Point Set Topology

Book 1

Let X be a set. A <u>topology</u> on X is a collection $\mathcal{T}$ of subsets of X (called the <u>open</u> sets) such that

(i) $\emptyset, X \in \mathcal{T}$

(ii) $\mathcal{T}$ is closed under finite intersection and arbitrary union, i.e.

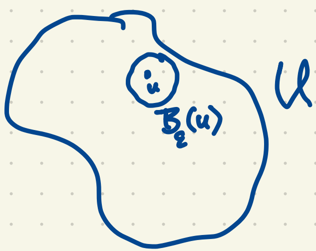
if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$;

if $\mathcal{U} \subseteq \mathcal{T}$ then $\bigcup \mathcal{U} \in \mathcal{T}$.

(So for $U, V \in \mathcal{T}$, $U \cup V \in \mathcal{T}$. If $\{U_\alpha : \alpha \in I\}$ is an indexed collection of open sets, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.)

<u>Example</u>

The <u>standard</u> topology on $\mathbb{R}^n$: $X = \mathbb{R}^n$. A set $U \subseteq \mathbb{R}^n$ is open (standard open set) if for all $u \in U$, there exists $\varepsilon > 0$ such that



$$B_\varepsilon(u) \subseteq U.$$

Here $B_\varepsilon(u) = \{ x \in \mathbb{R}^n : \underbrace{d(x, u)}_{\text{Euclidean distance}} < \varepsilon \}$.

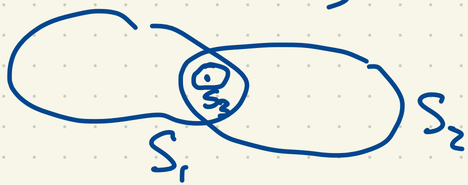
$$d(x, u) = \sqrt{(x_1 - u_1)^2 + \ldots + (x_n - u_n)^2}$$

(the open ε-ball centered at u).

In other words, a standard open set in $\mathbb{R}^n$ is a union of open balls.

Eg. (More generally) Let X be any set and let $\mathcal{S}$ be a collection of subsets of X which cover X, i.e. $\bigcup \mathcal{S} = X$. Then the collection of all unions of finite intersections $S_1 \cap S_2 \cap \cdots \cap S_k$, $S_1, \ldots, S_k \in \mathcal{S}$ is a topology on X. The members of $\mathcal{S}$ are called a sub-basis for this topology and the topology is said to be <u>generated</u> by $\mathcal{S}$.

$\mathcal{S}$ is called a <u>base</u> (or a <u>basis</u>) for the topology if the topology is the collection of arbitrary unions of elements of $\mathcal{S}$. This holds iff



for all $S_1, S_2 \in \mathcal{S}$,
and all $u \in S_1 \cap S_2$,
there exists $S_3 \in \mathcal{S}$ such that
$u \in S_3 \subseteq S_1 \cap S_2$.

<u>Eg.</u> Let X be any set. The <u>discrete topology</u> on X is the collection of <u>all</u> subsets of X. (2^X)
The <u>indiscrete</u> topology on X is $\{\emptyset, X\}$.

If $X = \{0, 1\}$ then there are four possible topologies on X: $\{\emptyset, X\}$, $\{\emptyset, \{0\}, \{1\}, X\}$, $\{\emptyset, \{0\}, X\}$, $\{\emptyset, \{1\}, X\}$.

Let X be an infinite set. Let $\mathcal{T}$ be the collection of complements of finite sets, and $\emptyset$

i.e. $\mathcal{T} = \{\emptyset\} \cup \{X \setminus A : A \subseteq X, |A| < \infty\}$, $X \setminus A = \{x \in X : x \notin A\}$.

set difference

This is a topology on X, called the <u>finite complement topology</u>.

$X - A$, $X \setminus A$, $X \backslash A$

$\emptyset$, ϕ, $\varnothing$, 0

\varnothing

A <u>topological space</u> is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on a set X.

Note: $\bigcup \mathcal{T} = X$. By abuse of language, we often say that X is a topological space.

Let X be a set. A <u>distance function</u> (or <u>metric</u>) on X is a function $d : X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

$d(x,y) = d(y,x)$

$d(x,y) \geq 0$ and equality holds iff $x = y$.

$d(x,z) \leq d(x,y) + d(y,z)$

The standard topology on $\mathbb{R}^n$ is a metric topology.

The metric $d_2(x,y) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$ (the Euclidean metric)

$d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

$d_\infty(x,y) = \max\{|x_1-y_1|, \dots, |x_n-y_n|\}$ all give the standard topology on $\mathbb{R}^n$.

In $\mathbb{R}^2$, open balls with respect to d_2, d_1, d_∞ look like

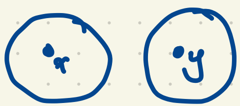
respectively. These three metrics define the same topology.

The metric $d(x,y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$ defines the discrete topology.

A topological space is **metrizable** if its topology can be given by some metric. (not uniquely however)

If X is an infinite set, then its finite complement topology is not metrizable.

A topology is **Hausdorff** if for any two points $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

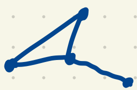


Every metric space is Hausdorff since if $x \neq y$, $d = d(x,y) > 0$. Take $U = B_{\delta/3}(x)$, $V = B_{\delta/3}(y)$

An **open neighbourhood** of a point $x \in X$ is an open set containing x.

nbhd

A **basic open nbhd** of a point $x \in X$ is an open nbhd of x which is basic (i.e. it's in the basis).



Even metric spaces can be rather surprising.
Consider $X = \mathbb{Q}$. A **norm** on $\mathbb{Q}$ is a function $\mathbb{Q} \to [0,\infty)$, $x \mapsto \|x\|$ satisfying

(i) $\|x\| \geq 0$; equality holds iff $x = 0$.

(ii) $\|xy\| = \|x\| \cdot \|y\|$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$.

From any norm on $\mathbb{Q}$, we obtain a metric $d(x,y) = \|x-y\|$.

One way to do this is with the usual absolute value $\|x\| = |x| = |x|_\infty = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0 \end{cases}$

This gives the standard topology on $\mathbb{Q}$.

An alternative is: given $x \in \mathbb{Q}$, if $x = 0$ define $\|0\|_2 = 0$.

If $x \neq 0$, write $x = 2^k \cdot \frac{a}{b}$, $a,b,k \in \mathbb{Z}$, $b \neq 0$; a,b odd. Then define $\|x\|_2 = 2^{-k}$.

This is the 2-adic norm on $\mathbb{Q}$. In fact it satisfies a stronger form of (iii), the ultrametric inequality $\|x+y\| \leq \max\{\|x\|, \|y\|\} \leq \|x\| + \|y\|$.

E.g. $\left\|\frac{20}{21}+\frac{5}{14}\right\|_2 = \left\|\frac{40+15}{42}\right\|_2 = \left\|\frac{55}{42}\right\|_2 = 2. = \max\left\{\underbrace{\left\|\frac{20}{21}\right\|_2}_{\frac{1}{4}}, \underbrace{\left\|\frac{5}{14}\right\|_2}_{2}\right\} = 2$

$\left\|\frac{20}{21}\right\|_2 = \frac{1}{4}$, $\left\|\frac{5}{14}\right\|_2 = 2$

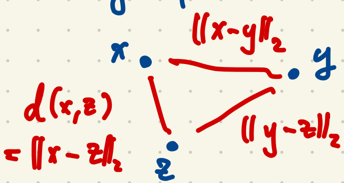
Compare: $\left\|\frac{20}{21}\right\|_2 + \left\|\frac{5}{14}\right\|_2 = 2\frac{1}{4} = 2.25.$

A basic open nbhd of zero looks like

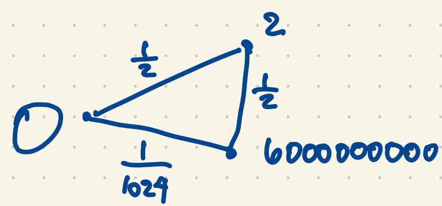
$$B_\varepsilon(0) = \{x \in \mathbb{Q} : \|x\|_2 < \varepsilon\}$$

$$B_1(0) = \{x \in \mathbb{Q} : \|x\|_2 < 1\} = \left\{\frac{a}{b} : a, b \in \mathbb{Z},\ a \text{ even},\ b \text{ odd}\right\}.$$

Every point in the ball is a centre of the ball i.e. if $c \in B_1(0)$ then $B_1(c) = B_1(0)$.



Then two of the sides of this triangle have the same length, i.e. the triangle is isosceles.



$$1+2+4+8+16+32+64+\cdots = -1$$

The partial sums $1, 3, 7, 15, 31, 63, \ldots$ converge to -1 in the 2-adic norm.

Note: If $(x_n)_n$ is a sequence of points in a top. space X, we say $(x_n)_n$ <u>converges</u> to $x \in X$ if for every open nbhd U of x, $x_n \in U$ for all n sufficiently large. (This means: for all U open nbhd of x, the exists N such that $x_n \in U$ whenever $n > N$.)

In place of arbitrary open nbhds of x, it suffices to check <u>basic</u> open nbhds. For metric topology, it suffices to check open balls. In this case, $x_n \to x$ provided that for all $\varepsilon > 0$, there exists N such that

$$\left.\begin{array}{l} x_n \in B_\varepsilon(x) \\ \text{i.e. } d(x_n, x) < \varepsilon \end{array}\right\} \text{ whenever } n > N.$$

In our example above, $d(x_n, x) = 2^n \to 0$ as $n \to \infty$.

$$\|2^n\| = \frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

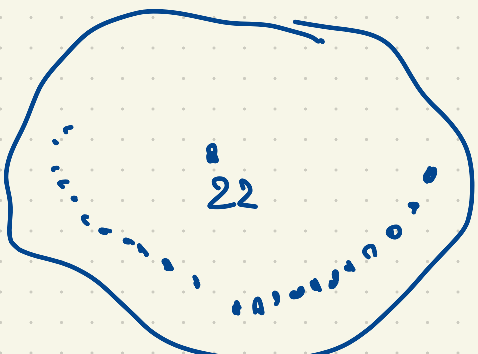
Find the inverse of 5 mod 64.

In $\mathbb{Z}/64\mathbb{Z}$,
$$\frac{1}{5} = \frac{1}{1+4} = 1 - 4 + 16 - \underbrace{64 + 256 - 1024 + \cdots}_{\text{zero}}$$
$$= 1 - 4 + 16$$
$$= 13.$$

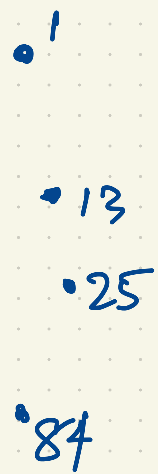
Eg. in $\mathbb{Z}$ with the finite complement topology, the sequence $(n)_n = (1, 2, 3, \ldots)$ converges. It converges to 22.

$$(n)_n \to 22.$$

In fact for every $a \in \mathbb{Z}$,
$$(a)_n \to a.$$



1, 13, 25, 84



Theorem If X is Hausdorff, then every sequence in X has at most one limit. (it converges to at most one point.)

Proof Suppose $a \neq b$ in a Hausdorff space X where a sequence $(x_n)_n \to a$ and $(x_n)_n \to b$. Choose disjoint open nbhds U, V of a, b respectively.

There exists N_1 such that $x_n \in U$ for all $n > N_1$; also N_2 such that $x_n \in V$ for all $n > N_2$.

then pick $n > \max\{N_1, N_2\}$ to obtain a contradiction.

We prefer to write $(x_n)_n \to a$ rather than $\lim_{n\to\infty} x_n = a$ in general.

In any top. space, closed sets are the complements of open sets.
$\emptyset, X$ are closed
If K, K' are closed then $K \cup K'$ is closed. (So finite unions of closed sets are closed.)
Arbitrary intersections of closed sets are closed.
De Morgan laws:

$$X \smallsetminus \Big(\bigcup_{\alpha\in I} A_\alpha\Big) = \bigcap_{\alpha\in I} (X \smallsetminus A_\alpha)$$

$$X \smallsetminus \Big(\bigcap_{\alpha\in I} A_\alpha\Big) = \bigcup_{\alpha\in I} (X \smallsetminus A_\alpha)$$

Given an infinite set X, the finite complement topology has as its closed sets the finite sets and X itself.

Let X be a top. space. Given $A \subseteq X$, the <u>closure</u> of A is the (unique) smallest closed set containing A i.e. $\overline{A} = \bigcap \{K \subseteq X : K \text{ closed}, \ K \supseteq A\}$.
The <u>interior</u> of A is the largest open set contained in A, <u>i.e.</u>
$A^\circ = \bigcup \{U \subseteq A : U \text{ open in } X\}$. $\quad (X \smallsetminus A)^\circ = X \smallsetminus \overline{A}$; $\quad \overline{X \smallsetminus A} = X \smallsetminus A^\circ$.

Theorem There are infinitely many primes.

Known proofs: Euclid's proof (elementary)

Euler's proof (analytic proof: $\sum \frac{1}{p}$ diverges)

This proof is topological.

Proof Form a topology on $X = \mathbb{Z}$ whose basic open sets are the (infinite) arithmetic progressions

$\ldots, -6, -1, 4, 9, 14, 19, \ldots$ for example.

Every nonempty open set is infinite.

Suppose there are only finitely many primes: $|P| < \infty$ is the set of all primes.

$$\{-1, 1\} = \{a \in \mathbb{Z} : a \text{ is not divisible by any prime}\}.$$

$$= \bigcap_{p \in P} \{a \in \mathbb{Z} : a \text{ is not divisible by } p\}$$

$$= \bigcap_{p \in P} \left(U_{1,p} \cup U_{2,p} \cup \cdots \cup U_{p-1,p} \right) \qquad U_{a,p} = \{x \in \mathbb{Z} : x \equiv a \mod p\}$$

is open. However it has only 2 elements, a contradiction. $\square$

More generally, let G be a group. Consider the topology on G whose basic open sets are cosets of subgroups $H \leq G$ of finite index, i.e. $gH = \{gh : h \in H\}$, $[G:H] < \infty$.

T_2 : Hausdorff

T_1 : Points are closed

If $x \in X$ and $y \neq x$, then there is an open nbhd U of x with $y \notin U$.

$T_2 \Rightarrow T_1$. Exercise: Give an example of a top. space which is T_1 but not T_2.

One answer: the finite complement topology for an infinite set.

Let $f : X \to Y$ be any function. For any $B \subseteq Y$, the preimage of B in X under f is $f^{-1}(B) = \{x \in X : f(x) \in B\}$. Similarly if $A \subseteq X$, the image of A in Y is $f(A) = \{f(a) : a \in A\}$. In general

$$f(f^{-1}(A)) \subseteq A \subseteq f^{-1}(f(A)).$$

Now let X and Y be top. spaces, i.e. $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$.

A function $f : X \to Y$ is <u>continuous</u> if the preimage of every open set (in Y) is open (in X); i.e. for every $U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

Exercise: Convince yourself that the "standard" definition of continuity for functions $\mathbb{R}^m \to \mathbb{R}^n$ is just a special case of this. (for the standard topologies on $\mathbb{R}^m$ and $\mathbb{R}^n$).

Theorem If $f: X \to Y$ and $g: Y \to Z$ are continuous, so is $g\circ f: X \to Z$.

Proof If $U \subseteq Z$ is open then $g^{-1}(U) \subseteq Y$ is open so $\underbrace{f^{-1}(g^{-1}(U))}_{=(g\circ f)^{-1}(U)} \subseteq X$ is open. $\square$

When are two topological spaces X, Y "the same"? ($X \simeq Y$: X, Y are homeomorphic)
This means there is a bijection $X \to Y$ taking one topology to the other.
I.e. there is a bijection $f: X \to Y$ such that f, f^{-1} are continuous.

Eg. X is $\mathbb{R}$ with the standard topology;
Y is $\mathbb{R}$ with the finite complement topology;
Z is $\mathbb{R}$ with the discrete topology;
W is $\mathbb{R}$ with the indiscrete topology $\{\emptyset, \mathbb{R}\}$.

$$\underset{\substack{\wedge \\ \text{finest} \\ \text{topology} \\ \text{on } \mathbb{R}}}{Z} \xrightarrow{\iota} X \xrightarrow{\iota} Y \xrightarrow{\iota} \underset{\substack{\wedge \\ \text{coarsest} \\ \text{topology} \\ \text{on } \mathbb{R}}}{W}$$

where $\iota(x) = x$.

If $\mathcal{T}, \mathcal{T}'$ are two topologies on X, we say
$\mathcal{T}'$ is <u>finer than</u> $\mathcal{T}$ if $\mathcal{T}' \supset \mathcal{T}$
($\mathcal{T}'$ is a <u>refinement</u> of $\mathcal{T}$)
$\mathcal{T}'$ is coarser than $\mathcal{T}$ if $\mathcal{T}' \subset \mathcal{T}$.

Eg. The finite complement topology on X is the coarsest topology for which points are closed.

i.e. any topology in which points are closed is a refinement of the finite complement topology.

Subspace Topology

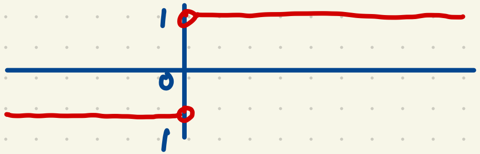
Let $A \subseteq X$ where X is a topological space $X = (X, \mathcal{T})$.
The topology A inherits from X in the most natural way is the subspace topology $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$.

Eg. $[0,1) = \{a \in \mathbb{R} : 0 \leq a < 1\}$ is neither open nor closed in $\mathbb{R}$ but it is closed in $[0,1]$ and in $[0,\infty)$ since

$$[0,1) = (-1,1) \cap [0,1] = (-1,1) \cap [0,\infty).$$

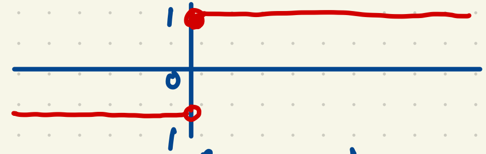
If $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$, we say f is continuous if it is continuous relative to the standard topology of $\mathbb{R}^m$ and the subspace topology on $A \subseteq \mathbb{R}^n$.

Eg.



continuous

$f: (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$



not continuous

$f: \mathbb{R} \to \mathbb{R}$

If $f: A \to \mathbb{R}^m$ has $f(A) \subseteq B$ we might as well think of f as $f: A \to B$. To say $f: A \to \mathbb{R}^m$ is continuous is equivalent to saying $f: A \to B$ is continuous.

Suppose $f: A \to B$ is continuous and let $U \subseteq \mathbb{R}^m$. Then $f^{-1}(U) = f^{-1}(U \cap B)$ is open in A. Similarly one proves the converse.

Given $A \subseteq X$ where X is a top. space, there is an inclusion map $\iota: A \to X$, $\iota(a) = a$. (one-to-one; not onto in general).

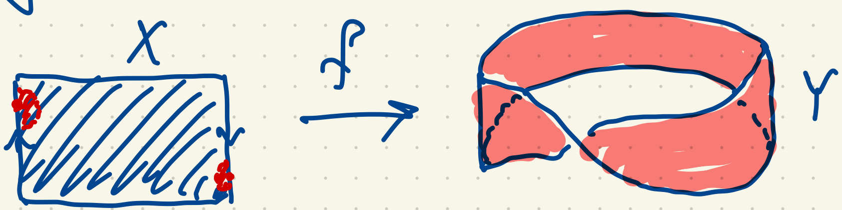
The subspace topology on A is the coarsest topology for which the inclusion map ι is continuous.

Given $U \subseteq X$ open. $i^{-1}(U) = U \cap A$.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

<u>Quotient Topology</u> Suppose $f: X \to Y$ is onto. Given a topology on $X = (X, \mathcal{T})$, the most natural way this gives a topology on Y is by taking the finest topology on Y for which f is continuous.

A Möbius strip



There are three ways to think of this situation.

(i) Identify (collapse) certain points of X together.

(ii) We have an equivalence relation on X.

(iii) A partition of X.

The quotient topology on Y is the finest topology on Y for which the map $f: X \to Y$ is continuous.

The quotient topology on $Y = X/f$ or $X/\sim$ is $\{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$.