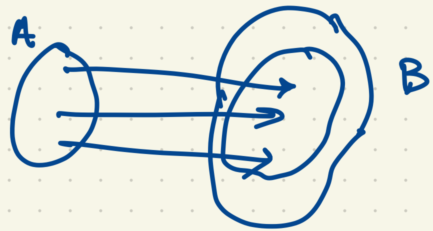
Point Set Topology

Book 2

Bernstein-Cantor-Schröder Theorem Let A, B be sets. If $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$. I.e. if there is an injection $A \to B$ and an injection $B \to A$ then there is a bijection $A \to B$.

Here $|A| \le |B|$ means there is an injection $A \to B$ i.e. A is in one-to-one correspondence with a subset of B. This is equivalent to the existence of a surjection $B \to A$ under the Axiom of Choice.

Bernstein-Cantor-Schröder Theorem uses ZF



Eg. $|(0,1)| = |[0,1]|$ but what is an explicit bijection?
There is an injection $(0,1) \to [0,1]$, $x \mapsto x$. So $|(0,1)| \le |[0,1]|$.
There is an injection $[0,1] \to (0,1)$, $x \mapsto \frac{1}{3}(x+1)$. So $|[0,1]| \le |(0,1)|$.

$$|\mathbb{R}| = |\mathbb{R}^3| = |[0,1]| = |[0,1]^3|$$

$[0,1] \to [0,1]^3$, $x \mapsto (x,0,0)$ is an injection.

$[0,1]^3 \to [0,1]$, $(x,y,z) \mapsto 0.x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 x_4 y_4 z_4 \cdots$

$$x = 0.x_1 x_2 x_3 x_4 \cdots$$
$$y = 0.y_1 y_2 y_3 y_4 \cdots$$
$$z = 0.z_1 z_2 z_3 z_4 \cdots$$

Theorem $X = \mathbb{R}^3 - \{0\}$ can be partitioned into lines.

Use transfinite induction.

$|X| = |\mathbb{R}| = 2^{\aleph_0}$

And how many lines do we need to (partition) cover X?

Let Σ be a set of lines partitioning X. Then $|\Sigma| = 2^{\aleph_0}$.

Pick a point on each $l \in \Sigma$. This gives an injection $\Sigma \to \mathbb{R}^3$ so $|\Sigma| \le |\mathbb{R}^3| = 2^{\aleph_0}$. An injection $\mathbb{R}^3 \to \Sigma$? $\mathbb{R}^3 \xrightarrow{1:1} \mathbb{R} \xrightarrow{1:1} l \xrightarrow{1:1} \Sigma$

Let l be any line in X which is not in Σ.

To construct Σ, we inductively construct a sequence sets of disjoint lines in X

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \cdots \quad ?$$

hoping that "in the limit" we cover all of X.

$\Sigma_0 = \emptyset$.

$\Sigma_1 = \{ l_0 \}$

$\Sigma_2 = \{ l_0, l_1 \}$

$\Sigma_3 = \{ l_0, l_1, l_2 \}$

$X = \mathbb{R}^3 - \{0\}$

Well-order the points of X as p_α, $\alpha \in A$ where A is well-ordered.
Actually we can take $A = \kappa$ the smallest ordinal such that $|\kappa| = 2^{\aleph_0}$

Inductively construct Σ_β, $\beta \in A$, a set of disjoint lines in X, such that

- Σ_β covers p_α whenever $\alpha < \beta$.
- $|\Sigma_\beta| \le |\beta| < |\kappa| = 2^{\aleph_0}$.
- $\Sigma_\beta \subseteq \Sigma_\gamma$ whenever $\beta \le \gamma$

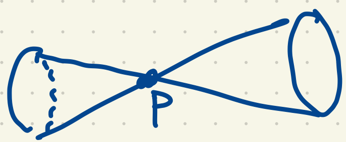
Take $\Sigma = \bigcup_{\beta \in A} \Sigma_\beta$

Key Lemma: (inductive step)

Given a set Σ of disjoint lines in X with $|\Sigma| < |\mathbb{R}| = 2^{\aleph_0}$ with $P \in X$ not covered by Σ $\quad (P \notin \underbrace{\bigcup \Sigma}_{\text{union of lines in } \Sigma})$,

there exists line ℓ in X disjoint from all lines in Σ passing through P.

Consider a cone with vertex P. Every line of Σ hits this cone in at most 2 points. There are $2^{\aleph_0}$ lines in this cone passing through P, at most $|\Sigma| < 2^{\aleph_0}$ hit lines of Σ.



By the Pigeonhole Principle, ℓ exists.

Where are we headed? (Rough plan)

- Product spaces. Tychonoff's Theorem.
- Separation axioms. Urysohn's Lemma.
- Examples: Tychonoff's corkscrew, Tychonoff's Plank
- Metrizatizability?
- Stone-Cech Compactification
- Ultrafilters

Given top. spaces X, Y, we have the disjoint union $X \sqcup Y$ which can be viewed as $\underbrace{(X \times \{0\})}_{\{(x,0):\, x \in X\}} \cup \underbrace{(Y \times \{1\})}_{\{(y,1)\,:\, y \in Y\}}$

eg. $\mathbb{R} \sqcup \mathbb{R} = \mathbb{R} \times \{0,1\} \subset \mathbb{R}^2$ $\longleftrightarrow$ $\mathbb{R} \times \{1\} =$ the line $y=1$

$\longleftrightarrow$ $\mathbb{R} \times \{0\} = x$-axis $(y=0)$

WLOG I will assume X and Y are already disjoint (in order to avoid excessive notation of ordered pairs).

Open sets in $X \sqcup Y$ are of the form $U \sqcup V$ where $U \subseteq X$ is open and $V \subseteq Y$ is open. In fact $X \sqcup Y$ is the coproduct of X and Y in the category-theoretic sense. $X \sqcup Y$ enjoys the following universal property:

Given top. spaces X and Y, a <u>coproduct</u> of X and Y is a top. space $X \sqcup Y$ and two morphisms (continuous maps) $\iota_0 : X \to X \sqcup Y$, $\iota_1 : Y \to X \sqcup Y$ such that whenever Z is a top. space and $f : X \to Z$, $g : Y \to Z$ (note: f, g assumed to be continuous), there exists a morphism $f \sqcup g : X \sqcup Y \to Z$ such that this 'diagram commutes i.e. $(f \sqcup g) =$ see over

$$\begin{tikzcd}
 & Z & \\
X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dotted, "f \sqcup g"] & Y \arrow[ul, "g"'] \arrow[l, "\iota_1"]
\end{tikzcd}$$

$\iota_0(x) = (x,0)$, $\quad \iota_1(y) = (y,1)$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, "f \sqcup g"] & Y \arrow[l, "\iota_1"] \arrow[ul, "g"'] \end{tikzcd}$$

$$X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$$

$$(f \sqcup g)(x,0) = f(x) \in Z$$

$$(f \sqcup g)(y,1) = g(y) \in Z$$

Any $X \sqcup Y$ together with ι_0, ι_1 satisfying this universal property is a (the) coproduct of X and Y. It exists by our construction; and it is unique. If W also satisfies the same universal property then

$$\begin{tikzcd} & W & \\ X \arrow[ur, "j_0"] & & Y \arrow[ul, "j_1"'] \end{tikzcd}$$

$$\begin{tikzcd} & W \arrow[d, dashed, shift left] & \\ X \arrow[ur, "j_0"] \arrow[r, "\iota_0"'] & X \sqcup Y \arrow[u, dashed, shift left] & Y \arrow[ul, "j_1"'] \arrow[l, "\iota_1"] \end{tikzcd}$$

$$\begin{tikzcd} & W \arrow[dd] & \\ X \arrow[ur, "j_0"] \arrow[dr, "j_0"'] & & Y \arrow[ul, "j_1"'] \arrow[dl, "j_1"] \\ & W & \end{tikzcd}$$

Given top. spaces X, Y, a <u>product</u> is a top. space $X \times Y$ together with morphisms (cont. maps) $\pi_0 : X \times Y \to X$, $\pi_1 : X \times Y \to Y$ such that for every top. space Z and morphisms $f : Z \to X$, $g : Z \to Y$, there exists a unique $h : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

Existence of direct product: $X\times Y = \{(x,y) : x\in X,\ y\in Y\}$.

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ X & X\times Y \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & Y \end{tikzcd}$$

Topology: $U\times V \subseteq X\times Y$ $(U\subseteq X,\ V\subseteq Y$ open$)$
are a basis for top. on $X\times Y$.

$$\pi_0 : (X,Y) \to X \qquad\qquad \pi_1 : X\times Y \longrightarrow Y$$
$$(x,y) \mapsto x \qquad\qquad\qquad (x,y) \longmapsto y$$

Given
$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$
we have $h(z) = (f(z), g(z))$.

The product topology $X\times Y$ is the coarsest topology on the Cartesian product for which the two projections π_0, π_1 are continuous.

We require $\pi_0^{-1}(U) = U\times Y$ to be open in $X\times Y$ whenever $U\subseteq X$ is open. Also

" " $\pi_1^{-1}(V) = X\times V$. - - - - - - - . $V\subseteq Y$ . . . . .

Then $U\times V = (U\times Y)\cap(X\times V)$ must be open in $X\times Y$.

Eg. $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ has topology generated by  $U\times V$ $(U, V \in \mathbb{R}$ open$)$.
which is the standard topology.

A <u>topological group</u> is a group G endowed with a topology such that the maps

$$\begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

is continuous

and

$$\begin{aligned} G\times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned}$$

is also continuous.

Eg. Consider $f: \underbrace{\mathbb{R}^2}_{\mathbb{R}\times\mathbb{R}} \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0); \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

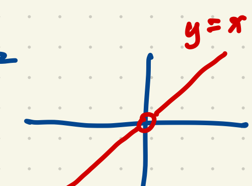
The map $\mathbb{R}\to\mathbb{R}$, $x\mapsto f(x,b)$ is continuous for every $b\in\mathbb{R}$.
" " " " " " " $y\mapsto f(a,y)$ " " " " " " $a\in\mathbb{R}$.

But f is not continuous.

$$f^{-1}(1) = \left\{(x,y)\in\mathbb{R}^2 : f(x,y) = \frac{2xy}{x^2+y^2} = 1\right\}$$

$$= \{(x,x)\in\mathbb{R}^2 : x\neq 0\}.$$ is not closed in $\mathbb{R}^2$.

$2xy = x^2+y^2$
$(x-y)^2 = 0$



$(\mathbb{R}, +)$ is a topological group.

$(\mathbb{R}^*, \times)$ · - — · · · ·

$+, \times$ are continuous maps $\mathbb{R}^2 \to \mathbb{R}$.

If $f, g : \mathbb{R} \to \mathbb{R}$ is continuous then so are $f+g$, fg.

One way to see this is

$$(f \times g) : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \mapsto (f(x), g(y))$$ is continuous.

$$\begin{array}{ccccccc} \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ x & \longmapsto & (x, x) & \longmapsto & (f(x), g(x)) & \longmapsto & f(x) + g(x). \end{array}$$

Similarly for multiplication.

diagonal embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

Given a top. space X, is the diagonal embedding $X \to X \times X$, $x \mapsto (x, x)$ always continuous?

Given a metric space (X, d), $d: X \times X \longrightarrow [0,\infty]$,
metric

d is continuous.

This description of product spaces generalizes easily to $X_1 \times X_2 \times \cdots \times X_n$ including $X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$ as a special case.

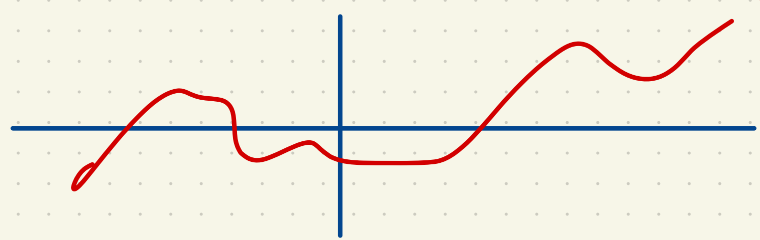
Infinite products are a little bit more subtle.

Notation: $\prod_{\alpha \in I} X_\alpha$ (I some index set)

Special case: $\mathbb{R}^\omega = \prod_{n=0}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \{(a_0, a_1, a_2, \ldots) : a_i \in \mathbb{R}\}$

Every function $\omega \longrightarrow \mathbb{R}$
$n \longmapsto a_n$

$\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$



The product topology for $\mathbb{R}^{\mathbb{R}} = \{\text{functions } \mathbb{R} \to \mathbb{R}\}$ is the coarsest topology for which the projections $f \mapsto f(a)$ $(a \in \mathbb{R})$ are continuous.
This means we require: for every $\underbrace{\varepsilon > 0, b \in \mathbb{R}}$, $\{f \in \mathbb{R}^{\mathbb{R}} : f(a) \in \underbrace{B_\varepsilon(b)}\}$ is open in $\mathbb{R}^{\mathbb{R}}$.

$\underbrace{B_\varepsilon(b)}$ or any open set in $\mathbb{R}$.

$$(\underbrace{\cdot - - - ,}_{\text{no restriction}} \underbrace{f(a)}, \underbrace{\cdot - - - -}_{\text{no restriction}}) \in \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times U \times \mathbb{R} \times \cdots$$

General product: Let X_α ($\alpha \in A$, some index set A) be top. spaces.
The <u>product space</u> $\prod_{\alpha \in A} X_\alpha$ has the Cartesian product as its underlying set.
As a set, an element $x = (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha$ is really a function
$A \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ subject to $x_\alpha \in X_\alpha$ for all $\alpha \in A$.
(Special case: all X_α isomorphic to X; $x \mapsto x_\alpha$ is a map $A \to X_\alpha = X$).
If $X_\alpha \neq \emptyset$ for all $\alpha \in A$, then $\prod_{\alpha \in A} X_\alpha \neq \emptyset$. This uses AC = Axiom of Choice

If all $X_\alpha = X$ for all $\alpha \in A$ then $\prod_{\alpha\in A} X_\alpha = X^A = \{\text{functions } A \to X\} \neq \emptyset$ assuming $X \neq \emptyset$. This holds in ZF without requiring AC. Let $x \in X$ and consider the constant function $f(\alpha) = x$ for all $\alpha \in A$. This gives the diagonal embedding $X \to X^A$.

Topology on $\prod_{\alpha\in A} X_\alpha$: A <u>subbasis</u> consists of the open cylinders

$$\{x = (x_\alpha)_\alpha : x_\alpha \in X_\alpha \text{ arbitrary for } \alpha \neq \beta;\ x_\beta \in U\} \quad \text{where } \beta \in A,\ U \subseteq X_\beta \text{ open}$$

$$= \underbrace{U}_{\text{in coordinate } \beta} \times \prod_{\substack{\alpha\in A \\ \alpha\neq\beta}} X_\alpha = \pi_\beta^{-1}(U) \quad \text{where } \pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta, \quad x = (x_\alpha)_{\alpha\in A} \mapsto x_\beta .$$

Under finite intersections, these generate a basis for the topology on the product space. Basic open sets have the form

$$\{x \in (x_\alpha)_{\alpha\in A} : x_{\alpha_i} \in U_{\alpha_i} \text{ for } i = 1, \dots, k\}$$

where $k \geq 1$ is a positive integer; $\alpha_1, \dots, \alpha_k \in A$; $U_{\alpha_i} \subseteq X_{\alpha_i}$ for each $i = 1, \dots, k$ are open sets

Arbitrary open sets are arbitrary unions of basic open sets. This is the <u>product topology</u> (or the Tychonoff topology).

If instead one takes as basic open sets $\prod_{\alpha\in A} U_\alpha$ ($U_\alpha \subseteq X_\alpha$ open), then one gets the <u>box topology</u>.

This is a refinement of the product topology.

Unless otherwise specified, the topology on $\prod_{\alpha\in A} X_\alpha$ is understood to be the product topology.

Eg. $\mathbb{R}^{\mathbb{R}} = \prod_{\alpha\in\mathbb{R}} \mathbb{R} = \{\text{functions } \mathbb{R}\to\mathbb{R}\}$

Each function $f:\mathbb{R}\to\mathbb{R}$ determines a point $(f(\alpha))_{\alpha\in\mathbb{R}}$ (a generalized sequence).

A basic open nbhd of $f\in\mathbb{R}^{\mathbb{R}}$ has the form

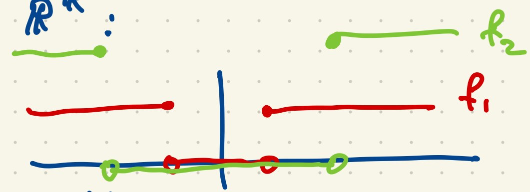
$$\{g\in\mathbb{R}^{\mathbb{R}} : g(x_i)\in U_i,\ i=1,2,\dots,k\}, \quad U_i \text{ is an open nbhd of } f(x_i) \text{ in } \mathbb{R}\}.$$

or specifically

$$\{g\in\mathbb{R}^{\mathbb{R}} : |g(x_i)-f(x_i)| < \varepsilon_i,\ i=1,\dots,k\}$$

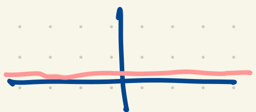
Varying $x_1,\dots,x_k$, k, $\varepsilon_1,\dots,\varepsilon_k$ we get a basis for the topology of $\mathbb{R}^{\mathbb{R}}$ in this way.

A convergent sequence of functions in $\mathbb{R}^{\mathbb{R}}$:



$$f_n(x) = \begin{cases} 0, & \text{if } |x| < n; \\ n, & \text{if } |x| \geq n. \end{cases}$$

$f_n \to \underbrace{0}_{\substack{\text{zero} \\ \text{function}}}$ i.e. for any basic open nbhd U of 0, $f_n \in U$ for all $n \gg 0$.



In usual language, $f_n \to 0$ <u>pointwise</u> meaning for all $x \in \mathbb{R}$,
$f_n(x) \longrightarrow 0$.

In the box topology, $f_n \not\to 0$.