

# Point Set Topology

Book 1

Let  $X$  be a set. A topology on  $X$  is a collection  $\mathcal{J}$  of subsets of  $X$  (called the open sets) such that

(i)  $\emptyset, X \in \mathcal{J}$

(ii)  $\mathcal{J}$  is closed under finite intersection and arbitrary union, i.e.

if  $U, V \in \mathcal{J}$  then  $U \cap V \in \mathcal{J}$ ;

if  $\mathcal{U} \subseteq \mathcal{J}$  then  $\bigcup \mathcal{U} \in \mathcal{J}$ .

(So for  $U, V \in \mathcal{J}$ ,  $U \cup V \in \mathcal{J}$ . If  $\{U_\alpha : \alpha \in I\}$  is an indexed collection of open sets, then  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$ .)

Example

The standard topology on  $\mathbb{R}^n$ :  $X = \mathbb{R}^n$ . A set  $U \subseteq \mathbb{R}^n$  is open if (standard open set)  
for all  $u \in U$ , there exists  $\varepsilon > 0$  such that



$$B_\varepsilon(u) \subseteq U.$$

Here  $B_\varepsilon(u) = \{x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon\}$ .

Euclidean distance

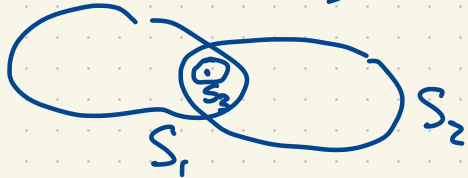
(the open  $\varepsilon$ -ball centered at  $u$ ).

$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \dots + (x_n - u_n)^2}$$

In other words, a standard open set in  $\mathbb{R}^n$  is a union of open balls.

Eg. (More generally) Let  $X$  be any set and let  $\mathcal{S}$  be a collection of subsets of  $X$  which cover  $X$ , i.e.  $\bigcup \mathcal{S} = X$ . Then the collection of all unions of finite intersections  $S_1 \cap S_2 \cap \dots \cap S_k$ ,  $S_1, \dots, S_k \in \mathcal{S}$  is a topology on  $X$ . The members of  $\mathcal{S}$  are called a sub-basis for this topology and the topology is said to be generated by  $\mathcal{S}$ .

$\mathcal{S}$  is called a base (or a basis) for the topology if the topology is the collection of arbitrary unions of elements of  $\mathcal{S}$ . This holds iff



for all  $S_1, S_2 \in \mathcal{S}$ ,  
and all  $u \in S_1 \cap S_2$ ,  
there exists  $S_3 \in \mathcal{S}$  such that  
 $u \in S_3 \subseteq S_1 \cap S_2$ .

Eg. Let  $X$  be any set. The discrete topology on  $X$  is the collection of all subsets of  $X$ . ( $2^X$ )

The indiscrete topology on  $X$  is  $\{\emptyset, X\}$ .

If  $X = \{0, 1\}$  then there are four possible topologies on  $X$ :  $\{\emptyset, X\}$ ,  $\{\emptyset, \{0\}, \{1\}, X\}$ ,  $\{\emptyset, \{0\}, X\}$ ,  $\{\emptyset, \{1\}, X\}$ .

Let  $X$  be an infinite set. Let  $\mathcal{J}$  be the collection of complements of finite sets, and  $\emptyset$   
 i.e.  $\mathcal{J} = \{\emptyset\} \cup \{X - A : A \subseteq X, |A| < \infty\}$ ,  $X - A = \{x \in X : x \notin A\}$ .  
 set difference

This is a topology on  $X$ , called the finite complement topology.

$X - A, X - A, X \setminus A$   
 $\emptyset, \emptyset, \emptyset, \emptyset$   
 \varnothing nothing

A topological space is a pair  $(X, \mathcal{J})$  where  $\mathcal{J}$  is a topology on a set  $X$ .

Note:  $\bigcup \mathcal{J} = X$ . By abuse of language, we often say that  $X$  is a topological space.

Let  $X$  be a set. A distance function (or metric) on  $X$  is a function

$d: X \times X \rightarrow [0, \infty]$  such that for all  $x, y, z \in X$ ,

$$d(x, y) = d(y, x)$$

$d(x, y) \geq 0$  and equality holds iff  $x = y$ .

$$d(x, z) \leq d(x, y) + d(y, z)$$

The standard topology on  $\mathbb{R}^n$  is a metric topology.

The metric  $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$  (the Euclidean metric)

$$d_1(x, y) = |x_1 - y_1| + \dots + |x_n - y_n|$$

$d_\infty(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$  all give the standard topology on  $\mathbb{R}^n$ .

In  $\mathbb{R}^2$ , open balls with respect to  $d_2$ ,  $d_1$ ,  $d_\infty$  look like



respectively.

These three metrics define the same topology.

The metric  $d(x,y) = \begin{cases} 0, & \text{if } x=y \\ 1, & \text{if } x \neq y \end{cases}$  defines the discrete topology.

A topological space is metrizable if its topology can be given by some metric. (not uniquely however)

If  $X$  is an infinite set, then its finite complement topology is not metrizable.

A topology is Hausdorff if for any two points  $x \neq y$ , there exist open sets  $U, V$  such that  $x \in U$ ,  $y \in V$ ,  $U \cap V = \emptyset$ .



Every metric space is Hausdorff since if  $x \neq y$ ,  $d = d(x,y) > 0$ . Take  $U = B_{d/3}(x)$ ,  $V = B_{d/3}(y)$

An open neighbourhood of a point  $x \in X$  is an open set containing  $x$ .

A basic open nbhd of a point  $x \in X$  is an open nbhd of  $x$  which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.

Consider  $X = \mathbb{Q}$ . A norm on  $\mathbb{Q}$  is a function  $\mathbb{Q} \rightarrow [0, \infty)$ ,  $x \mapsto \|x\|$  satisfying

- (i)  $\|x\| \geq 0$ ; equality holds iff  $x=0$ .
- (ii)  $\|xy\| = \|x\| \cdot \|y\|$ .
- (iii)  $\|x+y\| \leq \|x\| + \|y\|$ .

From any norm on  $\mathbb{Q}$ , we obtain a metric  $d(x,y) = \|x-y\|$ .

One way to do this is with the usual absolute value  $\|x\| = |x| = |x|_{\infty} = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$

This gives the standard topology on  $\mathbb{Q}$ .

An alternative is: given  $x \in \mathbb{Q}$ , if  $x=0$  define  $\|0\|_2 = 0$ .

If  $x \neq 0$ , write  $x = 2^k \frac{a}{b}$ ,  $a, b, k \in \mathbb{Z}$ ,  $b \neq 0$ ;  $a, b$  odd. Then define  $\|x\|_2 = 2^{-k}$ .

This is the 2-adic norm on  $\mathbb{Q}$ . In fact it satisfies a stronger form of (iii), the ultrametric inequality  $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$ .

E.g.  $\left\| \frac{20}{21} + \frac{5}{14} \right\|_2 = \left\| \frac{10+15}{42} \right\|_2 = \left\| \frac{55}{42} \right\|_2 = 2. = \max \left\{ \underbrace{\left\| \frac{20}{21} \right\|_2}_{\frac{1}{4}}, \underbrace{\left\| \frac{5}{14} \right\|_2}_2 \right\} = 2$

$\left\| \frac{20}{21} \right\|_2 = \frac{1}{4}, \left\| \frac{5}{14} \right\|_2 = 2$

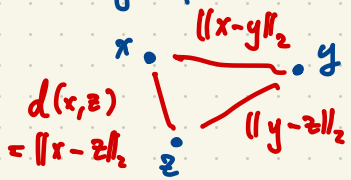
Compare:  $\left\| \frac{20}{21} \right\|_2 + \left\| \frac{5}{14} \right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

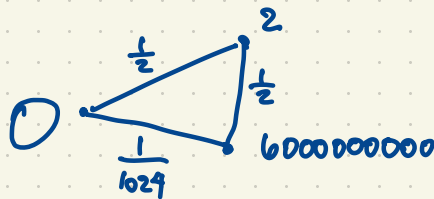
$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$

$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, a \text{ even}, b \text{ odd} \right\}.$

Every point in the ball is a centre of the ball i.e. if  $c \in B_1(0)$  then  $B_1(c) = B_1(0)$ .



Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \dots = -1$$

The partial sums  $1, 3, 7, 15, 31, 63, \dots$  converge to  $-1$  in the 2-adic norm.

Note: If  $(x_n)_n$  is a sequence of points in a top. space  $X$ , we say  $(x_n)_n$  converges to  $x \in X$  if for every open nbhd  $U$  of  $x$ ,  $x_n \in U$  for all  $n$  sufficiently large. (This means: for all  $U$  open nbhd of  $x$ , there exists  $N$  such that  $x_n \in U$  whenever  $n > N$ .)



In place of arbitrary open nbhds of  $x$ , it suffices to check basic open nbhds. For metric topology, it suffices to check open balls. In this case,  $x_n \rightarrow x$  provided that for all  $\varepsilon > 0$ , there exists  $N$  such that

$$\left. \begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array} \right\} \text{ whenever } n > N.$$

In our example above,  $d(x_n, x) = 2^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ .

$$\|2^{-n}\| = \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

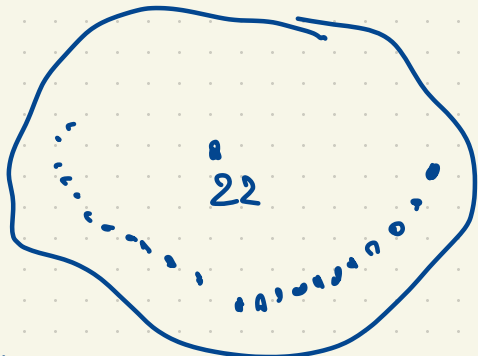
Find the inverse of 5 mod 64.



In  $\mathbb{Z}/64\mathbb{Z}$ ,  $\frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - 64 + 256 - 1024 + \dots$   
 $= 1 - 4 + 16$  (zero)  
 $= 13.$

Eg. in  $\mathbb{Z}$  with the finite complement topology, the sequence  $(n)_n = (1, 2, 3, \dots)$  converges. It converges to 22.

$(n)_n \rightarrow 22.$



1, 13, 25, 84

In fact for every  $a \in \mathbb{Z}$ ,  
 $(n)_n \rightarrow a.$

- 1
- 13
- 25
- 84

Theorem If  $X$  is Hausdorff, then every sequence in  $X$  has at most one limit. (it converges to at most one point.)

Proof Suppose  $a \neq b$  in a Hausdorff space  $X$  where a sequence  $(x_n)_n \rightarrow a$  and  $(x_n)_n \rightarrow b$ . Choose disjoint open nbhds  $U, V$  of  $a, b$ .



There exists  $N_1$  such that  $x_n \in U$  for all  $n > N_1$ ; also  $N_2$  such that  $x_n \in V$  for all  $n > N_2$ .

then pick  $n > \max\{N_1, N_2\}$  to obtain a contradiction.

We prefer to write  $(x_n)_n \rightarrow a$  rather than  $\lim_{n \rightarrow \infty} x_n = a$  in general.

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In any top. space, closed sets are the complements of open sets.

$\emptyset, X$  are closed

If  $K, K'$  are closed then  $K \cup K'$  is closed. (So finite unions of closed sets are closed.)

Arbitrary intersections of closed sets are closed.

De Morgan laws:  $X - \left( \bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X - A_\alpha)$

$$X - \left( \bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X - A_\alpha)$$

Given an infinite set  $X$ , the finite complement topology has as its closed sets the finite sets and  $X$  itself.

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Let  $X$  be a top. space. Given  $A \subseteq X$ , the closure of  $A$  is the (unique) smallest closed set containing  $A$  i.e.  $\bar{A} = \bigcap \{K \subseteq X : K \text{ closed, } K \supseteq A\}$ .

The interior of  $A$  is the largest open set contained in  $A$ , i.e.  $A^\circ = \bigcup \{U \subseteq A : U \text{ open in } X\}$ .  $(X - A)^\circ = X - \bar{A}$ ;  $\overline{X - A} = X - A^\circ$ .

Theorem There are infinitely many primes.

Known proofs: Euclid's proof (elementary)

Euler's proof (analytic proof:  $\sum \frac{1}{p}$  diverges)

This proof is topological.

Proof Form a topology on  $X = \mathbb{Z}$  whose basic open sets are the <sup>(finite)</sup> arithmetic progressions  
 $\dots, -6, -1, 4, 9, 14, 19, \dots$  for example.

Every nonempty open set is infinite.

Suppose there are only finitely many primes:  $|P| < \infty$  is the set of all primes.

$$\{-1, 1\} = \{a \in \mathbb{Z} : a \text{ is not divisible by any prime}\}.$$

$$= \bigcap_{p \in P} \{a \in \mathbb{Z} : a \text{ is not divisible by } p\}$$


$$= \bigcap_{p \in P} (U_{1,p} \cup U_{2,p} \cup \dots \cup U_{p-1,p})$$

$$U_{a,p} = \{x \in \mathbb{Z} : x \equiv a \pmod{p}\}$$

is open. However it has only 2 elements, a contradiction.  $\square$

More generally, let  $G$  be a group. Consider the topology on  $G$  whose basic open sets are cosets of subgroups  $H \leq G$  of finite index, i.e.  $gH = \{gh : h \in H\}$ ,  $[G:H] < \infty$ .

$T_2$ : Hausdorff 

$T_1$ : Points are closed 

If  $x \in X$  and  $y \neq x$ , then there is an open nbhd  $U$  of  $x$  with  $y \notin U$ .

$T_2 \Rightarrow T_1$ . Exercise: Give an example of a top. space which is  $T_1$  but not  $T_2$ .

One answer: the finite complement topology for an infinite set.

Let  $f: X \rightarrow Y$  be any function. For any  $B \subseteq Y$ , the preimage of  $B$  in  $X$  under  $f$  is  $f^{-1}(B) = \{x \in X : f(x) \in B\}$ . Similarly if  $A \subseteq X$ , the image of  $A$  in  $Y$  is  $f(A) = \{f(a) : a \in A\}$ . In general

$$f(f^{-1}(A)) \subseteq A \subseteq f(f(A)).$$

Now let  $X$  and  $Y$  be top. spaces, i.e.  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$ .

A function  $f: X \rightarrow Y$  is continuous if the preimage of every open set (in  $Y$ ) is open (in  $X$ ); i.e. for every  $U \subseteq Y$  open,  $f^{-1}(U) \subseteq X$  is open.

Exercise: Convince yourself that the "standard" definition of continuity for functions  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  is just a special case of this. (For the standard topologies on  $\mathbb{R}^m$  and  $\mathbb{R}^n$ ).

Theorem If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are continuous, so is  $g \circ f: X \rightarrow Z$ .

Proof If  $U \subseteq Z$  is open then  $g^{-1}(U) \subseteq Y$  is open so  $f(g^{-1}(U)) \subseteq X$  is open.

When are two topological spaces  $X, Y$  "the same"? ( $X \cong Y$ :  $X, Y$  are homeomorphic)  
This means there is a bijection  $X \rightarrow Y$  taking one topology to the other.  
I.e. there is a bijection  $f: X \rightarrow Y$  such that  $f, f^{-1}$  are continuous. □

Eg.  $X$  is  $\mathbb{R}$  with the standard topology;  
 $Y$  is  $\mathbb{R}$  with the finite complement topology;  
 $Z$  is  $\mathbb{R}$  with the discrete topology;  
 $W$  is  $\mathbb{R}$  with the indiscrete topology  $\{\emptyset, \mathbb{R}\}$ .

$Z \xrightarrow{\iota} X \xrightarrow{\iota} Y \xrightarrow{\iota} W$  where  $\iota(x) = x$ .

$\wedge$   
finest  
topology  
on  $\mathbb{R}$

$\wedge$   
coarsest  
topology  
on  $\mathbb{R}$

If  $\mathcal{J}, \mathcal{J}'$  are two topologies on  $X$ , we say

$\mathcal{J}'$  is finer than  $\mathcal{J}$  if  $\mathcal{J}' \supset \mathcal{J}$

( $\mathcal{J}'$  is a refinement of  $\mathcal{J}$ )

Eg. The finite complement topology  $\mathcal{J}'$  is coarser than  $\mathcal{J}$  if  $\mathcal{J}' \subset \mathcal{J}$ .  
on  $X$  is the coarsest topology for which points are closed.

i.e. any topology in which points are closed is a refinement of the finite complement topology.

### Subspace Topology

Let  $A \subseteq X$  where  $X$  is a topological space  $X = (X, \mathcal{T})$ .

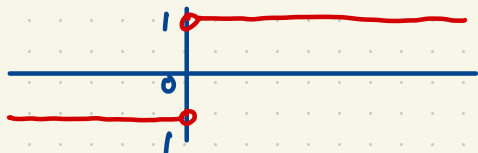
The topology  $A$  inherits from  $X$  in the most natural way is the subspace topology  $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$ .

Eg.  $[0, 1) = \{a \in \mathbb{R} : 0 \leq a < 1\}$  is neither open nor closed in  $\mathbb{R}$  but it is closed in  $[0, 1]$  and in  $[0, \infty)$  since

$$[0, 1) = (-1, 1) \cap [0, 1] = (-1, 1) \cap [0, \infty).$$

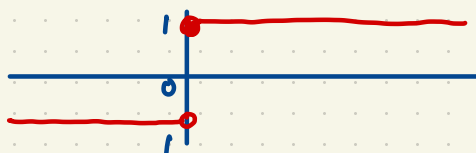
If  $f: A \rightarrow \mathbb{R}^m$  where  $A \subseteq \mathbb{R}^n$ , we say  $f$  is continuous if it is continuous relative to the standard topology of  $\mathbb{R}^m$  and the subspace topology on  $A \subseteq \mathbb{R}^n$ .

Fig.



continuous

$$f: (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{R}$$



not continuous

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

If  $f: A \rightarrow \mathbb{R}^m$  has  $f(A) \subseteq B$  we might as well think of  $f$  as  $f: A \rightarrow B$ . To say  $f: A \rightarrow \mathbb{R}^m$  is continuous is equivalent to saying  $f: A \rightarrow B$  is continuous.

Suppose  $f: A \rightarrow B$  is continuous and let  $U \subseteq \mathbb{R}^m$ . Then  $f^{-1}(U) = f^{-1}(U \cap B)$  is open in  $A$ . Similarly one proves the converse.

Given  $A \subseteq X$  where  $X$  is a top. space, there is an inclusion map  $\iota: A \rightarrow X$   $\iota(a) = a$ . (one-to-one; not onto in general). The subspace topology on  $A$  is the coarsest topology for which the inclusion map  $\iota$  is continuous.

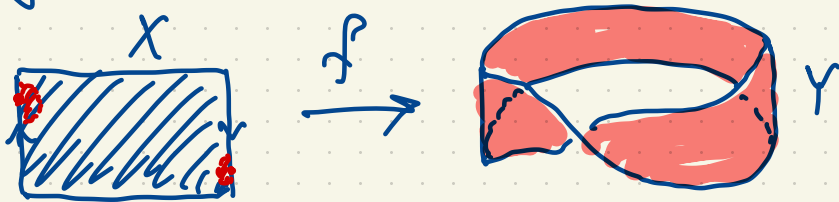
Given  $U \subseteq X$  open,  $i^{-1}(U) = U \cap A$ .

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Quotient Topology Suppose  $f: X \rightarrow Y$  is onto. Given a topology on  $X = (X, \mathcal{T})$ , the most natural way this gives a topology on  $Y$  is by taking the finest topology on  $Y$  for which  $f$  is continuous.

A Möbius strip



- There are three ways to think of this situation.
- (i) Identify (collapse) certain points of  $X$  together.
  - (ii) We have an equivalence relation on  $X$ .
  - (iii) A partition of  $X$ .

The quotient topology on  $Y$  is the finest topology on  $Y$  for which the map  $f: X \rightarrow Y$  is continuous.



The quotient topology on  $Y = X/\sim$   
or  $X/\sim$

is  $\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$ .

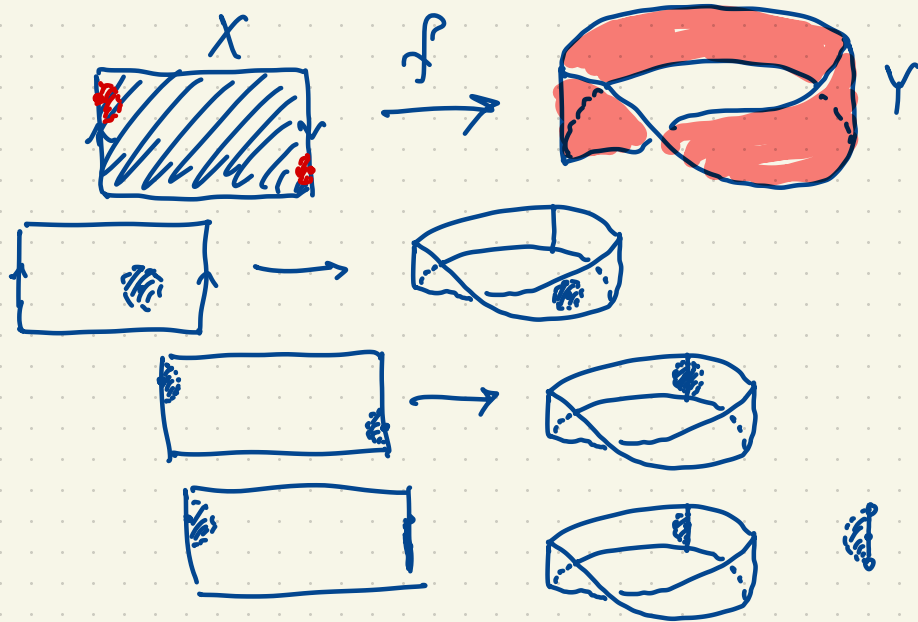
To show this is a topology, use

$$\bigcup_{\alpha} f(A_{\alpha}) = f\left(\bigcup_{\alpha} A_{\alpha}\right)$$

$$\bigcap_{\alpha} f(A_{\alpha}) \supseteq f\left(\bigcap_{\alpha} A_{\alpha}\right)$$

$$\bigcup_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left(\bigcup_{\alpha} A_{\alpha}\right)$$

$$\bigcap_{\alpha} f^{-1}(A_{\alpha}) = f^{-1}\left(\bigcap_{\alpha} A_{\alpha}\right)$$



$$\sin((-\infty, 0)) = [-1, 1]$$

$$\sin((0, \infty)) = [-1, 1]$$

$$\sin((-\infty, 0) \cap (0, \infty)) = \sin \emptyset = \emptyset$$

$$\sin((-\infty, 0) \cup (0, \infty)) = [-1, 1]$$

(closed) annulus



$\approx$



$\approx$



$\approx$



$\approx$



closed disk



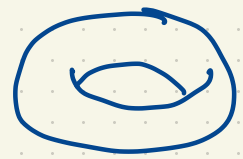
$\approx$



Möbius strip



$\approx$



torus



$\approx$

Klein bottle  
not embeddable in  $\mathbb{R}^3$ .



$\approx$

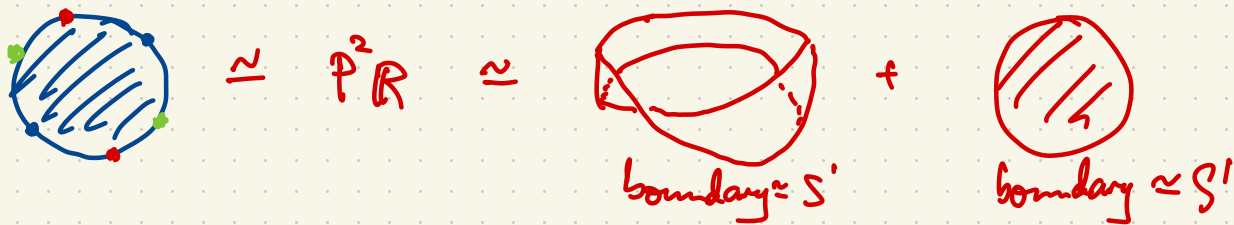
$\mathbb{P}^2_{\mathbb{R}}$  = real projective plane

identify all boundary points

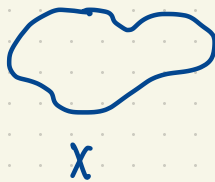


$S^2$  (2-sphere)

No two of the examples listed here are homeomorphic.



In  $\mathbb{R}^3$  consider the following two subspaces :



Is  $X \cong Y$  ? Yes.