Point Set Topology

Book 3

A <u>filter</u> on X is a collection $\mathcal{F}$ consisting of subsets of X such that

- $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$
- If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
- If $A, A' \in \mathcal{F}$ then $A \cap A' \in \mathcal{F}$.

Every ultrafilter is a filter, but not conversely.
A collection $\mathcal{S}$ of subsets of X has the <u>finite intersection property</u> (f.i.p.) if for all $A_1, \dots, A_n \in \mathcal{S}$, $A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$.
A filter has the f.i.p. If $\mathcal{S}$ is any collection of subsets of X having f.i.p. then $\mathcal{S}$ generates a filter:
$$\mathcal{F}_{\mathcal{S}} = \{\text{supersets of finite intersections of sets in } \mathcal{S}\}$$
$$= \{B \subseteq X : \ A_1 \cap A_2 \cap \cdots \cap A_n \subseteq B \text{ for some } A_1, A_2, \dots, A_n \in \mathcal{S}\}.$$
This is the (unique) smallest collection of subsets of X which contains $\mathcal{S}$ and is a filter.

If $\mathcal{F}, \mathcal{F}'$ are filters on X, we say $\mathcal{F}'$ <u>refines</u> $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$.
The collection of all filters on X is partially ordered by refinement.
Given a filter $\mathcal{F}_0$ on X, the collection of filters refining $\mathcal{F}_0$ has a maximal member by Zorn's Lemma. This is guaranteed to be an ultrafilter.

Assume we are given a nonprincipal ultrafilter $\mathcal{U}$ on $\omega = \{0,1,2,3,\dots\}$.
Construction of the nonstandard real numbers (hyperreals) ${}^*\mathbb{R}$ or $\mathbb{R}^*$ or $\hat{\mathbb{R}}$.
$\hat{\mathbb{R}}$ and $\mathbb{R}$ are examples of ordered fields. $\hat{\mathbb{R}}$ and $\mathbb{R}$ are very similar from first appearances.
eg. If $f(x) \in \mathbb{R}[x]$ or $\hat{\mathbb{R}}[x]$ (polynomial in x) of degree 3 then f has a root (in $\mathbb{R}$ or $\hat{\mathbb{R}}$ respectively).
If $f' > 0$ then this root is unique. Positive elements have a unique square root.

But: $\mathbb{R}$ is an Archimedean field: it has no infinite or infinitesmal elements. More precisely, if $a \in \mathbb{R}$ satisfies $0 \le a < \frac{1}{n}$ for all $n = 1,2,3,4,\dots$ then $a = 0$.

$\hat{\mathbb{R}}$ has infinitesimal elements (it is Non-Archimedian field).

Construction: Start with $\mathbb{R}^\omega = \{\underbrace{(a_0, a_1, a_2, a_3, \dots)}_{a=} : a_i \in \mathbb{R}\}$ (all sequences of real numbers).

Given $a, b \in \mathbb{R}^\omega$ we can add/multiply/subtract pointwise

$$a \pm b = (a_0 \pm b_0,\ a_1 \pm b_1,\ a_2 \pm b_2, \dots)$$
$$ab = (a_0 b_0,\ a_1 b_1,\ a_2 b_2, \dots)$$

making $\mathbb{R}^\omega$ into a commutative ring with identity $1 = (1,1,1,1,1,\dots)$. It's not a field; it has zero divisors e.g.

$$(1,0,1,0,1,0,\dots)(0,1,0,1,0,1,\dots) = (0,0,0,0,0,0,\dots) = 0 \in \mathbb{R}^\omega.$$

But take an ultrafilter $\mathcal{U}$ on ω ($\mathcal{U}$ nonprincipal).

If $a_i = b_i$ for all $i \in U \in \mathcal{U}$ then $a_i \sim b_i$ (equivalence mod $\mathcal{U}$).

In this case $(0,1,0,1,0,1,\dots) \sim (1,1,1,1,1,1,\dots) = 1$

$(1,0,1,0,1,0,\dots) \sim (0,0,0,0,0,0,\dots) = 0$

Given $a, b \in \mathbb{R}^\omega$, let $A = \{i \in \omega : a_i = b_i\}$. Either $A \in \mathcal{U}$ (in which case $a \sim b$) or $\omega \smallsetminus A \in \mathcal{U}$ (in which case $a \not\sim b$). $\hat{\mathbb{R}} = \mathbb{R}^\omega/\sim = \{[a]_\sim : a \in \mathbb{R}^\omega\}$, $[a]_\sim$ = equiv. class of a = $\{x \in \mathbb{R}^\omega : x \sim a\}$.

$\hat{\mathbb{R}}$ is a field. If $a \ne 0$ then actually $a \not\sim 0$ ($[a]_\sim \ne [0]_\sim$) so $\{i \in \omega : a_i \ne 0\} \in \mathcal{U}$. (most coordinates of a are nonzero). Then $\frac{1}{a} = (\frac{1}{a_i} : i \in \omega)$

$\leftarrow$ Anywhere that $a_i = 0$, ignore or replace by 1.

$a \cdot \frac{1}{a} = 1$

$\hat{\mathbb{R}}$ is an <u>ordered</u> <u>field</u>. Given $a, b \in \hat{\mathbb{R}}$, either $a<b$ or $a=b$ or $b<a$.

$$\omega = \{i \in \omega : a_i < b_i\} \sqcup \{i \in \omega : a_i = b_i\} \sqcup \{i \in \omega : b_i < a_i\}$$

Exactly one of these three sets is an ultrafilter set. Correspondingly, $a<b$ or $a=b$ or $b<a$.

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^\omega$$

Given $a \in \mathbb{R}$, identify with $(a,a,a,a,\dots) \in \mathbb{R}^\omega$. This way $\mathbb{R}$ is embedded in $\mathbb{R}^\omega$.

The standard topology on $\hat{\mathbb{R}}$ is the order topology: basic open sets are open intervals (a,b), $a,b \in \hat{\mathbb{R}}$.

Eg. $\varepsilon = [(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots)]_\sim \in \hat{\mathbb{R}}$ is an infinitesimal.

$\frac{1}{\varepsilon} = [(1,2,3,4,5,\cdots)]_\sim \in \hat{\mathbb{R}}$ is infinite.

$|\mathbb{Q}| = \aleph_0$, $|\mathbb{R}| = 2^{\aleph_0}$, $|\hat{\mathbb{R}}| = 2^{\aleph_0}$

$\mathbb{Q} \subset \mathbb{R} \subset \hat{\mathbb{R}}$

$|\mathbb{R}^\omega| = |\mathbb{R}|^{|\omega|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$

Every hyperreal is either infinite ($a \in \hat{\mathbb{R}}$, $|a| > n$ for every positive integer n) or it's bounded in which case a has a unique standard part $st(a) \in \mathbb{R}$ (the closest real number to a).

To compute $f'(x)$ where $f(x) = x^2+3x+7$ using nonstandard analysis, let $a \in \mathbb{R}$, and we want to compute $f'(a) \in \mathbb{R}$.
Pick $\hat{a} \in \hat{\mathbb{R}}$, $st(\hat{a}) = a$, $\hat{a} - a = \varepsilon$ is an infinitesimal. $f(\hat{a}) - f(a) = f(a+\varepsilon) - f(a) = (a+\varepsilon)^2 + 3(a+\varepsilon) + 7 - (a^2+3a+7) = 2\varepsilon a + \varepsilon^2 + 3a\varepsilon$,

$\frac{f(a+\varepsilon) - f(a)}{\varepsilon} = 2a+3+\varepsilon$, $st(2a+3) = 2a. = f'(a).$

Warm-up to the proof of Tychonoff's Theorem.

Let $\mathcal{S}$ be a collection of subsets of X. $\mathcal{S}$ has the <u>finite intersection property</u> (f.i.p.) if every finite intersection of sets in $\mathcal{S}$ is nonempty i.e.

$$S_1, S_2, \dots, S_n \in \mathcal{S} \implies S_1 \cap S_2 \cap \dots \cap S_n \neq \emptyset.$$

(Recall: if $\mathcal{S}$ has f.i.p. then supersets of finite intersections of sets in $\mathcal{S}$ is a filter.)

<u>Lemma 1.1</u> Let X be a top. space. Then the following are equivalent.

(i) X is compact. (Every open cover of X has a finite subcover.)

(ii) If $\mathcal{S}$ is any collection of closed sets with f.i.p. then $\bigcap \mathcal{S} \neq \emptyset$.

via complementation (use de Morgan's laws)

<u>Proof</u>: exercise.

filter such that for every $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$, not both.

An ultrafilter $\mathcal{U}$ on X <u>converges</u> to a point $x \in X$ if every (open?) nbhd of x is in $\mathcal{U}$. (A nbhd is a superset of an open nbhd.)

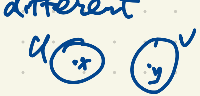
We write $\mathcal{U} \searrow x$ in this case. (Recall: The nbhds of x form a filter.)

Much topology is readily formulated in the language of ultrafilters e.g.

- X is Hausdorff iff every ultrafilter converges to at most one point.
- X is compact iff every ultrafilter converges to at least one point.
- A function $f: X \to Y$ is continuous iff it maps convergent ultrafilters to convergent ultrafilters.

<u>Theorem</u> 2.1(a) Let X be a top. space. Then X is Hausdorff iff every ultrafilter on X converges to at most one point of X.

<u>Proof</u> ($\Longrightarrow$) Suppose X is Hausdorff. Suppose $\mathcal{U}$ is an ultrafilter on X converging to two different points $x \neq y$ in X. There exist $U, V \subseteq X$ disjoint open sets with $x \in U$, $y \in V$.



Since $\mathcal{U} \searrow x$, $U \in \mathcal{U}$. Similarly $V \in \mathcal{U}$. Then $U \cap V = \emptyset \in \mathcal{U}$, contradiction.

($\Longleftarrow$) Suppose every ultrafilter on X converges to at most one point of X.

($\Leftarrow$) Suppose every ultrafilter on X converges to at most one point of X. Let $x \neq y$ in X. By way of contradiction, suppose that $U \cap V \neq \emptyset$ for every open nbhd U of x and every open nbhd V of y. Then

$$\{\text{open nbhds of } x\} \cup \{\text{open nbhds of } y\} \quad \text{has f.i.p.}$$

This generates a filter which in turn refines to an ultrafilter $\mathcal{U}$. $\mathcal{U} \searrow x$, $\mathcal{U} \searrow y$, a contradiction. So X must be Hausdorff.

Theorem 2.1(6) Let X be a top. space. Then X is compact iff every ultrafilter on X converges to at least one point of X.

Proof ($\Rightarrow$) Suppose X is compact. Let $\mathcal{U}$ be an ultrafilter on X. Suppose $\mathcal{U}$ does not converge to any point of X. So for each $x \in X$, there exists an open nbhd U_x of x such that $U_x \notin \mathcal{U}$. So $\{U_x : x \in X\}$ is an open cover of X. So there is a finite subcover

$$X = U_{x_1} \cup U_{x_2} \cup U_{x_3} \cup \dots \cup U_{x_n} \quad \text{for some } n \geq 1; \ x_1, \dots, x_n \in X.$$

So $U_{x_i} \in \mathcal{U}$ for some i, contradiction.

($\Leftarrow$) Suppose every ultrafilter on X converges to at least one point of X. We must show that X is compact. Let $\mathcal{S}$ be a collection of closed subsets of X with f.i.p.; we must show $\bigcap \mathcal{S} \neq \emptyset$. Now $\mathcal{S}$ generates a filter which refines to an ultrafilter $\mathcal{U} \supseteq \mathcal{S}$. By assumption, $\mathcal{U} \searrow x$ for some point $x \in X$. We will show $x \in \bigcap \mathcal{S}$. If not, then there exists $K \in \mathcal{S}$ such that $x \notin K$. Then $X \setminus K$ is an open nbhd of x. So $X \setminus K \in \mathcal{U}$. But also $K \in \mathcal{S} \subseteq \mathcal{U}$, contradiction. $\square$

Ultrafilters gives the following characterization of open sets.

<u>Theorem 2.2</u> Let X be a top. space, and let $U \subseteq X$. The following are equivalent:

(i) U is open.

(ii) Whenever an ultrafilter converges to a point $x \in U$, we have $U \in \mathcal{U}$.

<u>Proof</u> ($\Rightarrow$) Trivial. Suppose U is open. Suppose also $\mathcal{U}$ is an ultrafilter converging to a point $x \in U$.
Then $\mathcal{U} \searrow x \in U$ so $U \in \mathcal{U}$.

($\Leftarrow$) Suppose (ii) holds. We must prove U is open. If not, then there is some $x \in U$ such that every open nbhd of x meets $X \setminus U$ (i.e. has points outside U). The collection

$$\{\text{open nbhds of } x\} \cup \{X \setminus U\} \quad \text{has f.i.p.}$$

It generates a filter which refines to an ultrafilter $\mathcal{U} \searrow x \in U$. By (ii), $U \in \mathcal{U}$.
Also $X \setminus U \in \mathcal{U}$, contradiction. $\square$