University of Wyoming — Math 5600—Fall 2022 — Department of Mathematics

Point-Set Topology

HW1

(Due 5:00pm Wednesday, October 12, 2022 *on WyoCourses*)

Instructions: See the course syllabus for general expectations regarding homework.

The ordinal $\omega = \{0,1,2,3,\ldots\}$ has the usual total ordering '$<$'. For each $k = 1,2,3,\ldots$ we consider the cartesian product

$$\omega^k = \{(\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_{k-1}) : \alpha_0,\ldots,\alpha_{k-1} \in \omega\}$$

which is totally ordered by lex (lexicographic or dictionary) order; thus

$$\begin{aligned}(\alpha_0,\alpha_1,\ldots,\alpha_{k-1}) \leqslant (\beta_0,\beta_1,\ldots,\beta_{k-1}) \quad \text{iff} \quad & \alpha_0<\beta_0 \qquad \text{or}\\ & \alpha_0=\beta_0 \text{ and } \alpha_1<\beta_1 \qquad \text{or}\\ & (\alpha_0,\alpha_1)=(\beta_0,\beta_1) \text{ and } \alpha_2<\beta_2 \qquad \text{or}\\ & \quad\vdots \qquad \text{or}\\ & (\alpha_0,\alpha_1,\ldots,\alpha_{k-2})=(\beta_0,\beta_1,\ldots,\beta_{k-2}) \text{ and } \alpha_{k-1}<\beta_{k-1}.\end{aligned}$$

In other words, compare two sequences by first comparing first term, then comparing second term, etc. just as you would for words in a dictionary.

1. (*10 points*) Show that the Cartesian power ω^k (with lex order) is well ordered.

Although the Cartesian power ω^k is distinct from the ordinal ω^k, they are order-isomorphic (i.e. there is a one-to-one correspondence between them which preserves order), so for the time being we will identify them. Recall that ω^2 is topologically embeddable in $\mathbb{R}$, i.e. it is homeomorphic to a subspace of $\mathbb{R}$, for instance $\{m - \frac{1}{n} : m, n \text{ are positive integers}\}$.

2. (*20 points*) Show that ω^k (with the order topology) is homeomorphic to a subspace of $\mathbb{R}$ (with the standard topology).

It is natural to identify ω^k with a subset of ω^ℓ whenever $k < \ell$, by appending $\ell - k$ zeroes on the end of each sequence in ω^k. In this case ω^k is a topological subspace of ω^ℓ. We now have an increasing chain

$$\omega^1 \subset \omega^2 \subset \omega^3 \subset \cdots$$

so we can take the limit (or union)

$$\omega^\omega = \lim_{k\to\infty} \omega^k = \bigcup_{k=1}^{\infty} \omega^k$$

which is the set of all infinite sequences $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ such that each $\alpha_i \in \omega$, and $\alpha_i = 0$ for i sufficiently large (i.e. we only allow a finite number of nonzero entries in each sequence (α_i)). Since ω^ω is a countable union of countable sets, ω^ω is itself countable. Again, ω is totally ordered by lex.

3. (*10 points*) Show that ω^ω is well ordered by lex.

Once again, our lex-ordered set ω^ω is order-isomorphic with the ordinal ω^ω, so we may identify them for now.

4. (*20 points*) Show that ω^ω (with the order topology) is homeomorphic to a subspace of $\mathbb{R}$.

More generally, any countable ordinal (with the order topology) is homeomorphic to a subspace of $\mathbb{R}$. However

5. (*20 points*) Let λ be an uncountable ordinal. (Equivalently, you may take λ to be any uncountable well ordered set, since this is order-isomorphic to an ordinal.) Show that λ, with the order topology, is *not* homeomorphic to a subspace of $\mathbb{R}$.

Thus, for example, the first uncountable ordinal ω_1 (with the order topology) is not homeomorphic to a subspace of $\mathbb{R}$, despite the fact that $|\omega_1| \leqslant |\mathbb{R}|$.