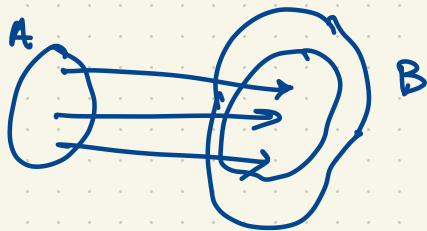# Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem

Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \leq |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.

Bernstein-Cantor-Schröder Theorem uses ZF

Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?

There is an injection $(0,1) \to [0,1]$, $x \mapsto x$. So $|(0,1)| \leq |[0,1]|$.

There is an injection $[0,1] \to (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \leq |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \longrightarrow [0,1]^3, \quad x \longmapsto (x,0,0) \quad$ is an injection.

$[0,1]^3 \longrightarrow [0,1], \quad (x,y,z) \longmapsto 0.x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 x_4 y_4 z_4 \cdots$

$$x = 0.x_1 x_2 x_3 x_4 \cdots$$
$$y = 0.y_1 y_2 y_3 y_4 \cdots$$
$$z = 0.z_1 z_2 z_3 z_4 \cdots$$

Theorem $X = \mathbb{R}^3 \setminus \{0\}$ can be partitioned into lines.

Use transfinite induction.

$|X| = |\mathbb{R}| = 2^{\aleph_0}$

And how many lines do we need to cover X? (partition)

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $\ell \in \Sigma$. This gives an injection $\Sigma \longrightarrow \mathbb{R}^3$ so

$|\Sigma| \leq |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \rightarrow \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} \ell \xrightarrow{1:1} \Sigma$

Let ℓ be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \ ?$$

hoping that "in the limit" we cover all of X.

$$\Sigma_0 = \emptyset.$$
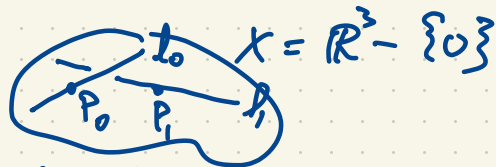
$$\Sigma_1 = \{ \ell_0 \}$$

$$\Sigma_2 = \{ \ell_0, \ell_1 \}$$

$$\Sigma_3 = \{ \ell_0, \ell_1, \ell_2 \}$$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers P_α whenever $\alpha < \beta$.

- $|\Sigma_\beta| \leq |\beta| < |K| = 2^{\aleph_0}$.

- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \leq \gamma$

$$t_0 \quad X = \mathbb{R}^3 - \{0\}$$

Well-order the points of X
as $\quad P_\alpha$, $\alpha \in A$
where A is well-ordered.
Actually we can take $A = K$
the smallest ordinal such that
$|K| = 2^{\aleph_0}$

Take $\Sigma = \bigcup\limits_{\beta \in A} \Sigma_\beta$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\kappa| = 2^{\aleph_0}$ with $P \in X$ not covered by Σ $\quad (P \notin \bigcup \Sigma)$,

$\underbrace{\qquad}_{\substack{\text{union of lines} \\ \text{in } \Sigma}}$

there exists line ℓ in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.
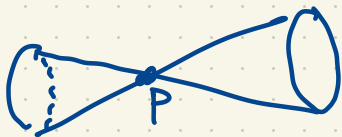
By the Pigeonhole Principle, ℓ exists.

Where are we headed? (Rough plan)
- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizability?

- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y, we have the disjoint union $X \sqcup Y$ which can be viewed as $\underbrace{(X \times \{0\})}_{\{(x,0) : x \in X\}} \cup \underbrace{(Y \times \{1\})}_{\{(y,1) : y \in Y\}}$

eg. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0, 1\} \subset \mathbb{R}^2 \longleftrightarrow \mathbb{R} \times \{1\} =$ the line $y = 1$

$\longleftrightarrow \mathbb{R} \times \{0\} = x$-axis $(y = 0)$

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).
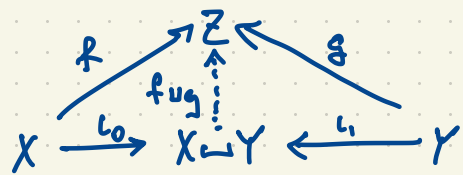
Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y, a coproduct of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \longrightarrow X \sqcup Y$, $\iota_1 : Y \longrightarrow X \sqcup Y$ such that whenever Z is a top. space and $f : X \to Z$, $g : Y \to Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \to Z$ such that this diagram commutes i.e. $(f \sqcup g) =$ see over

$X \xrightarrow{\iota_0} X \sqcup Y \xleftarrow{\iota_1} Y$, with f, g into Z and $f \sqcup g : X \sqcup Y \to Z$

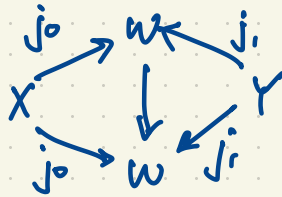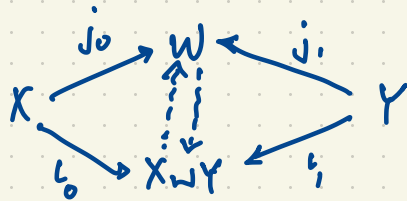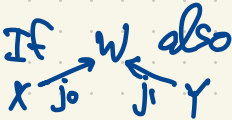$\iota_0(x) = (x, 0), \qquad \iota_1(y) = (y, 1)$

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \sqcup g)(x, 0) = f(x) \in Z$$

$$(f \sqcup g)(y, 1) = g(y) \in Z$$

Any $X \sqcup Y$ together with ι_0, ι_1 satisfying this universal property is a (the) coproduct of X and Y. It exists by our construction; and it is unique. If W also satisfies the same universal property then

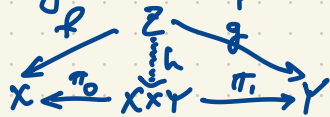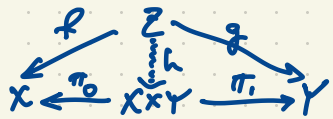(cont. maps)

Given top. spaces X, Y, a **product** is a top. space $X \times Y$ together with morphisms $\pi_0 : X \times Y \longrightarrow X$, $\pi_1 : X \times Y \longrightarrow Y$ such that for every top. space Z and morphisms $f : Z \longrightarrow X$, $g : Z \longrightarrow Y$, there exists unique $h : Z \longrightarrow X \times Y$ such that the following diagram commutes:
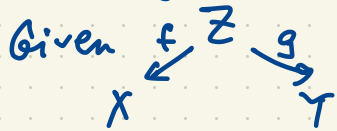
Existence of direct product: $X \times Y = \{(x,y) : x \in X, y \in Y\}$.

Topology: $U \times V \subseteq X \times Y$ $\quad (U \subseteq X, V \subseteq Y \text{ open})$
are a basis for top. on $X \times Y$.

$$\begin{array}{c}
Z \\
f \swarrow \downarrow h \searrow g \\
X \xleftarrow{\pi_0} X \times Y \xrightarrow{\pi_1} Y
\end{array}$$

$\pi_0 : (X, Y) \to X$
$\quad (x, y) \mapsto x$

$\pi_1 : X \times Y \to Y$
$\quad (x, y) \mapsto y$

Given $\begin{array}{c} Z \\ f \swarrow \quad \searrow g \\ X \qquad Y \end{array}$ we have $h(z) = (f(z), g(z))$.

The product topology $X \times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U \times Y$ to be open in $X \times Y$ whenever $U \subseteq X$ is open. Also

`` `` $\pi_1^{-1}(V) = X \times V$ `` `` `` `` `` `` `` `` `` `` `` `` $V \subseteq Y$ `` `` `` `` .

Then $U \times V = (U \times Y) \cap (X \times V)$ must be open in $X \times Y$.

Eg. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ has topology generated by [hatched rectangle] $U \times V$ $\quad$ ($U, V \in \mathbb{R}$ open).
which _is_ the standard topology.

A _topological group_ is a group G endowed with a topology
such that the maps $\quad G \longrightarrow G \qquad$ is continuous
$$g \longmapsto g^{-1}$$

and $\quad G \times G \longrightarrow G \qquad$ is also continuous.
$$(g, h) \longmapsto gh$$

Eg. Consider $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\quad f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$
$\quad \underset{\mathbb{R} \times \mathbb{R}}{}$

The map $\mathbb{R} \to \mathbb{R}$, $\quad x \longmapsto f(x, b)$ is continuous for every $b \in \mathbb{R}$.
$$\qquad\qquad\qquad y \longmapsto f(a, y) \qquad\qquad\qquad\qquad a \in \mathbb{R}.$$

But f is not continuous.
$$f^{-1}(1) = \left\{ (x, y) \in \mathbb{R}^2 : f(x, y) = \frac{2xy}{x^2 + y^2} = 1 \right\}$$
$$= \left\{ (x, x) \in \mathbb{R}^2 : x \neq 0 \right\} \text{ is not closed in } \mathbb{R}^2.$$

$2xy = x^2 + y^2$
$(x - y)^2 = 0$

$y = x$
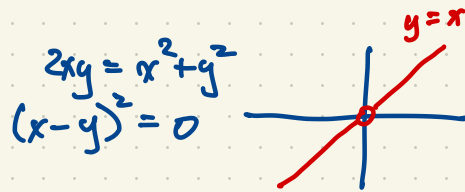
$(\mathbb{R}, +)$ is a topological group.

$(\mathbb{R}^*, \times)$. - --.. ..

$+, \times$ are continuous maps $\mathbb{R}^2 \to \mathbb{R}$.

If $f, g : \mathbb{R} \to \mathbb{R}$ is continuous then so are $f+g$, fg.

One way to see this is

$(f \times g) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$\qquad (x, y) \longmapsto (f(x), g(y))$ is continuous.

$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$x \longmapsto (x, x) \longmapsto (f(x), g(x)) \longmapsto f(x) + g(x).$$

diagonal
embedding of
$\mathbb{R}$ in $\mathbb{R}^2$.

Similarly for multiplication.

Given a top. space X, is the diagonal embedding $X \to X \times X$, $x \longmapsto (x, x)$ always continuous?

Given a metric space (X, d), $d: X \times X \longrightarrow [0, \infty]$,

d is continuous.   (↑ metric)

This description of product spaces generalizes easily to $X_1 \times X_2 \times \dots \times X_n$
including $X^n = \underbrace{X \times X \times \dots \times X}_{n \text{ times}}$ as a special case.

Infinite products are a little bit more subtle.

Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega = \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots = \{(a_0, a_1, a_2, \dots) : a_i \in \mathbb{R}\}$

Every function $\omega \longmapsto \mathbb{R}$
$n \longmapsto a_n$

$\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$

$f \in \mathbb{R}^{\mathbb{R}}$
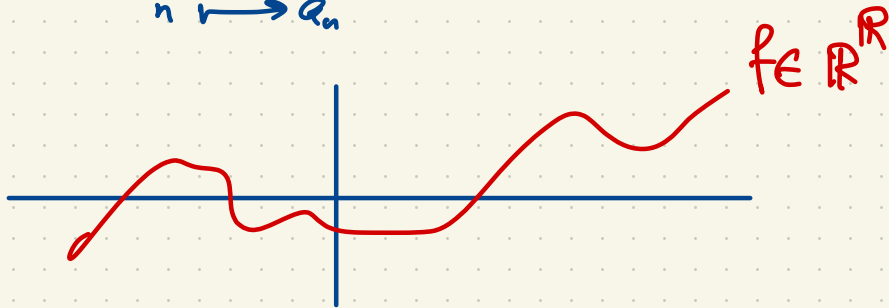
The product topology for $\mathbb{R}^{\mathbb{R}} = \{$ functions $\mathbb{R} \to \mathbb{R}\}$ is the coarsest topology for which the projections $f \mapsto f(a)$ $(a \in \mathbb{R})$ are continuous. This means we require: for every $\varepsilon > 0$, $b \in \mathbb{R}$, $\{f \in \mathbb{R}^{\mathbb{R}} : f(a) \in \underline{B_\varepsilon(b)}\}$ is open in $\mathbb{R}^{\mathbb{R}}$.

$\underline{B_\varepsilon(b)}$ or any open set in $\mathbb{R}$.

$$(\underbrace{\ . \ - - - \ }, \underbrace{f(a)}, \underbrace{\ . \ - - - - \ }) \ \in \ \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \cdots$$

no restriction   no restriction

___

General product: Let X_α $(\alpha \in A$, some index set A) be top. spaces. The <u>product</u> <u>space</u> $\prod_{\alpha \in A} X_\alpha$ has the Cartesian product as its underlying set. As a set, an element $x = (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$ is really a function $A \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ subject to $x_\alpha \in X_\alpha$ for all $\alpha \in A$. (Special case: all X_α isomorphic to X ; $x \mapsto x_\alpha$ is a map $A \to X_\alpha = X$). If $X_\alpha \neq \emptyset$ for all $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha \neq \emptyset$. This uses AC = Axiom of Choice.

If all $X_\alpha = X$ for all $\alpha \in A$ then $\prod_{\alpha \in A} X_\alpha = X^A = \{$ functions $A \to X \} \neq \emptyset$ assuming $X \neq \emptyset$. This holds in ZF without requiring AC. Let $x \in X$ and consider the constant function $f(\alpha) = x$ for all $\alpha \in A$. This gives the diagonal embedding $X \to X^A$.

Topology on $\prod_{\alpha \in A} X_\alpha$: A <u>subbasis</u> consists of the open cylinders

$$\{ x = (x_\alpha)_\alpha : x_\alpha \in X_\alpha \text{ arbitrary for } \alpha \neq \beta; \quad x_\beta \in U \} \qquad \text{where } \beta \in A, \; U \subseteq X_\beta \text{ open}$$

$$= \underbrace{U}_{\text{in coordinate } \beta} \times \underset{\substack{\alpha \in A \\ \alpha \neq \beta}}{\prod} X_\alpha \qquad = \pi_\beta^{-1}(U) \text{ where } \pi_\beta : \underset{\alpha \in A}{\prod} X_\alpha \to X_\beta$$
$$x = (x_\alpha)_{\alpha \in A} \longmapsto x_\beta.$$

Under finite intersections, these generate a basis for the topology on the product space. Basic open sets have the form

$$\{ x \in (x_\alpha)_{\alpha \in A} : x_{\alpha_i} \in U_{\alpha_i} \text{ for } i = 1, \cdots, k \} \qquad \begin{array}{l} \text{where } k \geq 1 \text{ is a positive integer,} \\ \alpha_1, \cdots, \alpha_k \in A; \\ U_{\alpha_i} \subseteq X_{\alpha_i} \text{ for each } i = 1, \cdots, k \end{array}$$

Arbitrary open sets are ~~unions~~ arbitrary unions of basic open sets. are open sets
This is the <u>product topology</u> (or the Tychonoff topology).

If instead one takes as basic open sets $\prod_{\alpha \in A} U_\alpha$ ($U_\alpha \subseteq X_\alpha$ open,

↖ Cartesian product

then one gets the <u>box</u> topology.

This is a refinement of the product topology. $\prod_{\alpha \in A} X_\alpha$ is understood to

Unless otherwise specified, the topology on

be the product topology.

Eg. $\mathbb{R}^{\mathbb{R}} = \prod_{\kappa \in \mathbb{R}} \mathbb{R} = \{$ functions $\mathbb{R} \longrightarrow \mathbb{R} \}$

Each function $f : \mathbb{R} \longrightarrow \mathbb{R}$ determines a point $(f(\alpha))_{\alpha \in \mathbb{R}}$ (a generalized sequence).

A basic open nbhd of $f \in \mathbb{R}^{\mathbb{R}}$ has the form

$$\{ g \in \mathbb{R}^{\mathbb{R}} : \quad g(x_i) \in U_i , \quad i = 1, 2, \cdots, k \} \quad , \qquad U_i \text{ is an open nbhd of } f(x_i) \text{ in } \mathbb{R} \}.$$
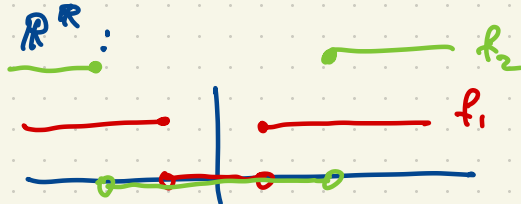
or specifically

$$\{ g \in \mathbb{R}^{\mathbb{R}} : \quad |g(x_i) - f(x_i)| < \varepsilon_i \quad , \quad i = 1, \cdots, k \}$$

Varying $x_1, \cdots, x_k$, k, $\varepsilon_1, \cdots, \varepsilon_k$ we get a basis for the topology of $\mathbb{R}^{\mathbb{R}}$ in this way.

A convergent sequence of functions in $\mathbb{R}^{\mathbb{R}}$:

$$f_n(x) = \begin{cases} 0, & \text{if } |x| < n; \\ n, & \text{if } |x| \geq n. \end{cases}$$

$f_n \longrightarrow \underset{\substack{\text{zero} \\ \text{function}}}{0}$ i.e. for any basic open nbhd U of 0, $f_n \in U$ for all $n \gg 0$.

In usual language, $f_n \longrightarrow 0$ ~~pointwise~~ meaning for all $x \in \mathbb{R}$,
$f_n(x) \longrightarrow 0$.

In the box topology, $f_n \not\longrightarrow 0$.

$\omega = \{0, 1, 2, 3, \dots\}$

Take $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{ \overbrace{(a_0, a_1, a_2, a_3, \dots)}^{x} : a_i \in \mathbb{R} \} = \mathbb{R}^{\omega}$
as a set (Cartesian product). Compare product topology, box topology,
and ~~topologies~~ from a few norms including

$$\|x\|_1 = \sum_{i \in \omega} |a_i| = |a_0| + |a_1| + |a_2| + \cdots \qquad \|x\|_\infty = \sup |a_i|.$$

$$\|x\|_2 = \left(\sum |a_i|^2 \right)^{1/2}$$

$$l^1 = \{x \in \mathbb{R}^\omega : \|x\|_1 < \infty\}$$
$$l^2 = \{x \in \mathbb{R}^\omega : \|x\|_2 < \infty\}$$
$$l^\infty = \{x \in \mathbb{R}^\omega : \|x\|_\infty < \infty\}$$

$$x_1 = (1, 1, 1, 1, 1, 1, \dots)$$
$$x_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \dots)$$
$$x_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots)$$
$$\vdots$$
$$x_n = (\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \dots) \in \mathbb{R}^\omega$$

$x_n \longrightarrow 0 = (0, 0, 0, \dots)$ in the product topology but not in the box topology.

In the uniform norm topology, $x_n \longrightarrow 0$ ($x_n \rightarrow 0$ in l^∞).

In the box topology, $\prod\limits_{n=0}^{\infty} (-\tfrac{1}{n+1}, \tfrac{1}{n+1})$ is a basic open nbhd of 0 and it contains no terms of the sequence $(x_n)_{n \in \omega}$.

Now consider
$$y_1 = (1, 0, 0, 0, 0, \ldots)$$
$$y_2 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, \ldots)$$
$$y_3 = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0, \ldots)$$
etc.

$$\|y_n\|_1 = 1 < \infty$$
$$\|y_n\|_2 = \frac{1}{\sqrt{n}} < \infty$$
$$\|y_n\|_\infty = \frac{1}{n} < \infty$$

$$y_n = (\underbrace{\tfrac{1}{n}, \tfrac{1}{n}, \tfrac{1}{n}, \ldots, \tfrac{1}{n}}_{n}, 0, 0, \ldots) \longrightarrow 0 \quad \text{in} \quad \ell^2, \ell^\infty, \text{ product topology}$$

but not in the box topology.

The box topology has $\prod\limits_{n=0}^{\infty} \left(-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right)$ as a basic open nbhd of 0 and it contains no term of the sequence of points $(y_n)_n$.

The product topology is sometimes called the topology of pointwise convergence.

The box topology is not usually as useful the other topologies.

A sequence f_n in $\mathbb{R}^A$ converges uniformly to f if for all $\varepsilon > 0$ there exists N such that $|f_n(a) - f(a)| < \varepsilon$ whenever $n > N$ for all $a \in A$

Basic open sets in the $\underset{\wedge}{\text{uniform}}$ topology look like $U^A = \prod_{a \in A} U$, $U \subseteq \mathbb{R}$ is open.
(finer than the product topology but coarser than the box topology).

If $|A| < \infty$ then the product topology on $\prod_{\alpha \in A} X_\alpha$ agrees with the box topology.

_____

If $|A| = |B|$ then products $\prod_{\alpha \in A} X_\alpha$ and $\prod_{\beta \in B} Y_\beta$ are essentially the same.
(The order of the factors does not alter the definition of the product or box topology.)

$\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R} \times \mathbb{R}}$ in the product topology

The Cantor Space   (as a topological space)

$K_1 = [0,1]$

$K_2 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$K_3 = [0,\frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

   etc.

$C = \bigcap_{n=1}^{\infty} K_n$ is a compact top. space. $C \subset \mathbb{R}$ and
we take the standard topology.   It is a metric space.
It is totally disconnected: given $x \neq y$ in C, there exists a partition

$C = U \sqcup V$, $U, V \subset C$ open, $x \in U, y \in V$.

Equivalently, $C = \{0,1\}^\omega = 2^\omega$ with the product topology.   ($\{0,1\}$ is discrete)
Points of C have the form $(a_0, a_1, a_2, a_3, \ldots)$ where $a_i \in \{0,1\}$.

$|C| = |\mathbb{R}| = 2^{\aleph_0}$

A set of basic open nbhds of $a = (a_0, a_1, a_2, a_3, \dots) \in C$ is the set of
$$\{ b \in C : \quad b_i = a_i \text{ for } i \leq n \}.$$

A metric defining this topology is
$$d(a, b) = \begin{cases} 0, & \text{if } a = b \\ \frac{1}{2^n}, & \text{if } a_n \neq b_n \text{ for some } n \text{ and we take the smallest such } n. \end{cases}$$

This is really $\mathbb{Z}_2 = $ 2-adic integers
$$= \{ a \in \mathbb{Q} : \quad \|a\|_2 \leq 1 \}.$$