm.

Note: If $(x_n)_n$ is a sequence of points in a top. space X , we say $(x_n)_n$ converges to $x \in X$ if for every open nbhd U of x , $x_n \in U$ for all n sufficiently large. (This means: for all U open nbhd of x , there exists N such that $x_n \in U$ whenever $n > N$.)



In place of arbitrary open nbhds of x , it suffices to check basic open nbhds. For metric topology, it suffices to check open balls. In this case, $x_n \rightarrow x$ provided that for all $\varepsilon > 0$, there exists N such that

$$\left. \begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array} \right\} \text{ whenever } n > N.$$

In our example above, $d(x_n, x) = 2^{-n} \rightarrow 0$ as $n \rightarrow \infty$.

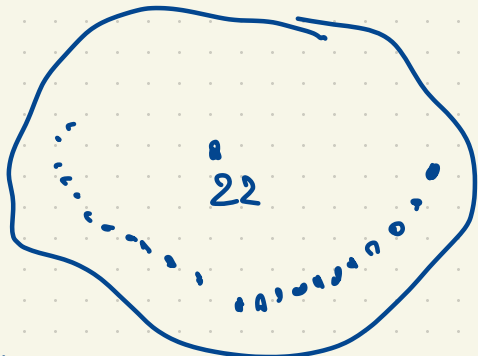
$$\|2^{-n}\| = \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Find the inverse of 5 mod 64.

In $\mathbb{Z}/64\mathbb{Z}$, $\frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - 64 + 256 - 1024 + \dots$
 $= 1 - 4 + 16$ (zero)
 $= 13.$

Eg. in \mathbb{Z} with the finite complement topology, the sequence $(n)_n = (1, 2, 3, \dots)$ converges. It converges to 22.

$(n)_n \rightarrow 22.$



1, 13, 25, 84

In fact for every $a \in \mathbb{Z}$,
 $(n)_n \rightarrow a.$

- 1
- 13
- 25
- 84

Theorem If X is Hausdorff, then every sequence in X has at most one limit. (it converges to at most one point.)

Proof Suppose $a \neq b$ in a Hausdorff space X where a sequence $(x_n)_n \rightarrow a$ and $(x_n)_n \rightarrow b$. Choose disjoint open nbhds U, V of a, b .



There exists N_1 such that $x_n \in U$ for all $n > N_1$; also N_2 such that $x_n \in V$ for all $n > N_2$.

then pick $n > \max\{N_1, N_2\}$ to obtain a contradiction.

We prefer to write $(x_n)_n \rightarrow a$ rather than $\lim_{n \rightarrow \infty} x_n = a$ in general.

In any top. space, closed sets are the complements of open sets.

\emptyset, X are closed

If K, K' are closed then $K \cup K'$ is closed. (So finite unions of closed sets are closed.)

Arbitrary intersections of closed sets are closed.

De Morgan laws: $X - \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X - A_\alpha)$

$$X - \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X - A_\alpha)$$

Given an infinite set X , the finite complement topology has as its closed sets the finite sets and X itself.

Let X be a top. space. Given $A \subseteq X$, the closure of A is the (unique) smallest closed set containing A i.e. $\bar{A} = \bigcap \{K \subseteq X : K \text{ closed, } K \supseteq A\}$.

The interior of A is the largest open set contained in A , i.e. $A^\circ = \bigcup \{U \subseteq A : U \text{ open in } X\}$. $(X - A)^\circ = X - \bar{A}$; $\overline{X - A} = X - A^\circ$.

Theorem There are infinitely many primes.

Known proofs: Euclid's proof (elementary)

Euler's proof (analytic proof: $\sum \frac{1}{p}$ diverges)

This proof is topological.

Proof Form a topology on $X = \mathbb{Z}$ whose basic open sets are the ^(finite) arithmetic progressions
 $\dots, -6, -1, 4, 9, 14, 19, \dots$ for example.

Every nonempty open set is infinite.

Suppose there are only finitely many primes: $|P| < \infty$ is the set of all primes.

$$\{-1, 1\} = \{a \in \mathbb{Z} : a \text{ is not divisible by any prime}\}.$$

$$= \bigcap_{p \in P} \{a \in \mathbb{Z} : a \text{ is not divisible by } p\}$$


$$= \bigcap_{p \in P} (U_{1,p} \cup U_{2,p} \cup \dots \cup U_{p-1,p})$$

$$U_{a,p} = \{x \in \mathbb{Z} : x \equiv a \pmod{p}\}$$

is open. However it has only 2 elements, a contradiction. \square

More generally, let G be a group. Consider the topology on G whose basic open sets are cosets of subgroups $H \leq G$ of finite index, i.e. $gH = \{gh : h \in H\}$, $[G:H] < \infty$.

T_2 : Hausdorff 

T_1 : Points are closed  y

If $x \in X$ and $y \neq x$, then there is an open nbhd U of x with $y \notin U$.

$T_2 \Rightarrow T_1$. Exercise: Give an example of a top. space which is T_1 but not T_2 .

One answer: the finite complement topology for an infinite set.

Let $f: X \rightarrow Y$ be any function. For any $B \subseteq Y$, the preimage of B in X under f is $f^{-1}(B) = \{x \in X : f(x) \in B\}$. Similarly if $A \subseteq X$, the image of A in Y is $f(A) = \{f(a) : a \in A\}$. In general

$$f(f^{-1}(A)) \subseteq A \subseteq f(f(A)).$$

Now let X and Y be top. spaces, i.e. (X, \mathcal{T}) and (Y, \mathcal{T}') .

A function $f: X \rightarrow Y$ is continuous if the preimage of every open set (in Y) is open (in X); i.e. for every $U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

Exercise: Convince yourself that the "standard" definition of continuity for functions $\mathbb{R}^m \rightarrow \mathbb{R}^n$ is just a special case of this. (For the standard topologies on \mathbb{R}^m and \mathbb{R}^n).

Theorem If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous, so is $g \circ f: X \rightarrow Z$.

Proof If $U \subseteq Z$ is open then $g^{-1}(U) \subseteq Y$ is open so $f(g^{-1}(U)) \subseteq X$ is open.

When are two topological spaces X, Y "the same"? ($X \cong Y$: X, Y are homeomorphic)
This means there is a bijection $X \rightarrow Y$ taking one topology to the other.
I.e. there is a bijection $f: X \rightarrow Y$ such that f, f^{-1} are continuous. □

Eg. X is \mathbb{R} with the standard topology;
 Y is \mathbb{R} with the finite complement topology;
 Z is \mathbb{R} with the discrete topology;
 W is \mathbb{R} with the indiscrete topology $\{\emptyset, \mathbb{R}\}$.

$Z \xrightarrow{\iota} X \xrightarrow{\iota} Y \xrightarrow{\iota} W$
 \wedge finest topology on \mathbb{R}
 \wedge coarsest topology on \mathbb{R}

where $\iota(x) = x$.

If $\mathcal{J}, \mathcal{J}'$ are two topologies on X , we say

\mathcal{J}' is finer than \mathcal{J} if $\mathcal{J}' \supset \mathcal{J}$

(\mathcal{J}' is a refinement of \mathcal{J})

Eg. The finite complement topology on X is the coarsest topology for which points are closed. (\mathcal{J}' is coarser than \mathcal{J} if $\mathcal{J}' \subset \mathcal{J}$.)

i.e. any topology in which points are closed is a refinement of the finite complement topology.

Subspace Topology

Let $A \subseteq X$ where X is a topological space $X = (X, \mathcal{T})$.

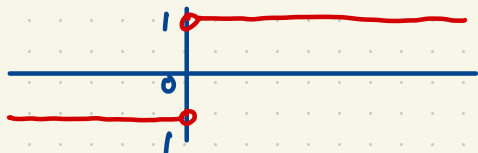
The topology A inherits from X in the most natural way is the subspace topology $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$.

Eg. $[0, 1) = \{a \in \mathbb{R} : 0 \leq a < 1\}$ is neither open nor closed in \mathbb{R} but it is closed in $[0, 1]$ and in $[0, \infty)$ since

$$[0, 1) = (-1, 1) \cap [0, 1] = (-1, 1) \cap [0, \infty).$$

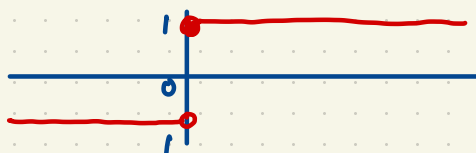
If $f: A \rightarrow \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$, we say f is continuous if it is continuous relative to the standard topology of \mathbb{R}^m and the subspace topology on $A \subseteq \mathbb{R}^n$.

Fig.



continuous

$$f: (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$$



not continuous

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

If $f: A \rightarrow \mathbb{R}^m$ has $f(A) \subseteq B$ we might as well think of f as $f: A \rightarrow B$. To say $f: A \rightarrow \mathbb{R}^m$ is continuous is equivalent to saying $f: A \rightarrow B$ is continuous.

Suppose $f: A \rightarrow B$ is continuous and let $U \subseteq \mathbb{R}^m$. Then $f^{-1}(U) = f^{-1}(U \cap B)$ is open in A . Similarly one proves the converse.

Given $A \subseteq X$ where X is a top. space, there is an inclusion map $\iota: A \rightarrow X$ $\iota(a) = a$. (one-to-one; not onto in general). The subspace topology on A is the coarsest topology for which the inclusion map ι is continuous.

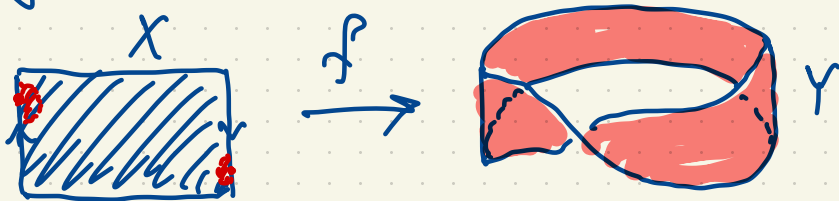
Given $U \subseteq X$ open, $i^{-1}(u) = U \cap A$.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Quotient Topology Suppose $f: X \rightarrow Y$ is onto. Given a topology on $X = (X, \mathcal{T})$, the most natural way this gives a topology on Y is by taking the finest topology on Y for which f is continuous.

A Möbius strip



- There are three ways to think of this situation.
- (i) Identify (collapse) certain points of X together.
 - (ii) We have an equivalence relation on X .
 - (iii) A partition of X .

The quotient topology on Y is the finest topology on Y for which the map $f: X \rightarrow Y$ is continuous.

The quotient topology on $Y = X/\sim$
or X/\sim

is $\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$.

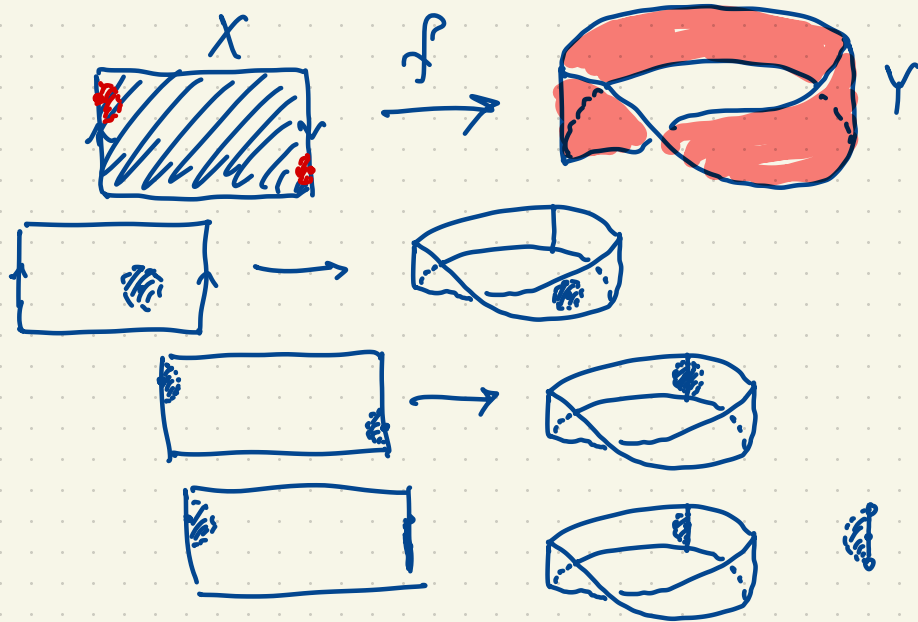
To show this is a topology, use

$$\bigcup_{\alpha} f(A_{\alpha}) = f\left(\bigcup_{\alpha} A_{\alpha}\right)$$

$$\bigcap_{\alpha} f(A_{\alpha}) \supseteq f\left(\bigcap_{\alpha} A_{\alpha}\right)$$

$$\bigcup_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left(\bigcup_{\alpha} A_{\alpha}\right)$$

$$\bigcap_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left(\bigcap_{\alpha} A_{\alpha}\right)$$



$$\sin((-\infty, 0)) = [-1, 1]$$

$$\sin((0, \infty)) = [-1, 1]$$

$$\sin((-\infty, 0) \cap (0, \infty)) = \sin \emptyset = \emptyset$$

$$\sin((-\infty, 0) \cup (0, \infty)) = [-1, 1]$$

(closed) annulus



\approx



\approx



\approx



\approx



closed disk



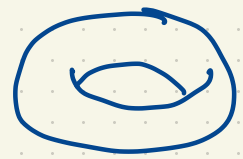
\approx



Möbius strip



\approx



torus



\approx

Klein bottle
not embeddable in \mathbb{R}^3 .



\approx

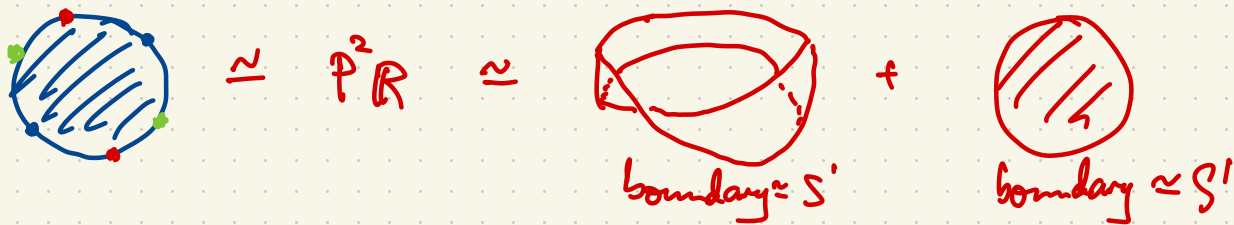
$\mathbb{P}^2_{\mathbb{R}}$ = real projective plane

identify all boundary points



S^2 (2-sphere)

No two of the examples listed here are homeomorphic.



In \mathbb{R}^3 consider the following two subspaces :



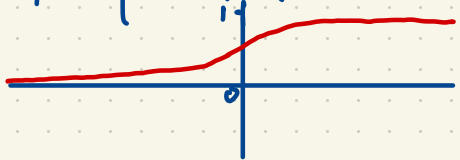
Is $X \cong Y$? Yes.

$S^n = n\text{-sphere} \cong \text{unit sphere in } \mathbb{R}^{n+1} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$.

$S^0 = \cdot \cdot$ $S^1 = \bigcirc$ $S^2 = \text{circle with diagonal lines}$ $S^3 = \mathbb{R}^3 \cup \{\infty\}$

$\mathbb{R} \cong (0,1) \cong (a,b) \stackrel{(0,\infty)}{\cong}$ for $a < b$
(open interval)

An example of a homeomorphism $f: \mathbb{R} \rightarrow (0,\infty)$ is $f(x) = \frac{e^x}{1+e^x}$.



$\mathbb{R} \cong (0,1) \not\cong \begin{cases} [0,1] \\ [0,1) \end{cases}$ Why is $(0,1) \not\cong [0,1)$?

If we remove any point of $(0,1)$, what's left is disconnected. This is not true in $[0,1)$ which has a point 0 whose removal leaves a connected set $(0,1)$. $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ is disconnected since it is a disjoint union of two open sets.

Def. A top. space X is disconnected if $X = U \sqcup V$ where U, V are nonempty open sets in X . If X is not disconnected, then it is connected. In other words, X is connected iff its only clopen sets are \emptyset and X ("clopen" means both open and closed).

$[0, 1]$ is connected. This is a theorem in analysis.

Outline of argument: Suppose $[0, 1] = U \sqcup V$ where U, V are nonempty open. $0 \in U$ without loss of generality. So $[0, \varepsilon) \subseteq U$ for some $\varepsilon > 0$. How large can ε be?

$\{ \varepsilon : [0, \varepsilon) \subseteq U \}$ is a nonempty set with upper bound 1.

So there is a least upper bound. (supremum)

Is this supremum in U or in V ? Either way leads to a contradiction.

If we remove any point from $(0, 1) \cong \mathbb{R} \sqcup \mathbb{R}$, we get a subspace $\cong \mathbb{R} \sqcup \mathbb{R}$ which is disconnected.

This is not true for $[0, 1]$.

\mathbb{Q} is disconnected (in the standard topology in \mathbb{R})

$$\mathbb{Q} = U \sqcup V \quad \text{where} \quad U = \{ x \in \mathbb{Q} : \dots x < \sqrt{2} \} = \mathbb{Q} \cap (-\infty, \sqrt{2})$$

$$V = \{ x \in \mathbb{Q} : \dots x > \sqrt{2} \} = \mathbb{Q} \cap (\sqrt{2}, \infty)$$

\mathbb{Q} is totally disconnected: